

Math 135, Section C7

Solutions review problems for Exam #2 - July 7, 2010

#1 Find  $\frac{dy}{dx}$  and if  $2x + e^{xy} = 0$ .

**Solution:** Differentiating both sides of the equation with respect to  $x$  gives:

$$2 + e^{xy}(y + x\frac{dy}{dx}) = 0.$$

Then solving for  $\frac{dy}{dx}$  gives

$$\frac{dy}{dx} = -\frac{2 + ye^{xy}}{xe^{xy}}.$$

#2 Find an equation of the tangent line to the graph of  $x^2y - 2xy^3 = 0$  at the point  $(2, 1)$ .

**Solution:** We first find  $\frac{dy}{dx}$  by implicit differentiation. Differentiating both sides of the equation with respect to  $x$  gives:

$$2xy + x^2\frac{dy}{dx} - 2y^3 - 6xy^2\frac{dy}{dx} = 0.$$

Then solving for  $\frac{dy}{dx}$  gives

$$\frac{dy}{dx} = \frac{2y^3 - 2xy}{x^2 - 6xy^2}.$$

Therefore, at the point  $(2, 1)$ , the value of  $\frac{dy}{dx}$  is  $\frac{1}{4}$ . Then, using the point slope form of the equation for a straight line we see that

$$(y - 1) = \left(\frac{1}{4}\right)(x - 2)$$

is an equation of the tangent line.

#3 Find  $\frac{dy}{dx}$  at the point  $(2, 1)$  on the graph of  $(x - y)^3 + y^2 = 2$ .

**Solution:** Differentiating both sides of the equation with respect to  $x$  gives

$$3(x - y)^2\left(1 - \frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0.$$

Setting  $x = 2, y = 1$  in gives

$$3 - \frac{dy}{dx} + 2\frac{dy}{dx} = 0$$

and so

$$\frac{dy}{dx} = 3.$$

#4 Find the derivative of the function  $(2x + 1)^{(3x+1)}$  (where  $x > 0$ ).

**Solution:** Set  $y = (2x + 1)^{(3x+1)}$ . Then  $\ln(y) = (3x + 1)\ln(2x + 1)$  and so

$$\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = 3\ln(2x + 1) + 2\left(\frac{3x + 1}{2x + 1}\right).$$

Multiplying both sides by  $y$  and substituting  $(2x + 1)^{(3x+1)}$  for  $y$  gives

$$\frac{dy}{dx} = (2x + 1)^{(3x+1)}\left(3\ln(2x + 1) + 2\left(\frac{3x + 1}{2x + 1}\right)\right).$$

\$5 If  $y = 2\sqrt{x} - 9$  and  $\frac{dy}{dt} = 5$  find  $\frac{dx}{dt}$  when  $x = 9$ .

**Solution:** Differentiating both sides with respect to  $t$  gives

$$\frac{dy}{dt} = x^{-\frac{1}{2}} \frac{dx}{dt}.$$

Setting  $x = 9$ ,  $\frac{dy}{dt} = 5$  in this equation gives

$$5 = \left(\frac{1}{3}\right) \frac{dx}{dt}$$

and so

$$\frac{dx}{dt} = 15.$$

#6 One end of a rope is fastened to a boat and the other end is wound around a windlass located on a dock at a point 5 feet above the level of the boat. If the boat is drifting away from the dock at the rate of 7 feet/minute, how fast is the rope unwinding at the instant when the length of the rope is 13 feet?

**Solution:** Let  $x$  be the distance from the boat to the dock and  $l$  be the length of the rope. Then, as shown in the diagram, we have a right triangle with sides 5 and  $x$  and hypotenuse  $l$ . By the Pythagorean Theorem

$$l^2 = 5^2 + x^2$$

and so

$$2l \frac{dl}{dt} = 2x \frac{dx}{dt}.$$

Now we are given  $\frac{dx}{dt} = 7$  and we want  $\frac{dl}{dt}$  when  $l = 13$ . Note that when  $l = 13$  we have  $x^2 + 5^2 = (13)^2$  and so  $x^2 = 169 - 25 = 144 = (12)^2$ . Thus when  $l = 13$  we have  $x = 12$  and so

$$26 \frac{dl}{dt} = (24)(7).$$

Hence

$$\frac{dl}{dt} = 7 \left( \frac{24}{26} \right) = \frac{84}{13}.$$

#7 A car is travels north from the city of Centralia at the rate of 30 miles per hour, starting at 11 AM. A truck travels east from Centralia at the rate of 45 miles per hour, starting at noon. How fast is the distance between the truck and the car changing at 1 PM?

**Solution:** Let  $x$  be the distance from the truck to Centralia. Thus  $\frac{dx}{dt} = 45$ . Let  $y$  be the distance from the car to Centralia. Thus  $\frac{dy}{dt} = 30$ . Let  $D$  be the distance between the truck and the car. Thus we want  $\frac{dD}{dt}$ . Now

$$D^2 = x^2 + y^2$$

and so

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.$$

Also, at 1 PM we have  $x = 45$  and  $y = 60$ . Therefore, at 1 PM,

$$D = \sqrt{(45)^2 + (60)^2} = \sqrt{(15)^2(3^2 + 4^2)} = \sqrt{(15)^2(5)^2} = 75.$$

Thus, at 1 PM,

$$2(75) \left( \frac{dD}{dt} \right) = 2(45)(45) + 2(60)(30) = 7650.$$

Thus  $\frac{dD}{dt} = 51$  miles per hour.

#8 Find  $d(x\sqrt{x^2 - 1})$ .

**Solution:** In general,  $df = \left(\frac{df}{dt}\right)dt$ , so, in this case

$$d(x\sqrt{x^2 - 1}) = ((x^2 - 1)^{1/2} + x(x^2 - 1)^{-1/2}(2x))dx = ((x^2 - 1)^{1/2} + x^2(x^2 - 1)^{-1/2})dx.$$

#9 Use differentials to approximate  $\sqrt{9.04}$ .

**Solution:** We use the approximation  $f(x_0 + dx) \approx f(x_0) + f'(x_0)dx$ . Here  $f(x) = \sqrt{x}$ ,  $x_0 = 9$ ,  $dx = .04$ . Then  $f'(x) = (1/2)x^{-1/2}$  and so

$$\sqrt{9.04} \approx \sqrt{9} + (1/2)9^{-1/2}(.04) = 3 + \frac{1}{3}(.02) = 3.0066\dots$$

#10 The radius of a circle has been measured as 15 inches, but there is a possible error of 0.05 inch in the measurement. Give an approximate value for the error in the computed area.

**Solution:** Let  $A$  be the area of the circle and  $r$  be its radius. Then  $A = \pi r^2$ . We are asked for to approximate  $\Delta A$  when  $A = 15$  and  $\Delta r = dr = \pm 0.05$ . Then

$$\Delta A \approx dA = \left(\frac{dA}{dr}\right)(dr) = 2\pi(15)(\pm 0.05) = \pm(1.5)\pi.$$

#11 For each of the following functions:

- (i) find all critical numbers;
- (ii) find the intervals where the function is increasing;
- (iii) find the intervals where the function is decreasing;
- (iv) determine whether each critical point is a relative maximum, a relative minimum, or neither;
- (v) find the intervals where the graph of the function is concave up and the intervals where the graph of the function is concave down ;
- (vi) find all points of inflection;
- (vii) find all horizontal and vertical asymptotes (realizing that there may be none);
- (viii) sketch the graph of the function.

(a)  $f(x) = x - x^2$ ;

**Solution:** (i)  $f'(x) = 1 - 2x$  so the only critical number is  $1/2$ .

(ii)  $f'(x) > 0$  if  $x < 1/2$  so the function is increasing on the interval  $(-\infty, 1/2]$ .

(iii)  $f'(x) < 0$  if  $x > 1/2$  so the function is decreasing on the interval  $[1/2, \infty)$ .

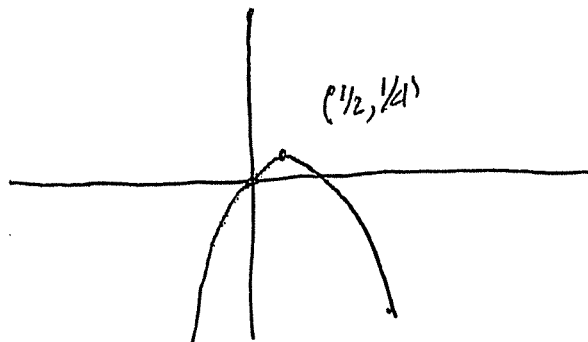
(iv) Since  $f(x)$  is increasing to the left of  $x = 1/2$  and decreasing to the right of  $x = 1/2$ , the first derivative test shows that there is a relative maximum at  $x = 1/2$ . Alternatively,  $f''(x) = -2$  and in particular  $f''(1/2) = -2 < 0$  so the second derivative test also shows that there is a relative maximum at  $x = 1/2$ . Note that the maximum value of  $f(x)$  is  $f(1/2) = (1/2) - (1/2)^2 = 1/4$

(v) Since  $f''(x) = -2 < 0$  for all  $x$ , the graph of the function is always concave down.

(vi) There are no points of inflection.

(vii) There are no horizontal or vertical asymptotes.

(viii)



(b)  $f(x) = x^3 - 3x^2 + 3$ ;

**Solution:** (i)  $f'(x) = 3x^2 - 6x = 3x(x - 2)$ . so the only critical numbers are 0, 2.

(ii) If  $x < 0$  or  $2 < x$  then  $f'(x) > 0$ , so the function is increasing on the intervals  $(-\infty, 0]$  and  $[2, \infty)$ .

(iii) If  $0 < x < 2$  then  $f'(x) < 0$  so the function is decreasing on the interval  $[0, 2]$ .

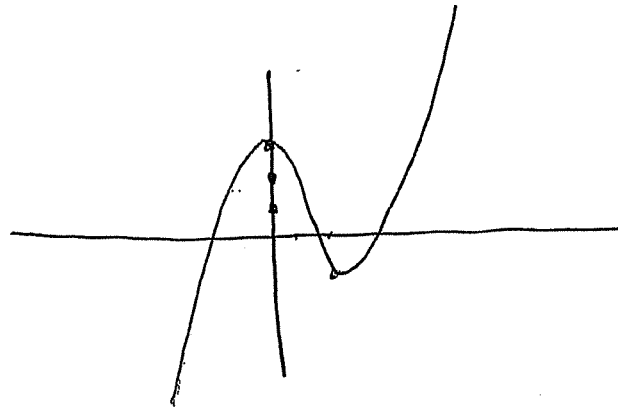
(iv) Since  $f(x)$  is increasing to the left of  $x = 0$  and decreasing at  $x$  when  $x$  close to 0 and to the right of 0, the first derivative test shows that there is a relative maximum at  $x = 0$ . Note that the value of the function at this point is  $f(0) = 3$ . Since  $f(x)$  is decreasing when  $x$  is close to 2 and to the left of 2, and is increasing to the right of  $x = 2$ , the first derivative test shows that there is a relative minimum at  $x = 2$ . Note that the value of the function at this point is  $f(2) = -1$ . Alternatively,  $f''(x) = 6x - 6$ . Then  $f''(0) = -6 < 0$  so the second derivative test also shows that there is a relative maximum at  $x = 0$ . Also  $f''(2) = 6 > 0$  so the second derivative test shows that there is a relative minimum at  $x = 2$ .

(v) Since  $f''(x) = 6x - 6$ ,  $f''(x) < 0$  for all  $x < 1$  and  $f''(x) > 0$  for all  $x > 1$ . Thus the graph of the function is concave down on  $(-\infty, 1)$  and concave up on  $(1, \infty)$ .

(vi) The only point of inflection is  $(1, 1)$ .

(vii) There are no horizontal or vertical asymptotes.

(viii)



$$(c) f(x) = \frac{1}{3-x};$$

**Solution:** (i)  $f'(x) = (3-x)^{-2}$ . Thus  $f'(x) > 0$  for all  $x \neq 3$ . Since 3 is not in the domain of  $f(x)$  there are no critical numbers.

(ii)  $f'(x) > 0$  if  $x < 3$  or if  $x > 3$ . Thus the function is increasing on the intervals  $(-\infty, 3)$  and  $(3, \infty)$ .

(iii)  $f'(x)$  is never negative, so the function is never decreasing.

(iv) Since there are no critical numbers, there can be no relative maximum or relative minimum.

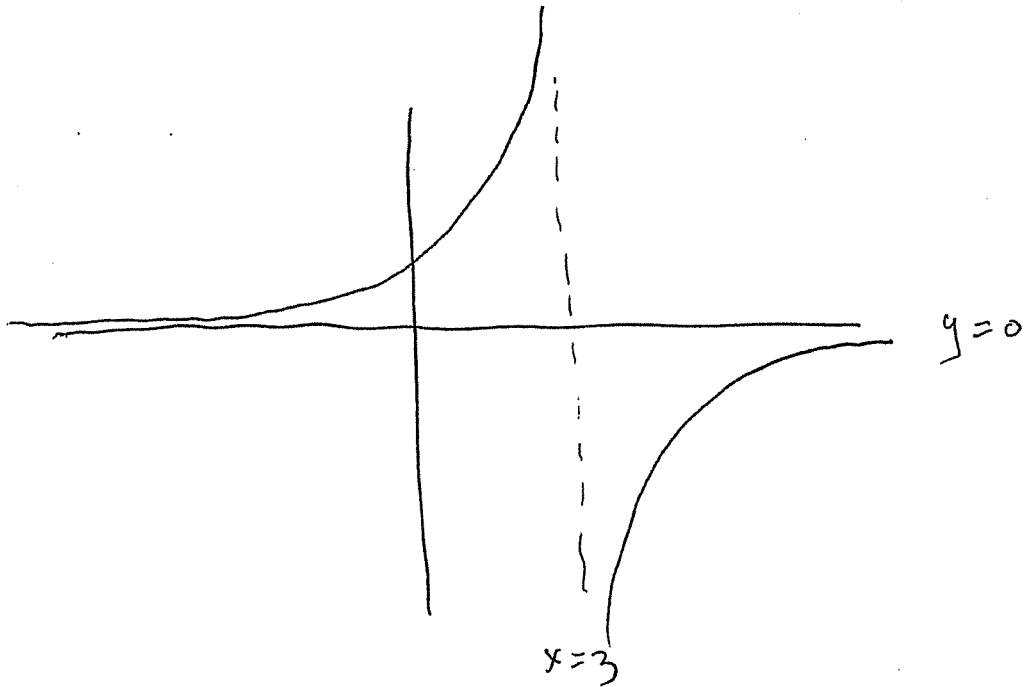
(v)  $f''(x) = 2(3-x)^{-3}$ . Thus  $f''(x) > 0$  if  $x < 3$  and  $f''(x) < 0$  if  $x > 3$ . Thus the graph of the function is concave up on  $(-\infty, 3)$  and is concave down on  $(3, \infty)$ .

(vi) The only place where the graph changes concavity is at  $x = 3$ . But 3 is not in the domain of  $f(x)$ . Thus there is no point of inflection.

(vii)  $\lim_{x \rightarrow -\infty} f(x) = 0 = \lim_{x \rightarrow \infty} f(x)$ . So there is one horizontal asymptote:  $y = 0$ .

Also  $\lim_{x \rightarrow 3^-} f(x) = \infty$  and  $\lim_{x \rightarrow 3^+} f(x) = -\infty$ . Thus there is one vertical asymptote,  $x = 3$ .

(viii)





$$(d) f(x) = \frac{x+1}{2-x};$$

**Solution:** (i) Using the quotient rule we see that  $f'(x) = \frac{(1(2-x) - (x+1)(-1))}{(2-x)^2} = 3(2-x)^{-2}$ . Thus  $f'(x) > 0$  for all  $x \neq 2$ . Since 2 is not in the domain of  $f(x)$  there are no critical numbers.

(ii)  $f'(x) > 0$  if  $x < 2$  or if  $x > 2$ . Thus the function is increasing on the intervals  $(-\infty, 2)$  and  $(2, \infty)$ .

(iii)  $f'(x)$  is never negative, so the function is never decreasing.

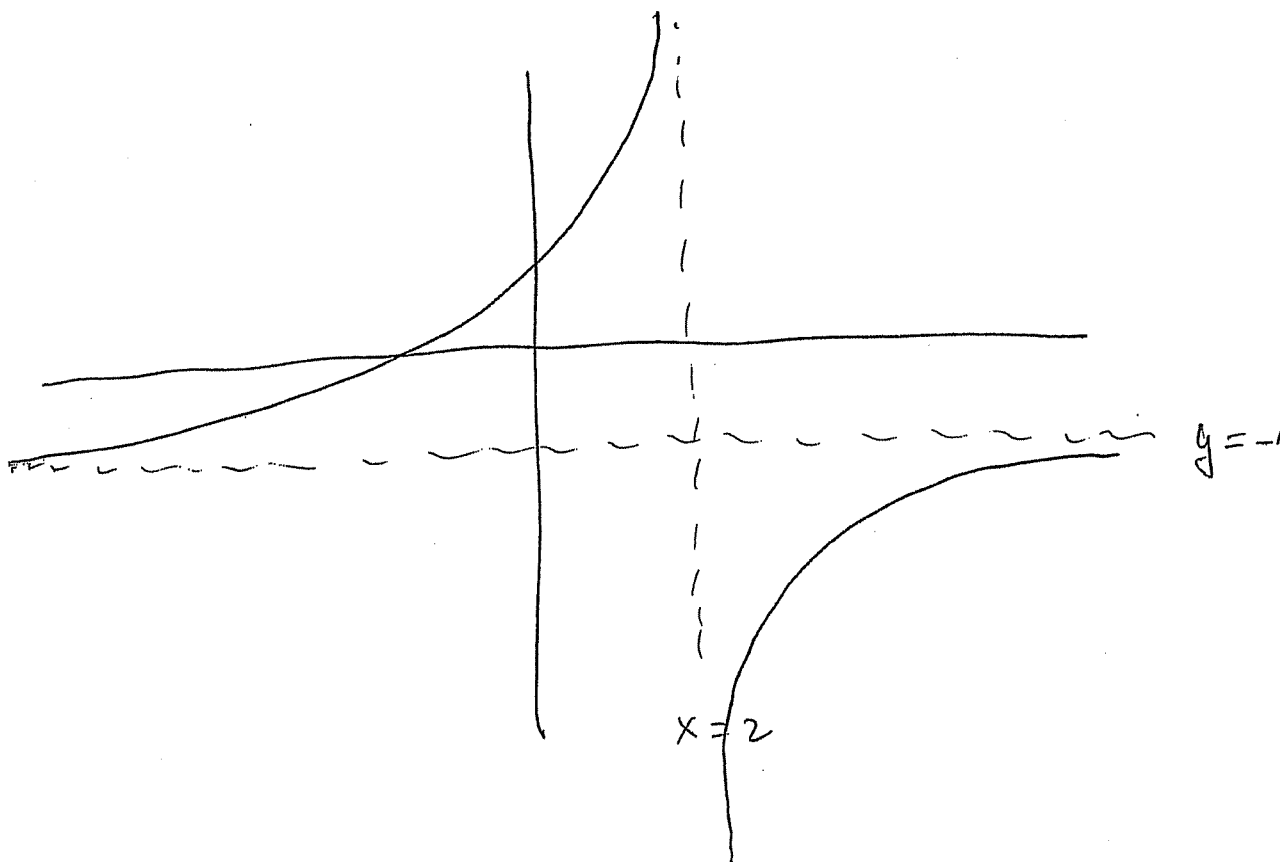
(iv) Since there are no critical numbers, there can be no relative maximum or relative minimum.

(v)  $f''(x) = 6(2-x)^{-3}$ . Thus  $f''(x) > 0$  if  $x < 2$  and  $f''(x) < 0$  if  $x > 2$ . Thus the graph of the function is concave up on  $(-\infty, 2)$  and is concave down on  $(2, \infty)$ .

(vi) The only place where the graph changes concavity is at  $x = 2$ . But 2 is not in the domain of  $f(x)$ . Thus there is no point of inflection.

(vii)  $\lim_{x \rightarrow -\infty} f(x) = -1 = \lim_{x \rightarrow \infty} f(x)$ . So there is one horizontal asymptote:  $y = -1$ . Also  $\lim_{x \rightarrow 2^-} f(x) = \infty$  and  $\lim_{x \rightarrow 2^+} f(x) = -\infty$ . Thus there is one vertical asymptote,  $x = 2$ .

(viii)



#12 Sketch the graph of a function satisfying the following conditions:

$$\lim_{x \rightarrow -\infty} f(x) = 1,$$

$$\lim_{x \rightarrow \infty} f(x) = -1,$$

$$\lim_{x \rightarrow 1^-} f(x) = -\infty,$$

$$\lim_{x \rightarrow 1^+} f(x) = \infty,$$

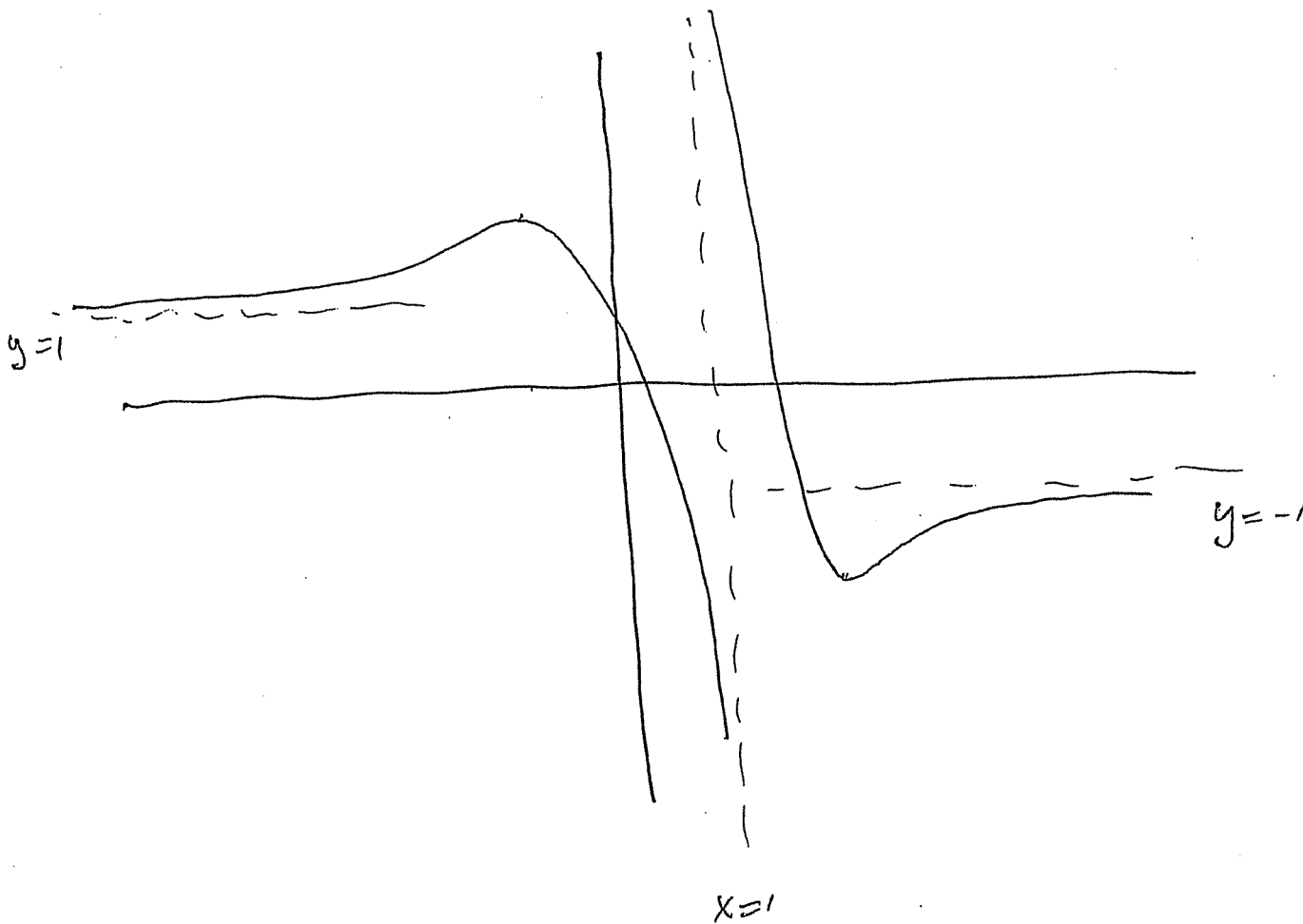
$$f'(x) > 0 \text{ if } x < -1, \text{ or if } 3 < x,$$

$$f'(x) < 0 \text{ if } -1 < x < 1, \text{ or if } 1 < x < 3,$$

$$f''(x) > 0 \text{ if } x < -3 \text{ or if } 1 < x < 4,$$

$$f''(x) < 0 \text{ if } -3 < x < 1 \text{ or if } 4 < x.$$

**Solution:** We see that there are two horizontal asymptotes:  $y = 1$  (to the left) and  $y = -1$  (to the right). Also, there is one vertical asymptote:  $x = 1$ . The function is increasing on  $(-\infty, -1]$  and on  $[3, \infty)$ . It is decreasing on  $[-1, 1)$  and on  $(1, 3]$ . The function is concave up on  $(-\infty, -3)$  and on  $(1, 4)$ . It is concave down on  $(-3, 1)$  and on  $(4, \infty)$ .



#13 Find the absolute maximum and minimum values of the following functions on the given intervals. Give the values of  $x$  for which the absolute maximum and absolute minimum are attained.

(a)  $f(x) = x^2 - 6x + 1$ , on  $[1, 4]$ ,

**Solution:**  $f'(x) = 2x - 6$ , so the only critical number is  $x = 3$ . Thus the only possible  $x$  values for an absolute extremum are 1, 3, 4 (the endpoints and the critical numbers). Since  $f(1) = -4$ ,  $f(3) = -8$ ,  $f(4) = -7$ , we see that the absolute maximum is  $-4$  at  $x = 1$  and the absolute minimum is  $-8$  at  $x = 3$ .

(b)  $f(x) = \frac{x^3}{3} - x^2 + 1$  on  $[-3, 3]$ ,

**Solution:**  $f'(x) = x^2 - 2x$ , so the only critical numbers are  $x = 0, 2$ . Thus the only possible  $x$  values for an absolute extremum are  $-3, 0, 2, 3$  (the endpoints and the critical numbers). Since  $f(-3) = -17$ ,  $f(0) = 1$ ,  $f(2) = -1/3$ ,  $f(3) = 1$ , we see that the absolute maximum is 1 at  $x = 0$  and  $x = 3$  and the absolute minimum is  $-17$  at  $x = -3$ .

(c)  $f(x) = |2x - 1|$  on  $[0, 2\pi]$ .

**Solution:** Note that if  $x > 1/2$  then  $f(x) = 2x - 1$  and so  $f'(x) = 2 > 0$ . Also, if  $x < 1/2$ , then  $f(x) = -(2x - 1)$  so  $f'(x) = -2 < 0$ . Finally,  $\lim_{x \rightarrow \frac{1}{2}^+} f(x) = \lim_{x \rightarrow \frac{1}{2}^+} \frac{2x-1}{x-1/2} = 2$  and  $\lim_{x \rightarrow \frac{1}{2}^-} f(x) = \lim_{x \rightarrow \frac{1}{2}^-} \frac{-2x+1}{x-1/2} = -2$ . Thus  $\lim_{x \rightarrow \frac{1}{2}} f(x)$  does not exist and so  $f(x)$  is not differentiable at  $x = 1/2$ . Thus  $x = 1/2$  is the only critical number. Therefore the only possible  $x$  values for an absolute extremum are  $0, 1/2, 2\pi$  (the endpoints and the critical numbers). Since  $f(0) = 1$ ,  $f(1/2) = 0$ ,  $f(2\pi) = 2\pi - 1$  we see that the absolute maximum is  $2\pi - 1$  at  $x = 2\pi$  and the absolute minimum is 0 at  $x = 1/2$ .

(d)  $f(x) = \sin^2(x) + \cos(x)$  on  $[0, \pi]$ .

**Solution:**  $f'(x) = 2\sin(x)\cos(x) - \sin(x) = (\sin(x)(2\cos(x) - 1))$ . Thus the only critical numbers are the values of  $x$  for which  $\sin(x) = 0$  (which are integer multiples of  $\pi$  and the values of  $x$  for which  $\cos(x) = 1/2$  (which are numbers of the form  $\pi/3$  plus a multiple of  $2\pi$  or  $2\pi/3$  plus a multiple of  $2\pi$ ). The only numbers of this form in the interval  $[0, \pi]$  are 0 and  $\pi/3$ . Thus the only possible  $x$  values for an absolute extremum are  $0, \pi/3, \pi$  (the endpoints and the critical numbers). Since  $f(0) = 1$ ,  $f(\pi/3) = 5/4$ ,  $f(\pi) = -1$  we see that the absolute maximum is  $5/4$  at  $x = \pi/3$  and the absolute minimum is  $-1$  at  $x = \pi$ .

#14 Find the value of each of the following limits:

(a)  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$ ,

**Solution:** By l'Hopital's rule, this is  $\lim_{x \rightarrow 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1} = \frac{1}{2}$ .

(b)  $\lim_{x \rightarrow 0} \frac{x - \sin(x)}{x^3}$ ,

**Solution:** By l'Hopital's rule, this is  $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{3x^2}$ . Using l'Hopital's rule again, this is  $\lim_{x \rightarrow 0} \frac{\sin(x)}{6x}$ . One more application of l'Hopital's rule gives  $\lim_{x \rightarrow 0} \frac{\cos(x)}{6} = 1/6$ .

$$(c) \lim_{x \rightarrow 0^+} x^{-5} \ln(x),$$

**Solution:** As  $x$  goes to  $\infty$  the first factor goes to  $\infty$  and the second goes to  $-\infty$ . Thus the limit does not exist (as a finite number). Note that this problem does not require l'Hopital's rule.

$$(d) \lim_{x \rightarrow \infty} \frac{\ln(\ln(x))}{x},$$

**Solution:** By l'Hopital's rule this is  $\lim_{x \rightarrow \infty} \frac{\frac{1}{x \ln(x)}}{1} = 0$ .

$$(e) \lim_{x \rightarrow 0} \left( \frac{1}{\sin(3x)} - \frac{1}{3x} \right),$$

**Solution:** First add the two fractions to get

$$\lim_{x \rightarrow 0} \left( \frac{3x - \sin(3x)}{3x(\sin(3x))} \right).$$

By l'Hopital's rule this is

$$\lim_{x \rightarrow 0} \left( \frac{3 - 3\cos(3x)}{3\sin(3x) + 9x\cos(3x)} \right).$$

Using l'Hopital's rule again, this is

$$\lim_{x \rightarrow 0} \left( \frac{9\sin(3x)}{9(\cos(3x) + 9\cos(3x) - 27x\sin(3x))} \right) = 0.$$

$$(f) \lim_{x \rightarrow 0^+} \left( \frac{2\cos(x)}{\sin(2x)} - \frac{1}{x} \right).$$

**Solution:** First add the two fractions to get

$$\lim_{x \rightarrow 0} \left( \frac{2x\cos(x) - \sin(2x)}{x(\sin(2x))} \right).$$

By l'Hopital's rule this is

$$\lim_{x \rightarrow 0} \frac{2\cos(x) - 2x\sin(x) - 2\cos(2x)}{(\sin(2x) + 2x\cos(2x))}.$$

Using l'Hopital's rule again, this is

$$\lim_{x \rightarrow 0} \left( \frac{2\sin(x) - 2\sin(x) - 2x\cos(x) + 4\sin(2x)}{2(\cos(2x) + 2\cos(2x) - 4x\sin(2x))} \right) = 0.$$

#15 A cylindrical can is to have volume  $108\pi$  cubic inches. If the material used to make the top and bottom of the can costs twice as much as the material used to make the side, what are the dimensions of the can that is least expensive to produce.

**Solution:** Let  $h$  be the height of the can and let  $r$  be the radius of the top. Then the volume of the can is  $\pi r^2 h$  so

$$r^2 h = 108$$

so

$$h = 108/r^2.$$

Now the area of the side of the can is  $2\pi r h$  and the area of the top (or of the bottom) is  $\pi r^2$ . If the material used for the side costs  $C$  cents per square inch then the material used for the top and bottom costs  $2C$  cents per square inch. Thus the cost of the side is  $2\pi r h C$ , the cost of the top is  $\pi r^2(2C)$  and the cost of the bottom is the same. Thus the total cost is

$$K = 2\pi r h C + 4\pi r^2 C = 2\pi C(rh + 2r^2).$$

Substituting the value  $108/r^2$  for  $h$  we obtain

$$K = 2\pi C(2r^2 + 108r^{-1}).$$

Now

$$\frac{dK}{dr} = 2\pi C(4r - 108r^{-2})$$

and so when  $K$  takes its minimum value we must have

$$0 = 4r - 108r^{-2}.$$

Thus

$$r^3 = 27$$

and so

$$r = 3.$$

#16 A rectangular box (without a top) is to be made from a sheet of cardboard with width 5 inches and length 8 inches by cutting squares of equal size out of each corner of the sheet of cardboard and folding up the resulting flaps to make the sides of the box. Find the dimensions of the box of largest possible volume.

**Solution:** Let the squares to be cut out of each corner of the sheet of cardboard have side  $x$ . Note that we must have  $0 \leq x \leq 5/2$  (since  $x$  can be at most one-half of the width). Then the volume of the box is

$$V = (8 - 2x)(5 - 2x)(x) = 4x^3 - 26x^2 + 40x.$$

Then

$$\frac{dV}{dx} = 12x^2 - 52x + 40 = 4(3x^2 - 13x + 10) = 4(x - 1)(3x - 10).$$

If  $\frac{dV}{dx} = 0$  we have  $x = 1$  or  $x = 10/3$ . But  $10/3 > 5/2$ , so  $x$  cannot equal  $10/3$ . Thus we must have  $x = 1$ .