Practice problems - Math 552

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#1. Let R be a ring and, $A \in mod - R$, and $B \in R - mod$. Let A' be a submodule of A and B' be a submodule of B. Show that $(A/A') \otimes_R (B/B')$ is isomorphic to $(A \otimes_R B)/C$ where C is the subgroup of $A \otimes_R B$ generated by all $a' \otimes b$ and $a \otimes b'$ for $a \in A, b \in B, a' \in A', b' \in B'$.

#2. Let R be a ring and $M \in R - mod$ be both artinian and noetherian. Let $f \in End_R(M)$. Recall that f^∞M is defined to be $\cap_{n \geq 1} f^i(M)$ and $f^{-\infty}0$ is defined to be $\cup_{n \geq 1} ker(f^n)$. Prove that

$$M = f^\infty M \oplus f^{-\infty}0.$$

(This is Fitting's Lemma.)

#3 State the definition of a projective resolution of an R-module M and show that any module has a projective resolution.

#4 Let $(C', d'), (C, d)$, and (C'', d'') be complexes. Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence (where the chain homomorphism from C' to C is denoted α and the chain homomorphism from C to C'' is denoted β). Suppose that there exist module homomorphisms $S_i : C'_i \to C_{i+1}$ for all $i \in \mathbf{Z}$ such that

$$\alpha_i = d_{i+1} s_i + s_{i-1} d'_i$$

for all i. Prove that if C'' is exact then C and C' are exact.

#5 Show that the ideal $(9, 3x + 3)$ has infinitely many primary decompositions.

#6 If R is a commutative ring, $B \neq 0$ is an R-module, and P is maximal in the set of ideals

$$\{ann\ x\ | 0 \neq x \in B\}$$

then P is prime. (Recall that $ann\ x\ = \{r \in r \mid rx = 0\}$.)

#7 Let R be noetherian and let S be a submonoid of the multiplicative monoid of R. Show that R_S is noetherian.

#8 Determine the Galois groups of $x^5 - 6x + 3$ and of $(x^3 - 2)(x^2 - 5)$ over the rational numbers.

#9 Let $E \subseteq F$ be fields and $u, v \in E$. Suppose that v is algebraic over $F(u)$, and that v is transcendental over F. Show that u is algebraic over $F(v)$.

#10 Let E, K, L, and F be fields with $E \subseteq K \subseteq F$, $E \subseteq L \subseteq F$ and $K \cap L = F$. Assume that E is generated by $K \cup L$. Suppose $[K : F] = n_1$, $[L : F] = n_2$, and that K is a Galois extension of F. What is $[E : F]$? Why?