INVARIANT POLYNOMIALS IN THE FREE SKEW FIELD

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Introduction

The free associative algebra on $\{x_1, ..., x_n\}$ over a field F, denoted $F < x_1, ..., x_n >$, has a universal quotient ring $F \not< x_1, ..., x_n \not>$ called the *free skew field* on $\{x_1, ..., x_n\}$ over F (cf. [C1, C2]). The symmetric group on n letters, S_n, acts on $F < x_1, ..., x_n >$ and hence on $F \not< x_1, ..., x_n \not>$. Gelfand and Retakh [GR1 - GR3] have constructed, using quasideterminants, an important set of elements $\{y_1, ..., y_n\}$ contained in $F \not< x_1, ..., x_n \not>$. They define, for $k > 1$, the *Vandermonde* quasideterminant

$$V(x_1, ..., x_k) = \begin{vmatrix} x_1^{k-1} & ... & x_k^{k-1} \\ & ... & \\ x_1 & ... & x_k \\ 1 & ... & 1 \end{vmatrix}_{1k} .$$

They then define $y_1 = x_1,\ y_k = V(x_1, ..., x_k)x_k(V(x_1, ..., x_k))^{-1}$ for $k > 1$, and

$$\Lambda_{i,j} = \sum_{j \geq l_1 > ... > l_i \geq 1} y_{l_1} ... y_{l_i}$$

for $1 \leq i \leq j \leq n$. Gelfand and Retakh prove [GR3] that, for $1 \leq i \leq n$, $\Lambda_{i,n}$ is S_n-invariant. They also conjecture that any S_n-invariant polynomial in $\{y_1, ..., y_n\}$ is, in fact, a polynomial in $\{\Lambda_{1,n}, ..., \Lambda_{n,n}\}$.

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This is in contrast to the fact [BC] that the algebra of S_n-invariant elements in $F < x_1, ..., x_n >$ is not finitely generated. The present paper is devoted to proving the Gelfand-Retakh conjecture.

We begin, in Section 1, by recalling an important universal property of $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ and by proving some technical results about certain division rings, in particular, the left quotient rings of the universal enveloping algebras of certain finite-dimensional Lie algebras. In Section 2 we use these results to prove that certain subsets of $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ are algebraically independent. In Section 3 we characterize the S_n-invariant elements in any associative algebra A with an S_n-action and appropriate independence properties. Finally, in Section 4 we combine the results of Sections 2 and 3 to obtain the proof of the Gelfand-Retakh conjecture (Theorem 4.1).

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1. Preliminaries

§1.1 The universal property of the free field $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$

The free associative algebra, $F < x_1, ..., x_n >$, on the set $\{x_1, ..., x_n\}$ is a *free ideal ring*, i.e., every left (respectively, right) ideal is a left (respectively, right) $F < x_1, ..., x_n >$-module of unique rank. Consequently, $F < x_1, ..., x_n >$ has a universal field of fractions, denoted $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ and called the *free skew field* over F on $\{x_1 ., , , .x_n\}$ (cf. [C1, C2]).

This means that there is a homomorphism

$$\mu : F < x_1, ..., x_n > \longrightarrow F\langle\!\langle x_1, ..., x_n \rangle\!\rangle,$$

and that if D is any division ring and

$$\alpha : F < x_1, ..., x_n > \longrightarrow D$$

is a homomorphism, then there is a subring A of $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ containing $F < x_1, ..., x_n >$ and a homomorphism

$$\beta : A \longrightarrow D$$

such that $\alpha = \beta\mu$ and that if $a \in A$ and $\beta(a) \neq 0$, then $a^{-1} \in A$.

Note that the symmetric group S_n acts on $F < x_1, ..., x_n >$ by permuting subscripts and so S_n acts on $F \lessdot x_1, ..., x_n \gtrdot$.

§1.2 Abelian valuations

Let A be an associative algebra over F. Recall (e.g., [C2, p. 83]) that a *valuation* on A is a function

$$\|\cdot\| : A \to \mathbf{Z} \cup \{-\infty\}$$

such that

(1.2.1) if $a \in A$, then $\|a\| = -\infty$ if and only if $a = 0$;
(1.2.2) if $a \in A$ and $0 \neq r \in F$ then $\|ra\| = \|a\|$;
(1.2.3) if $a, b \in A$, then $\|a + b\| \leq max\{\|a\|, \|b\|\}$; and
(1.2.4) if $a, b \in A$, then $\|ab\| = \|a\| + \|b\|$.

Definition. Let A be an associative algebra over F and $\|\cdot\|$ be a valuation on A. We say that $\|\cdot\|$ is an abelian valuation if it satisfies:

(1.2.5) if $a, b \in A, a, b \neq 0$, *then* $\|[a, b]\| < \|a\| + \|b\|$.

Note that, in view of (1.2.4), if A has a valuation, then A must be a domain, i.e., A can have no nonzero zero divisors. It is easy to see that if $\|\cdot\|$ is a valuation

$$A_i = \{a \in A | \|a\| \leq i\}$$

for all $i \in \mathbf{Z}$, then

$$... \subseteq A_{-1} \subseteq A_0 \subseteq A_1 \subseteq ... \subseteq A$$

is an increasing filtration of the associative algebra A. In this case (1.2.5) holds if and only if the associative graded algebra of A, grA is abelian.

Recall that a domain A is a left Ore domain if whenever $0 \neq a, b \in A$, there exist $0 \neq c, d \in A$ such that $ca = db$. Recall also that a left Ore domain A may be imbedded in a left ring of quotients D, that is, there is a division ring $D \supseteq A$ such that every element of D has the form $a^{-1}b$ where $a, b \in A, a \neq 0$.

Lemma 1.1. *Let A be a left Ore domain and let $\|\cdot\|$ be a valuation on A. Let D be a left ring of quotients of A. Then $\|\cdot\|$ has a unique extension to a valuation on D, again denoted $\|\cdot\|$, and*

$$\|a^{-1}b\| = \|b\| - \|a\| \tag{1.2.6}$$

for all $a, b \in A, a \neq 0$. Furthermore, if $\|\cdot\|$ is an abelian valuation on A, then its extension is an abelian valuation on D.

Proof Suppose $\|\cdot\|$ extends to a valuation on D. Then, by (1.2.4), for $a, b \in A, a \neq 0$ we have $\|b\| = \|a(a^{-1}b)\| = \|a\| + \|a^{-1}b\|$, and so $\|a^{-1}b\| = \|b\| - \|a\|$.

Now let $u \in D$ and suppose $u = a^{-1}b = c^{-1}d$, where $a, b, c, d \in A, a, c \neq 0$. Then there exist $e, f \in A$ such that $0 \neq g = ea = fc$. Thus, by (1.2.4), $\|g\| = \|e\| + \|a\| = \|f\| + \|c\|$. Furthermore, $a^{-1} = g^{-1}e$ and $c^{-1} = g^{-1}f$. Consequently $a^{-1}b = g^{-1}eb = c^{-1}d = g^{-1}fd$ and so $eb = fd$. Therefore $\|e\| + \|b\| = \|f\| + \|d\|$ and hence $\|b\| - \|a\| = (\|b\| + \|e\|) - (\|e\| + \|a\|) = (\|d\| + \|f\|) - (\|f\| + \|c\|) = \|d\| - \|c\|$. Thus we may extend $\|\cdot\|$ from A to D by setting

$$\|a^{-1}b\| = \|b\| - \|a\|$$

for $a, b \in A, a \neq 0$.

We next show that $\|\cdot\|$ satisfies conditions (1.2.1) - (1.2.4) on D.

Suppose $u \in D, u = a^{-1}b, a, b \in A, a \neq 0$. Then $\|u\| = \|b\| - \|a\|$, and so $\|u\| = -\infty$ if and only if $\|b\| = -\infty$. But $\|b\| = -\infty$ if and only if $b = 0$, and $u = 0$ if and only if $b = 0$. Hence (1.2.1) holds for D. Also, if $0 \neq r \in F$, then $\|ru\| = \|r(a^{-1}b)\| = \|a^{-1}(rb)\| = \|rb\| - \|a\| = \|b\| - \|a\| = \|u\|$. Thus (1.2.2) holds for D.

Now let $u, v \in D, u = a^{-1}b, v = c^{-1}d, a, b, c, d \in A, 0 \neq a, c$. Then there exist $0 \neq e, f, g \in A$ such that $g = ea = fc$. By (1.2.4) for A this gives $\|g\| = \|e\| + \|a\| = \|f\| + \|c\|$. Furthermore, $a^{-1} = g^{-1}e$ and $c^{-1} = g^{-1}f$. Thus $u + v = a^{-1}b + c^{-1}d = g^{-1}(eb + fd)$. Then $\|u + v\| = \|eb + fd\| - \|g\|$ and, by (1.2.3) for A, this is $\leq max\{\|eb\|, \|fd\|\} - \|g\|$. By (1.2.4) for A, this is equal to

$$max\{\|e\| + \|b\|, \|f\| + \|d\|\} - \|g\| = max\{\|e\| - \|g\| + \|b\|, \|f\| - \|g\| + \|d\|\}$$

$$= max\{\|b\| - \|a\|, \|d\| - \|c\|\} = max\{\|u\|, \|v\|\}.$$

Thus (1.2.3) holds for D.

To prove (1.2.4) for D note that if $u = 0$ or $v = 0$, then (1.2.4) holds for u and v. Hence we may assume $u, v \neq 0$, and so $a, b \neq 0$. Then there exist $0 \neq r, s, t \in A$ such that $t = rb = sc$. Then we have $\|s\| - \|r\| = \|b\| - \|c\|$. Now $b = r^{-1}t$ and $c = s^{-1}t$. Thus $uv = a^{-1}bc^{-1}d = a^{-1}r^{-1}sd$, and so $\|uv\| = -\|a\| - \|r\| + \|s\| + \|d\| = \|b\| - \|a\| + \|d\| - \|c\| = \|u\| + \|v\|$.

Finally, assume that $\|\cdot\|$ satisfies (1.2.5) for A. To prove (1.2.5) for D note that

$$[u, v] = [a^{-1}b, c^{-1}d]$$

$$= [a^{-1}, c^{-1}]bd + a^{-1}[b, c^{-1}]d + c^{-1}[a^{-1}, d]b + c^{-1}a^{-1}[b, d].$$

Then since

$$[b, c^{-1}] = -c^{-1}[b, c]c^{-1},$$

$$[a^{-1}, d] = -a^{-1}[a, d]a^{-1}$$

and

$$[a^{-1}, c^{-1}] = c^{-1}a^{-1}[a, c]a^{-1}c^{-1}$$

the result follows from (1.2.4) and (1.2.5) for A and (1.2.2) and (1.2.3) for D.

§1.3 A filtration of $U(L)$

Given a filtration of a Lie algebra L, we will define a corresponding filtration of the universal enveloping algebra $U(L)$ and prove several properties of this filtration.

Let

$$... \subseteq A_i \subseteq A_{i+1} \subseteq ... \subseteq A_{-1} \subseteq A_0 = A$$

be a filtration of the (not necessarily associative) algebra A. Recall that grA, the *associated graded algebra*, is defined by setting

$$(grA)_{[i]} = A_i / A_{i-1}$$

for $i \leq 0$, setting

$$grA = \sum_{i \leq 0} (grA)_{[i]}$$

and defining a bilinear product on grA by

$$(a + A_{i-1})(b + A_{j-1}) = ab + A_{i+j-1}$$

for $i, j \leq 0, a \in A_i, b \in A_j$. Note that if A is a Lie algebra (respectively, an associative algebra), then grA is a Lie algebra (respectively, an associative algebra).

Now let

$$... \subseteq L_i \subseteq L_{i+1} \subseteq ... \subseteq L_{-1} \subseteq L_0 = L \tag{1.3.1}$$

be a filtration of the Lie algebra L satisfying

$$\cap_i L_i = (0), \tag{1.3.2}$$

and let $U(L)$ denote the universal enveloping algebra of L. For $i \leq 0$, let

$$U(L)_i = \sum L_{i_1} L_{i_2} ... L_{i_t} \tag{1.3.3}$$

where the sum is taken over all $t \geq 0, i_1, ..., i_t \leq 0, i_1 + ... + i_t = i$. Clearly

$$... \subseteq U(L)_i \subseteq U(L)_{i+1} \subseteq ... \subseteq U(L)_{-1} \subseteq U(L)_0 = U(L) \tag{1.3.4}$$

is a filtration of $U(L)$.

Define a function

$$v : L \to \mathbf{Z} \cup \{-\infty\}$$

by

$$v(a) = \inf\{i | a \in L_i\}. \tag{1.3.5}$$

By (1.3.2) we have that $v(a) = -\infty$ if and only if $a = 0$.

Let B be a basis of the Lie algebra L. We say that B is compatible with the filtration (1.3.1) if $B \cap L_i$ is a basis for L_i for every $i \leq 0$. If B is a compatible ordered basis of L define, for $i \leq 0, j \geq 0$, $P(B)_{i,j} = \{b_1...b_t | 0 \leq t \leq j, b_1, ..., b_t \in B, b_1 \leq ... \leq b_t, v(b_1) + ... + v(b_t) \leq i\}$ and $P(B)_i = \cup_{j=0}^{\infty} P(B)_{i,j}$. Note that, by the Poincaré-Birkhoff-Witt Theorem, $P(B)_0$ is a basis for $U(L)$.

Lemma 1.2. *Let L be a Lie algebra with filtration (1.3.1). Let B be a compatible ordered basis of L. Then, for $i \leq 0, P(B)_i$ is a basis for $U(L)_i$.*

Proof. $P(B)_i$ is linearly independent since it is a subset of the basis $P(B)_0$ of $U(L)$. It is contained in $U(L)_i$ by (1.3.3). Thus it is sufficient to show that $P(B)_i$ spans $U(L)_i$. Now it is immediate from (1.3.3) that

$$\{c_1...c_s | s \geq 0, c_1, ..., c_s \in B, v(c_1) + ... + v(c_s) \leq i\}$$

spans $U(L)_i$. Thus it is sufficient to show that if $s \geq 0, c_1, ..., c_s \in B$, and $v(c_1) + ... + v(c_s) = i$, then $c_1...c_s$ is in the span of $P(B)_{i,s}$. This is vacuously true for $s = 0, 1$. Assume the result holds for $s - 1$. Then $c_1...c_s = (c_1...c_{s-1})c_s$ and, applying the induction assumption to $c_1...c_{s-1}$, we may assume that $c_1 \leq ... \leq c_{s-1}$. If $c_{s-1} \leq c_s$ we are done, so we may assume that there is some $j, 1 \leq j \leq s - 1$ so that $c_l \leq c_s$ if and only if $l < j$. Then

$$c_1...c_s = c_1...c_{j-1}c_sc_jc_{j+1}...c_{s-1} + \sum_{l=j}^{s-1} c_1...c_{l-1}[c_l, c_s]c_{l+1}...c_{s-1}.$$

The first summand is in $P(B)_{i,s}$ (by the choice of j). Since (1.3.1) is a filtration of the Lie algebra L, we have that $[c_l, c_s]$ is a linear combination of elements $b \in B$ with $v(b) \leq v(c_l) + v(c_s)$. The induction assumption then shows that for each $l, j \leq l \leq s - 1, c_1...c_{l-1}[c_l, c_s]c_{l+1}...c_{s-1}$ is a linear combination of elements of $P(B)_{i,s-1} \subseteq P(B)_{i,s}$. This completes the proof of the lemma.

Corollary 1.3. *Let L be a Lie algebra with filtration (1.3.1) satisfying (1.3.2). Assume that there is a compatible basis of L. Then $L_i = L \cap U(L)_i$ for all i and $\cap_i U(L)_i = (0)$.*

Note that, in view of Corollary 1.3, we may extend the function v defined in (1.3.5) to $U(L)$ by setting

(1.3.6) $$v(u) = inf\{i | u \in U(L)_i\}$$

for all $u \in U(L)$.

Corollary 1.4. *Let L be a Lie algebra with filtration (1.3.1) satisfying (1.3.2). Assume that there is a compatible basis of L. Then $U(grL) \cong grU(L)$.*

Proof. The linear map $\phi : grL \to grU(L)$ defined by $\phi(a + L_{i-1}) = a + U(L)_{i-1}$ for $i \leq 0, a \in L_i$ is a Lie homomorphism. Hence it extends to a homomorphism of associative algebras

$$\phi : U(grL) \to grU(L).$$

Let B be an ordered basis for L which is compatible with (1.3.1). For $b \in B$, let $b' = b + L_{v(b)-1} \in (grL)_{v(b)}$ and $b" = b + U(L)_{v(b)-1} \in (grU(L))_{v(b)-1}$. Then $\{b' | b \in B\}$ is an ordered basis for grL, and so $\{b'_1...b'_t | t \geq 0, b_1, ..., b_t \in B, b_1 \leq ... \leq b_t\}$ is a basis for $U(grL)$. Clearly $\phi(b + L_{v(b)-1}) = b"$ for each $b \in B$ and so $\phi(b'_1...b'_t) = b''_1...b''_t$. By Lemma 1.1, $\{b''_1...b''_t | t \geq 0, b_1, ..., b_t \in B, b_1 \leq ... \leq b_t\}$ is a basis for $grU(L)$. Thus ϕ is an isomorphism.

Now let $L = \sum_{k \leq 0} L_{[k]}$ be a graded Lie algebra over F and $U(L)$ be its universal enveloping algebra. For $k \leq 0$, we define $U(L)_{[k]}$ to be the span of all products $a_1...a_t$ where $t \geq 0, a_i \in L_{[s_i]}$ for $1 \leq i \leq t$ and $\sum_{i=1}^{t} s_i = k$. This gives $U(L) = \sum_{k \leq 0} U(L)_{[k]}$ the structure of a graded associative algebra.

Now assume $L = \sum_{k<0} L_{[k]}$ and define a filtration of L by

$$L_i = \sum_{i-1 \leq k < 0} L_{[k]} \tag{1.3.7}$$

for $i \leq 0$. This is not the most natural way to define a filtration of L; setting $L_i = \sum_{i \leq k < 0} L_{[k]}$ defines a filtration which is more closely related to the graded algebra L (in the sense that its associated graded algebra is isomorphic to L). However, as the following lemma shows, the definition (1.3.7) has properties which make it useful for our purposes.

Note that there exist bases compatible with (1.3.7). Indeed, if B_k is a basis of $L_{[k]}$ for each $k \leq 0$, then the basis $B = \cup_k B_k$ is compatible with (1.3.7).

Lemma 1.5. *Let $L = \sum_{k \leq 0} L_{[k]}$ be a graded Lie algebra. Define a filtration by (1.3.6) and a function v on $U(L)$ by (1.3.6). Then v is an abelian valuation.*

Proof. It is immedate from (1.3.7) that grL is abelian. Corollary 1.3 then shows that $grU(L)$ is isomorphic to a polynomial algebra and hence is an integral domain. It is then immediate that v is an abelian valuation.

Let L be a finite-dimensional graded Lie algebra. It is well-known (cf. [J]) that $U(L)$, the universal enveloping algebra of L, is a left Ore domain. Let $Q(L)$ denote the left ring of quotients of $U(L)$. Thus $Q(L)$ is a division ring which contains $U(L)$ and every element of $Q(L)$ is of the form $a^{-1}b$ where $a, b \in U(L)$ and $a \neq 0$.

Corollary 1.6. *Let L be a finite-dimensional graded Lie algebra. Then $Q(L)$ has an abelian valuation satisfying (1.2.6) and (1.3.2).*

Proof. This is immediate from Lemmas 1.1 and 1.5.

Lemma 1.7. *Let $L = \sum_{i<0} L_{[i]}$ and $M = \sum_{i<0} M_{[i]}$ be graded Lie algebras. Let $\phi : L \longrightarrow M$ be a surjective homorphism of graded Lie algebras and let $\Phi : U(L) \longrightarrow U(M)$ be the unique homomorphism extending ϕ. Suppose ker $\phi \subseteq L_k$. Then if $u \in U(L)$ and $\|u\| > k$, we have $\|\Phi(u)\| = \|u\|$.*

Proof. *Ker* Φ is the ideal of $U(L)$ generated by *ker* ϕ. Then, since *ker* $\phi \subseteq L_k \subseteq U(L)_k$ and each $U(L)_i$ is an ideal in $U(L)$, we have *ker* $\Phi \subseteq U(L)_k$.

Clearly $\|\Phi(u)\| \geq \|u\|$. If $\|\Phi(u)\| > \|u\|$ then, setting $i = \|u\|$, we have $u \in U(L)_i, u \notin U(L)_{i-1}, \Phi(u) \in U(M)_{i-1}$. Since ϕ (and hence Φ) is surjective, there exists $u' \in U(L)_{i-1}$ so that $\Phi(u') = \Phi(u)$. Hence $u - u' \in ker\ \Phi \subseteq U(L)_k$. Thus $u \in u' + U(L)_k \subseteq U(L)_{i-1} + U(L)_k \subseteq U(L)_{i-1}$, a contradiction.

Let $L = S \oplus I$ have filtration (1.3.1) where I is an ideal of L, S is a subalgebra of L and $L_i = S \cap L_i + I \cap L_i$ for all i. Then L, S and I are all filtered Lie algebras, and so $U(L)_i, U(S)_i$ and $U(I)_i$ are all defined and the following relations among these filtrations are obvious.

Lemma 1.8. *Let $L = S \oplus I$ have filtration (1.3.1) where I is an ideal of L, S is a subalgebra of L and $L_i = S \cap L_i + I \cap L_i$ for all i.*

(a) $U(I)_i S_j \subseteq S_j U(I)_i + U(I)_{i+j}$;

(b) $U(I)_i U(S)_j \subseteq \sum_{k \leq 0} U(S)_{j-k} U(I)_{i+k}$;

(c) $U(L)_i = \sum_{i \leq j \leq 0} U(S)_j U(I)_{i-j}$;

(d) $U(I)_i = U(I) \cap U(L)_i$.

§1.4 The free Lie algebra $\mathcal{F}(n)$

Let $\mathcal{F}(n)$ denote the free Lie algebra on $\{x_1, ..., x_n\}$. For $k \geq 0$ define $\mathcal{M}(n)_{[k]}$ to be the set of all products

$$(ad\ x_{i_1})(ad\ x_{i_2})...(ad\ x_{i_{k-1}})x_{i_k}, 1 \leq i_1, ..., i_k \leq n,$$

and define $\mathcal{F}(n)_{[-k]}$ to be the span of $\mathcal{M}(n)_{[k]}$. Then

$$\mathcal{F}(n) = \sum_{k<0} \mathcal{F}(n)_{[k]} \tag{1.4.1}$$

is a graded Lie algebra.

Now assume $n \geq 2$ and, for $k \geq 0$, define $\mathcal{M}(n)_{<k>}$ to be the span of all products

$$(ad\ x_{i_1})(ad\ x_{i_2})...(ad\ x_{i_{l-1}})x_{i_l}, 1 \leq i_1, ..., i_l \leq n,$$

where

$$\sum_{j=1}^{l} \delta_{2,i_j} = k$$

and define $\mathcal{F}(n)_{<k>}$ to be the span of $\mathcal{M}(n)_{<k>}$. Then

$$\mathcal{F}(n) = \sum_{k\geq 0} \mathcal{F}(n)_{<k>} \tag{1.4.2}$$

is a graded Lie algebra.

Let $\mathcal{I}(n)$ denote the ideal of $\mathcal{F}(n)$ generated by $\{x_2, ..., x_n\}$. Then

$$\mathcal{F}(n) = Fx_1 + \mathcal{I}(n)$$

and so

$$U(\mathcal{F}(n)) = \sum_{i\geq 0} x_1^i U(\mathcal{I}(n))$$

is a vector space grading. Define $S(n)_{\{j\}} = \sum_{0\leq i\leq j} x_1^i U(\mathcal{I}(n))$. Then $S(n)_{\{j_1\}} S(n)_{\{j_2\}} \subseteq S(n)_{\{j_1+j_2\}}$ and $U(\mathcal{I}(n)) = S(n)_{\{0\}} \subset S(n)_{\{1\}} \subset ...$ is an increasing filtration of the associative algebra $U(\mathcal{F}(n))$.

For $i \leq 0, j \geq 0$ define $G(i,j) = U(\mathcal{F}(n))_{i-1} + U(\mathcal{F}(n))_i \cap S(n)_{\{j\}}$. Note that $G(i_1,j_1) \subseteq G(i_2,j_2)$ when $i_1 < i_2$ or when $i_1 = i_2$ and $j_1 \leq j_2$.

As in Section 1.3, the grading (1.4.1) of $\mathcal{F}(n)$ gives a grading of $U(\mathcal{F}(n))$

$$U(\mathcal{F}(n)) = \sum_{k \leq 0} U(\mathcal{F}(n))_{[k]} \tag{1.4.3}$$

and the grading (1.4.2) of $\mathcal{F}(n)$ gives a grading of $U(\mathcal{F}(n))$

$$U(\mathcal{F}(n)) = \sum_{k \leq 0} U(\mathcal{F}(n))_{<k>}. \tag{1.4.4}$$

By the results of Section 1.3 there is an abelian valuation, denoted $\|\cdot\|$, on $U(\mathcal{F}(n))$ corresponding to the grading (1.4.1).

Now let A be a domain with an abelian valuation which we again denote by $\|a\|$.

Lemma 1.9. *Let $f \in U(\mathcal{F}(n)) \cong F < x_1, ..., x_n >, u_1, ..., u_n, v_1, ..., v_n \in A$, and $\|u_i\| \leq 0, \|v_i\| < 0$ for all i. Then*

$$\|f(u_1, ..., u_n)\| \leq \|f\| \tag{1.4.5}$$

and

$$\|f(u_1, ..., u_n) - f(u_1 + v_1, ..., u_n + v_n)\| \leq \|f\| - 1. \tag{1.4.6}$$

Proof. Note that, for each i, $\|u_i + v_i\| \leq max\{\|u_i\|, \|v_i\|\} = 0$.

We first prove two special cases of the lemma.

(1) The lemma holds if $f \in \mathcal{M}(n)_{[k]}$.

Proof of (1). In this case

$$f = (ad\ x_{i_1})(ad\ x_{i_2})...(ad\ x_{i_{k-1}})x_{i_k}$$

for some $1 \leq i_1, ..., i_k \leq n$. If $k > 1$, let

$$f' = (ad\ x_{i_2})...(ad\ x_{i_{k-1}})x_{i_k}.$$

Then $\|f\| = -k+1$ and $\|f'\| = -k+2$.

We will proceed by induction on k. If $k = 1$, then $f(u_1, ..., u_n) = u_i$ for some i and so, by hypothesis, $\|f(u_1, ..., u_n)\| = \|u_i\| \leq 0$ and

$$\|f(u_1, ..., u_n) - f(u_1+v_1, ..., u_n+v_n)\| = \|u_i - (u_i+v_i)\| = \|v_i\| < 0.$$

Thus, for $k = 1$, (1.4.5) and (1.4.6) hold.

Now assume that $k > 1$ and that the result holds for all $g \in \mathcal{M}(n)_{[l]}$, $l < k$. Then, in particular, the result holds for f', and so $\|f'(u_1, ..., u_n)\| \leq \|f'\| = -k+2$ and $\|f'(u_1, ..., u_n) - f'(u_1+v_1, ..., u_n+v_n)\| \leq \|f'\| - 1 = -k+1$. Moreover, since $\|u_i + v_i\| \leq 0$ for all i, we also have $\|f'(u_1+v_1, ..., u_n+v_n)\| \leq \|f'\| = -k+2$. Then, by (1.2.5),

$$\|f(u_1, ..., u_n)\| = \|[u_{i_1}, f'(u_1, ..., u_n)]\| \leq \|u_{i_1}\| + \|f'(u_1, ..., u_n)\| - 1$$

$$\leq \|f'(u_1, ..., u_n)\| - 1 \leq -k+2-1 = -k+1 = \|f\|.$$

Furthermore,

$$\|f(u_1, ..., u_n) - f(u_1+v_1, ..., u_n+v_n)\|$$

$$= \|[u_1, f(u_1, ..., u_n)] - [u_1+v_1, f'(u_1+v_1, ..., u_n+v_n)]\| =$$

$$\|[u_1, f'(u_1, ..., u_n) - f'(u_1+v_1, ..., u_n+v_n)] - [v_1, f'(u_1+v_1, ..., u_n+v_n)]\|.$$

As $\|[u_1, f'(u_1, ..., u_n) - f'(u_1+v_1, ..., u_n+v_n)]\| \leq \|u_1\| + \|f'(u_1, ..., u_n) - f'(u_1+v_1, ..., u_n+v_n)\| - 1 \leq -k$ and $\|[v_1, f'(u_1+v_1, ..., u_n+v_n)]\| \leq \|v_1\| + \|f'(u_1+v_1, ..., u_n+v_n)\| - 1 \leq -k$, we have

$$\|f(u_1, ..., u_n) - f(u_1+v_1, ..., u_n+v_n)\| \leq -k = \|f\| - 1.$$

Thus, by induction on k, (1.4.5) and (1.4.6) hold whenever $f \in \mathcal{M}(n)_{[k]}$.

(2) The lemma holds if $f = f_1 ... f_l, f_i \in \mathcal{M}(n)_{[k_i]}, k_1 + ... + k_l = k$.

Proof of (2). If $l > 1$, set $f' = f_2 ... f_l$. We have $\|f\| = -k+l$ and $\|f'\| = -k+l+k_1-1$.

We will proceed by induction of l. If $l = 1$, then (1.4.5) and (1.4.6) hold by case (1). Assume $l > 1$ and that the result holds for all $g = g_1 ... g_t$ where $t < l$ and each $g_i \in \mathcal{M}(n)_{[s_i]}$ for some s_i. Then, in particular, the result holds for f', and so $\|f'(u_1, ..., u_n)\| \leq \|f'\|$ and

$$\|f'(u_1, ..., u_n) - f'(u_1+v_1, ..., u_n+v_n)\| \leq \|f'\| - 1.$$

Then, by (1.2.4), $\|f(u_1,...,u_n)\| = \|f_1(u_1,...,u_n)f'(u_1,...,u_n)\| = \|f_1(u_1,...,u_n)\| + \|f'(u_1,...,u_n)\| \leq \|f_1\| + \|f'\| = \|f\|$. Furthermore, $f(u_1,...,u_n) - f(u_1+v_1,...,u_n+v_n) = (f_1(u_1,...,u_n) - f_1(u_1+v_1,...,u_n+v_n))(f'(u_1+v_1,...,u_n+v_n)) + (f_1(u_1,...,u_n)(f'(u_1,...,u_n) - f'(u_1+v_1,...,u_n+v_n))$. Thus $\|f(u_1,...,u_n) - f(u_1+v_1,...,u_n+v_n)\| \leq max\{\|f_1(u_1,...,u_n) - f_1(u_1+v_1,...,u_n+v_n)\|\|f'(u_1+v_1,...,u_n+v_n)\|, \|f_1(u_1,...,u_n)\|\|f'(u_1,...,u_n) - f'(u_1+v_1,...,u_n+v_n)\|\} \leq max\{(\|f_1\| - 1) + \|f'\|, \|f_1\| + \|f'\| - 1\} \leq \|f_1\| + \|f'\| - 1 = \|f\| - 1$. Thus, by induction on l, the lemma holds in case (2).

Finally, let f be an arbitrary element of $U(\mathcal{F}(n))$ and let $\|f\| = k$. Thus $f \in U(\mathcal{F}(n))_k$ and so f is a linear combination of elements of the form treated in case (2). The conclusion of the lemma then follows from (1.2.2), (1.2.3) and case (2).

§1.5 The Division Ring $Q_k(n)$

Let A be an associative algebra and $t \in A$. As usual, define $l_t : A \longrightarrow A$ by $l_t(u) = tu$, $r_t : A \longrightarrow A$ by $r_t(u) = ut$ and $ad(t) : A \longrightarrow A$ by $ad(t) = l_t - r_t$.

Lemma 1.10. *Let A be an associative algebra and let $t \in A$ be an invertible element such that $ad(t)$ is locally nilpotent. Let $s \in \mathbf{Z}$. Then*

(i) $(r_t)^s = (l_t)^s \sum_{j=0}^{\infty} (-1)^j \binom{s}{j} (l_t)^{s-j} (ad(t))^j$;

(ii) $ad(t^s) = \sum_{j=1}^{\infty} (-1)^{j+1} \binom{s}{j} (l_t)^{s-j} (ad(t))^j$.

Proof. We have $r_t = l_t - ad(t) = l_t(1 - (l_t)^{-1}(ad(t)))$. Since $(l_t)^{-1}(ad(t))$ is locally nilpotent, (i) follows. Furthermore, $ad(t^s) = (l_t)^s - (r_t)^s$. Then substituting the expression for $(r_t)^s$ from (i) gives (ii).

Let L be a Lie algebra. Recall that the sequence of ideals $L^1 \supseteq L^2 \supseteq ...$ (the *lower central series*) is defined inductively by $L^1 = L$ and $L^{i+1} = [L^i, L]$ for $i \geq 1$. Note that $\mathcal{F}(n)^k = \sum_{j \leq -k} \mathcal{F}(n)_{[j]}$.

Define

$$\mathcal{FN}_k(n) = \mathcal{F}(n)/\mathcal{F}(n)^k.$$

This is the *free nilpotent Lie algebra of degree* k on $\{x_1,...,x_n\}$. Let $\alpha_k : \mathcal{F}(n) \longrightarrow \mathcal{FN}_k(n)$ denote the quotient map. This extends to a homomorphism, which we again denote α_k from $F < x_1,...,x_n >= U(\mathcal{F}(n))$ to $U(\mathcal{FN}_k(n))$. Since $ker\ \alpha_k = \mathcal{F}(n)^k = \sum_{j \leq -k} \mathcal{F}(n)_{[j]}$ is a graded ideal in $\mathcal{F}(n)$, we see that $\mathcal{F}N_k(n)$ has the structure of a graded Lie algebra.

Note that $\mathcal{FN}_k(n)$ is finite-dimensional. Thus $U(\mathcal{FN}_k(n))$ has a left ring of quotients which we denote by $Q_k(n)$. By Corollary 1.6, $Q_k(n)$ has an abelian valuation, denoted $\|\cdot\|$, satisfying (1.2.6) and (1.3.2). Note that $\alpha_k : F < x_1, ..., x_n > \longrightarrow Q_k(n)$.

Write $t_i = \alpha_k(x_i)$ for $1 \leq i \leq n$, and let $R_k(n)$ denote the subalgebra of $Q_k(n)$ generated by $t_1, ..., t_n$ and t_1^{-1}. Let $\mathcal{IN}_k(n)$ denote $\alpha_k(\mathcal{I}(n))$. Clearly $\mathcal{IN}_k(n)$ is the ideal of $\mathcal{FN}_k(n)$ generated by $t_2, ..., t_n$, $\mathcal{FN}_k(n) = Ft_1 + \mathcal{IN}_k(n)$, and

$$U(\mathcal{FN}_k(n)) = \sum_{i \geq 0} t_1^i U(\mathcal{IN}_k(n)) \tag{1.5.1}$$

is a vector space direct sum. Since the sum in (1.5.1) is direct, so is

$$\sum_{i \in \mathbf{Z}} t_1^i U(\mathcal{IN}_k(n)) = R_k(n) \tag{1.5.2}$$

Let $S_k(n)_{\{j\}} = \sum_{i \in \mathbf{Z}, i \leq j} t_1^i U(\mathcal{IN}_k(n))$. Then, by Corollary 1.8(ii), we have

$$S_k(n)_{\{j_1\}} S_k(n)_{\{j_2\}} \subseteq S_k(n)_{\{j_1+j_2\}}.$$

Thus $... \subset S_k(n)_{\{j\}} \subset S_k(n)_{\{j+1\}} \subset ...$ is a filtration of $R_k(n)$ with

$$\cap_{j \in \mathbf{Z}} S_k(n)_{\{j\}} = (0)$$

and

$$\cup_{j \in \mathbf{Z}} S_k(n)_{\{j\}} = R_k(n).$$

For $i \leq 0, j \in \mathbf{Z}$, define

$$G_k(i,j) = R_k(n)_{i-1} + R_k(n)_i \cap S_k(n)_{\{j\}}.$$

Note that $G_k(i_1, j_1) \subseteq G_k(i_2, j_2)$ when $i_1 < i_2$ or when $i_1 = i_2$ and $j_1 \leq j_2$.

Lemma 1.11. *(i)* $[t_1^{j_1}, G_k(i_2, j_2)] \subseteq G_k(i_2 - 1, j_1 + j_2 - 1)$;
(ii) $G(i_1, j_1)G(i_2, j_2) \subseteq G(i_1 + i_2, j_1 + j_2)$;
(iii) $G_k(i_1, j_1)G_k(i_2, j_2) \subseteq G_k(i_1 + i_2, j_1 + j_2)$;
(iv) $[G(i_1, j_1), G(i_2, j_2)] \subseteq G(i_1 + i_2 - 1, j_1 + j_2)$.

(v) $[G_k(i_1, j_1), G_k(i_2, j_2)] \subseteq G_k(i_1 + i_2 - 1, j_1 + j_2)$.

Proof. Clearly $(ad\ t_1)G_k(i, j) \subseteq G(i-1, j)$. Hence, by Lemma 1.10(ii),

$$(ad\ t_1^s)G(i, j) \subseteq \sum_{l=1}^{\infty} t_1^{s-l}(ad\ t_1)^l G(i, j)$$

$$\subseteq t_1^{s-1}G(i-1, j) + R_k(n)_{i-2} \subseteq G(i-1, j+s-1),$$

proving (i).

Since $U(\mathcal{F}(n))_p U(\mathcal{F}(n))_q \subseteq U(\mathcal{F}(n))_{p+q}$ and $S(n)_{\{j_1\}}S(n)_{\{j_2\}} \subseteq S(n)_{\{j_1+j_2\}}$ we have (ii). Similarly, since $R_k(n)_p R_k(n)_q \subseteq R_k(n)_{p+q}$ and $S_k(n)_{\{j_1\}}S_k(n)_{\{j_2\}} \subseteq S_k(n)_{\{j_1+j_2\}}$ we have (iii).

As $[U(\mathcal{F}(n))_p, U(\mathcal{F}(n))_q] \subseteq U(\mathcal{F}(n))_{p+q-1}$ and $[S(n)_{\{j_1\}}, S(n)_{\{j_2\}}] \subseteq S(n)_{\{j_1+j_2\}}$ we have (iv) Similarly, since $[R_k(n)_p, R_k(n)_q] \subseteq R_k(n)_{p+q-1}$ and $[S_k(n)_{\{j_1\}}, S_k(n)_{\{j_2\}}] \subseteq S_k(n)_{\{j_1+j_2\}}$ we have (v).

Lemma 1.12. *Let* $f(x_1, ..., x_n) \in U(\mathcal{F}(n))_{<k>} \cap G(i, j)$. *Then*

$$t_1^k f(t_1, t_2t_1^{-1}, t_3, ..., t_n) - f(t_1, ..., t_n) \in Gk(i, j-1). \tag{1.5.3}$$

Proof. Let $X(k, i, j)$ denote the set of all $f(x_1, ..., x_n) \in U(\mathcal{F}(n))_{<k>} \cap G(i, j)$ such that (1.5.3) holds. Note that if $f(x_1, ..., x_n) \in X_(k, i, j)$, then $f(t_1, ..., t_n) \in G_k(i, j)$ so $t_1^k f(t_1, t_2t_1^{-1}, t_3, ..., t_n) \in G_k(i, j)$, and $f(t_1, t_2t_1^{-1}, t_3, ..., t_n) \in G_k(i, j-k)$.

Clearly each $X(k, i, j)$ is a subspace of $U(\mathcal{F}(n))$. Furthermore, $1 \in X(0, 0, 0), x_1 \in X(0, 0, 1), x_2 \in X(1, 0, 0)$ and $x_j \in X(0, 0, 0)$ if $j > 2$.

Now let $f(x_1, ..., x_n) \in X(k_1, i_1, j_1)$ and $g(x_1, ..., x_n) \in X(k_2, i_2, j_2)$. Then

$$f(x_1, ..., x_n)g(x_1, ..., x_n) \in U(\mathcal{F}(n))_{<k_1>}U(\mathcal{F}(n))_{<k_2>} \cap G(i_1, j_1)G(i_2, j_2)$$

By Lemma 1.11(ii) this is contained in $U(\mathcal{F}(n))_{<k_1+k_2>} \cap G(i_1+i_2, j_1+j_2)$. Furthermore,

$$t_1^{k_1+k_2} f(t_1, t_2t_1^{-1}, t_3, ..., t_n)g(t_1, t_2t_1^{-1}, t_3, ..., t_n) - f(t_1, ..., t_n)g(t_1, ..., t_n)$$

$$= t_1^{k_1}[t_1^{k_2}, f(t_1, t_2t_1^{-1}, t_3, ..., t_n)]g(t_1, t_2t_1^{-1}, t_3, ..., t_n)$$

$$+(t_1^{k_1} f(t_1, t_2t_1^{-1}, t_3, ..., t_n))(t_1^{k_2} g(t_1, t_2t_1^{-1}, t_3, ..., t_n))$$

$$-f(t_1,...,t_n)g(t_1,...,t_n)$$
$$= t_1^{k_1}[t_1^{k_2}, f(t_1, t_2t_1^{-1}, t_3, ..., t_n)]g(t_1, t_2t_1^{-1}, t_3, ..., t_n)$$
$$-t_1^{k_1}(f(t_1, t_2t_1^{-1}, t_3, ..., t_n) - f(t_1, ..., t_n))(t_1^{k_2}g(t_1, t_2t_1^{-1}, t_3, ..., t_n))$$
$$+f(t_1, ..., t_n)(t_1^{k_2}g(t_1, t_2t_1^{-1}, t_3, ..., t_n) - g(t_1, ..., t_n)).$$

By Lemma 1.11, the first summand is contained in

$$G_k(0, k_1)G_k(i_1 - 1, j_1 - k_1 + k_2 - 1)G_k(i_2, j_2 - k_2)$$
$$\subseteq G_k(i_1 + i_2 - 1, j_1 + j_2 - 1).$$

The second summand is contained in $G_k(i_1, j_1 - 1)G_k(i_2, j_2) \subseteq G_k(i_1 + i_2, j_1+j_2-1)$ and the third summand is contained in $G_k(i_1, j_1)G_k(i_2, j_2-1) \subseteq G_k(i_1 + i_2, j_1 + j_2 - 1)$. Thus $X(k_1, i_1, j_1)X(k_2, i_2, j_2) \subseteq X(k_1 + k_2, i_1 + i_2, j_1 + j_2)$.

Similarly

$$[f(x_1, ..., x_n), g(x_1, ..., x_n)] \in U(\mathcal{F}(n))_{<k_1+k_2>} \cap [G(i_1, j_1), G(i_2, j_2)].$$

By Lemma 1.11(ii) this is contained in $U(\mathcal{F}(n))_{<k_1+k_2>} \cap G(i_1 + i_2 - 1, j_1 + j_2)$. Furthermore,

$$t_1^{k_1+k_2}[f(t_1, t_2t_1^{-1}, t_3, ..., t_n), g(t_1, t_2t_1^{-1}, t_3, ..., t_n)]$$
$$-[f(t_1, ..., t_n), g(t_1, ..., t_n)]$$
$$= -t_1^{k_1}[f(t_1, t_2t_1^{-1}, t_3, ..., t_n), t_1^{k_2}]g(t_1, t_2t_1^{-1}, t_3, ..., t_n)$$
$$-t_1^{k_2}[t_1^{k_1}, g(t_1, t_2t_1^{-1}, t_3, ..., t_n)]f(t_1, t_2t_1^{-1}, t_3, ..., t_n)$$
$$+[t_1^{k_1}f(t_1, t_2t_1^{-1}, t_3, ..., t_n), t_1^{k_2}g(t_1, t_2t_1^{-1}, t_3, ..., t_n)]$$
$$-[f(t_1, ..., t_n), g(t_1, ..., t_n)]$$
$$= -t_1^{k_1}[f(t_1, t_2t_1^{-1}, t_3, ..., t_n), t_1^{k_2}]g(t_1, t_2t_1^{-1}, t_3, ..., t_n)$$
$$-t_1^{k_2}[t_1^{k_1}, g(t_1, t_2t_1^{-1}, t_3, ..., t_n)]f(t_1, t_2t_1^{-1}, t_3, ..., t_n)$$
$$+[t_1^{k_1}f(t_1, t_2t_1^{-1}, t_3, ..., t_n) - f(t_1, ..., t_n), t_1^{k_2}g(t_1, t_2t_1^{-1}, t_3, ..., t_n)]$$
$$+[f(t_1, ..., t_n), t_1^{k_2}g(t_1, t_2t_1^{-1}, t_3, ..., t_n) - g(t_1, ..., t_n)].$$

Lemma 1.11 shows that the first summand is in

$$G_k(0,k_1)[G_k(i_1,j_1-k_1),t_1^{k_2}]G_k(i_2,j_2-k_2)$$

$$\subseteq G_k(0,k_1)G_k(i_1-1,j_1-k_1+k_2-1)G_k(i_2,j_2-k_2)$$

$$\subseteq G_k(i_1+i_2-1,j_1+j_2-1),$$

and similarly, the second summand is also in $G_k(i_1+i_2-1,j_1+j_2-1)$. The third summand is in $[G_k(i_1,j_1-1),G_k(i_2,j_2)] \subseteq G_k(i_1+i_2-1,j_1+j_2-1)$, and similarly the fourth summand is also in $G_k(i_1+i_2-1,j_1+j_2-1)$. Thus $[X(k_1,i_1,j_1),X(k_2,i_2,j_2)] \subseteq X(k_1+k_2,i_1+i_2-1,j_1+j_2)$.

It is then immediate, by induction on s, that

$$(ad\ x_{l_s})(ad\ x_{l_{s-1}})...(ad\ x_{l_2})x_{l_1} \in X(l,s-1,0)$$

whenever $l=\sum_{p=1}^{s}\delta_{2,l_p}$ and $s\geq 2$. Thus every element of $\mathcal{M}(n)_{<l>}$ belongs to $X(l,p,0)$ for some $p\leq 0$ or to $X(l,0,1)$. Now $U(\mathcal{F}(n))_{<l>}\cap G(i,j)$ is spanned by products $b_1...b_q$ where $b_i\in\mathcal{M}(n)_{<l_i>}$ and $l_1+...+l_q=l$. But such a product belongs to $X(lk,i,j)$ (by Lemma 1.11), and so the lemma is proved.

Lemma 1.13. *(i)* $G(i,j)=\sum_{l>j}x_1^lU(\mathcal{I}(n))_{i-1}+\sum_{0\leq l\leq j}x_1^lU(\mathcal{I}(n))_i$;
(ii) $G_k(i,j)=\sum_{l>j}t_1^lU(\mathcal{IN}_k(n))_{i-1}+\sum_{l\leq j}t_1^lU(\mathcal{IN}_k(n))_i$.

Proof. Since $\mathcal{F}(n)=Fx_1+\mathcal{I}(n)$, we have

$$U(\mathcal{F}(n))_p=\sum_q U(Fx_1)_qU(\mathcal{I}(n))_{p-q}.$$

But $U(Fx_1)_q=(0)$ if $q<0$, so $U(\mathcal{F}(n))_p=U(Fx_1)_0U(\mathcal{I}(n))_p=\sum_{l=0}^{\infty}x_1^lU(\mathcal{I}(n))_p$. This gives (i) and (ii) is similar.

Corollary 1.14. *Let* $f\in U(\mathcal{F}(n))$ *and* $\alpha_k(f)\in G_k(i,j)$ *where* $k>|i|$. *Then* $f\in G(i,j)$.

Proof. We have $f=\sum_{l=0}^{\infty}x_1^lf_l$ where $f_l\in U(\mathcal{I}(n))$. Since $\alpha_k(f)=\sum_{l=0}^{\infty}t_1^l(\alpha_k(f_l))\in G_k(i,j)$, the lemma shows that $\alpha_k(f_l)\in U(\mathcal{IN}(n))_i$ for $0\leq l\leq j$ and $\alpha_k(f_l)\in U(\mathcal{IN}(n))_{i-1}$ for $l>j$. Then Lemma 1.7 implies $f_l\in U(\mathcal{I}(n))_i$ for $0\leq l\leq j$ and $f_l\in U(\mathcal{I}(n))_{i-1}$ for $l>j$, so that $f\in G(i,j)$.

Lemma 1.15. *Let A be an Ore domain with an abelian valuation $\|\cdot\|$. Suppose $u, v \in A, \|u\| = 0, \|v\| < 0$. Then $\|(u+v)^{-1} - u^{-1}\| < 0$.*

Proof: There exist $r, s, p \in A$ such that $p = r(u+v) = su \neq 0$. Then $(u+v)^{-1} = p^{-1}r, u^{-1} = p^{-1}s$ and $(u+v)^{-1} - u^{-1} = p^{-1}(r-s)$. Since $\|u\| = \|u+v\| = 0$ we have, $\|p\| = \|r\| = \|s\|$. Also $r(u+v) = su$ so $(r-s)u = -rv$, and hence $\|r-s\| = \|r\| + \|v\| - \|u\| < \|r\|$. Thus $\|(u+v)^{-1} - u^{-1}\| = \|p^{-1}(r-s)\| = \|r-s\| - \|p\| < \|r\| - \|r\| = 0$.

§1.6 Quasideterminants

We begin by recalling (from [GR3]) the definition of the *quasideterminants* of an n by n matrix $A = (a_{ij}), i \in I, j \in J$ with entries in a ring R. For $n = 1$, we set $|A|_{ij} = a_{ij}$.

For $n > 1$, and $\alpha \in I, \beta \in J$ we denote by $A^{\alpha\beta}$, the matrix of order $n-1$ constructed by deleting the row with the index α and the column with the index β in the matrix A. Suppose that, for $p \in I, q \in J$, the expressions $|A^{pq}|_{ij}^{-1}, i \in I, j \in J, i \neq p, j \neq q$, are defined. Set

$$|A|_{pq} = a_{pq} - \sum a_{pj}|A^{pq}|_{ij}^{-1}a_{iq}.$$

Here the sum is taken over all $i \in I \setminus \{p\}, j \in J \setminus \{q\}$. The expression $|A|_{pq}$, if it is defined, is called the quasideterminant of indices p and q of the matrix A.

By the well-known nonvanishing of Vandermonde determinants over commutative domains, the Vandermonde quasideterminants that occur in the definition of the elements $y_1, ..., y_n$ exist in $Q_2(n)$, hence in $Q_k(N)$ for all $k > 1$ and also in $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$.

2. Independence results

We will show that certain subsets of $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ are algebraically independent. Our first lemma records a well-known result (cf. [C1, C2]).

Lemma 2.1. *The set $\{x_1, ..., x_n\} \subset F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ is algebraically independent.*

Let $\alpha_k : F < x_1, ..., x_n > \longrightarrow Q_k(n)$ be the homomorphism defined in Section 1.4. Let A_k denote the corresponding subalgebra of $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ and let $\beta_k : A_k \longrightarrow Q_k(n)$ be the extension of α_k which exists by the universal property of $F\langle\!\langle x_1, ..., x_n \rangle\!\rangle$ as described in Section 1.1.

Proposition 2.2. *Assume that* $u_1, ..., u_n \in F\langle\!\!\langle x_1, ..., x_n \rangle\!\!\rangle$ *satisfy* $u_1, ..., u_n \in A_k$ *for all* $k \geq 1$*. Assume further that for each* i *and* k $\|\alpha_k(u_i)\| < 0$*. Then* $\{x_1+u_1, ..., x_n+u_n\}$ *is an algebraically independent subset of* $F\langle\!\!\langle x_1, ..., x_n \rangle\!\!\rangle$*.*

Proof: Assume that $\{x_1 + u_1, ..., x_n + u_n\}$ is not algebraically independent. Then there is some $0 \neq f \in F < x_1, ..., x_n >$ such that $0 = f(x_1+u_1, ..., x_n+u_n)$. Take $k < \|f\|$ and write $\alpha_k(x_i) = t_i, \alpha_k(u_i) = v_i$ for $1 \leq i \leq n$. Then Lemma 1.7 shows that $\|f\| = \|f(t_1, ..., t_n)\|$ and we have $f(t_1, ..., t_n) - f(t_1 + v_1, ..., t_n + v_n) = \alpha_k(f(x_1, ..., x_n) - f(x_1 + u_n, ..., x_n + u_n)) = \alpha(f(x_1, ..., x_n)) = f(t_1, ..., t_n)$. However, $f(x_1, ..., x_n) \neq 0$ by Lemma 2.1. Then (1.4.6) shows that $\|f(t_1, ..., t_n) - f(t_1 + v_1, ..., t_n + v_n)\| \|f\|$, a contradiction.

Corollary 2.3. *The set* $\{y_1, ..., y_n\} \subset F\langle\!\!\langle x_1, ..., x_n \rangle\!\!\rangle$ *is algebraically independent.*

Proof: Recall that $y_i = V(x_1, ..., x_i)x_i(V(x_1, ..., x_i))^{-1} = x_i + [V(x_1, ..., x_i), x_i](V(x_1, ..., x_i))^{-1}$. Set

$$u_i = [V(x_1, ..., x_i), x_i](V(x_1, ..., x_i))^{-1}$$

and $w_i = \alpha_k(V(x_1, ..., x_i))$. Then $\|\alpha_k(u_i)\| = \|[w_i, t_i]w_i^{-1}\|$. Then (1.2.5) and (1.2.6) show that $\|\alpha_k(u_i)\| > \|w_i\| + \|t_i\| - \|w_i\| = 0$. Hence Proposition 2.2 applies and gives the result.

For $1 \leq i \leq n-1$ let σ_i denote the permutation of $\{1, ..., n\}$ which interchanges i and $i+1$ and fixes all $j \neq i, i+1$. Then σ_i induces an automorphism, again denoted σ_i, of $F\langle\!\!\langle x_1, ..., x_n \rangle\!\!\rangle$. Define $s_i = (y_i + \sigma_i(y_i))(y_i - \sigma_i(y_i))^{-1}$ and $z_i = (y_i - \sigma(y_i))/2$.

Lemma 2.4. *Let* $1 \leq i \leq n-1$*. The set*

$$\{y_1, ..., y_{i-1}, z_i, s_i, y_{i+2}, ..., y_n\} \subset F\langle\!\!\langle x_1, ..., x_n \rangle\!\!\rangle$$

is algebraically independent.

Proof: The automorphism τ of the vector space $Fx_1 + ... + Fx_n$ defined by

$$\tau((x_i - x_{i+1})/2) = x_1$$

$$\tau(x_i + x_{i+1}) = x_2$$

$$\tau(x_j) = x_{j+2}, if \quad 1 \le j \le i-1$$
$$\tau(x_j) = x_j, if \quad j \ge i+2$$

extends to an automorphism, again denoted τ, of $F\langle x_1, ..., x_n \rangle$.

Note that $\alpha_k\tau(y_j)$

$$= \alpha_k\tau(x_j + [V(x_1, ..., x_j), x_j](V(x_1, ..., x_j))^{-1}) \in \alpha_k\tau(x_j) + Q_k(n)_{-1}.$$

Also $\alpha_k\tau(z_i) \in \alpha_k\tau((x_i - x_{i+1})/2) + Q_k(n)_{-1} = t_1 + Q_k(n)_{-1}$. Finally,

$$\alpha_k\tau(s_i) = \alpha_k\tau((y_i + \sigma_i(y_i))(y_i - \sigma_i(y_i))^{-1})$$
$$= \{\alpha_k\tau(x_i + [V(x_1, ..., x_i), x_i](V(x_1, ..., x_i))^{-1} + x_{i+1}$$
$$+[\sigma_i(V(x_1, ..., x_i)), x_{i+1}]\sigma_i((V(x_1, ..., x_i))^{-1}))\} \cdot$$
$$\{\alpha_k\tau(x_i + [V(x_1, ..., x_i), x_i](V(x_1, ..., x_i))^{-1} - x_{i+1}$$
$$-[\sigma_i(V(x_1, ..., x_i)), x_{i+1}]\sigma_i((V(x_1, ..., x_i))^{-1}))\}^{-1}$$
$$\in \{\alpha_k\tau(x_i + x_{i+1}) + Q_k(n)_{-1}\}\{\alpha_k\tau(x_i - x_{i+1}) + Q_k(n)_{-1}\}^{-1}.$$

By Lemma 1.15 this is contained in

$$\{\alpha_k\tau(x_i + x_{i+1}) + Q_k(n)_{-1}\}\{(\alpha_k\tau(x_i - x_{i+1}))^{-1} + Q_k(n)_{-1}\}$$
$$= \alpha_k\tau((x_i + x_{i+1})(x_i - x_{i+1})^{-1}) + Q_k(n)_{-1} = t_2t_1^{-1} + Q_(k)n_{-1}.$$

Now suppose that $\{y_1, ..., y_{i-1}, z_i, s_i, y_{i+2}, ..., y_n\}$ is not algebraically independent. Then there is some $0 \ne f \in F < x_1, ..., x_n >= U(\mathcal{F}(n))$ such that

$$f(z_i, 2s_i, y_1, ..., y_{i-1}, y_{i+2}, ..., y_n) = 0.$$

We may assume, without loss of generality, that $f \in U(\mathcal{F}(n))_{<l>}$ for some l, and we may find $i \le 0, j \ge 0$ so that $f \in G(i,j), f \ne G(i, j-1)$. Take $k > |i|$. We then have $0 = \alpha_k\tau f(z_i, 2s_i, y_1, ..., y_{i-1}, y_{i+2}, ..., y_n)$ $= f(t_1 + v_1, t_2t_1^{-1} + v_2, t_3 + v_3, ..., t_n + v_n)$ where $\|v_i\| < 0$ for all i. Then Lemma 1.9 shows that $\|f(t_1, t_2t_1^{-1}, t_3, ..., t_n)\| =$ $\|f(t_1, t_2t_1^{-1}, t_3, ..., t_n) - f(t_1 + v_1, t_2t_1^{-1} + v_2, t_3 + v_3, ..., t_n + v_n)\| \le i-1$. But Lemma 1.12 shows that

$$t_1^k f(t_1, t_2t_1^{-1}, t_3, ..., t_n) - f(t_1, ..., t_n) \in G(i, j-1).$$

Thus $f(t_1, ..., t_n) \in G_k(i, j-1) + U(\mathcal{FN}(n))_{i-1} = G_k(i, j-1)$. But, by Corollary 1.14, this implies that $f(x_1, ..., x_n) \in G(i, j-1)$, contradicting our choice of i and j.

3. Invariant elements

Let A be an associative algebra over F and $\sigma \in Aut\ A$. For any subset $X \subset A$, let $F[X]$ denote the F-subalgebra of A generated by X. Let $\{a_1, ..., a_k, s, z\}$ be an algebraically independent subset of A. Assume that $\sigma z = -z, \sigma s = -s$ and $\sigma a_i = a_i$ for all i.

Let $b_1 = (sz + z)/2$ and $b_2 = (zs - z)/2$.

Proposition 3.1. *Let $0 \neq f \in B = F[b_1, b_2, a_1, ..., a_k]$ satisfy $\sigma f = cf$ for some $c \in F$. Then $c = 1$ and $f \in F[b_1 + b_2, b_2 b_1, a_1, ..., a_k]$.*

Proof: We may assume, without loss of generality, that f is homogeneous of degree l (as a polynomial in $b_1, b_2, a_1, ..., a_k$). The result clearly holds if $l = 0$. We will proceed by induction on l. Thus we assume $l \geq 1$ and write

$$f = b_1 f_1 + b_2 f_2 + \sum_{j=1}^{k} a_j g_j$$

where $f_1, f_2, g_1, ..., g_k \in B$ are homogeneous of degree $l-1$. Then $2f = (s+1)zf_1 + z(s-1)f_2 + 2\sum_{j=1}^{k} a_j g_j = s(zf_1) + z(f_1 + sf_2 - f_2) + 2\sum_{j=1}^{k} a_j g_j$ and so

$$0 = 2(\sigma f - cf) \in sz(\sigma f_1 - cf_1) + 2\sum_{j=1}^{k} a_j(\sigma g_j - cg_j) + zB.$$

As $\{a_1, ..., a_k, s, z\}$ is algebraically independent, we have $0 = \sigma f_1 - cf_1 = \sigma g_1 - cg_1 = ... = \sigma g_k - cg_k$. Then the induction assumption implies that $f_1, g_1, ..., g_k \in F[b_1 + b_2, b_2 b_1, a_1, ..., a_k]$. Replacing f by $f - (b_1 + b_2)f_1 - \sum_{j=1}^{k} a_j g_j$, we may assume that $0 = f_1 = g_1 = ... = g_k$. Note that (as $\sigma b_2 \notin F b_2$) this proves the proposition in the case $l = 1$.

Now assume $l \geq 2$ and write

$$f = b_2(b_1 h_1 + b_2 h_2 + \sum_{j=1}^{k} a_j p_j)$$

where $h_1, h_2, p_1, ..., p_k \in B$. Then $4f = (z(s^2-1)z - z^2)h_1 + (z(s-1)z(s-1))h_2 + 2(zs-z)\sum_{j=1}^{k} a_j p_j)$ and so

$$0 = 4(\sigma f - cf) \in zs^2 z(\sigma h_1 - ch_1)$$

$$+2zs\sum_{j=1}^{k} a_j(\sigma p_j - cp_j) + 2s\sum_{j=1}^{k} a_j(\sigma p_j + cp_j) + z^2 B + zszB.$$

As $\{a_1, ..., a_k, s, z\}$ is algebraically independent, we have $0 = \sigma h_1 - ch_1 = \sigma p_1 - cp_1 = ... = \sigma p_k - cp_k = \sigma p_1 + cp_1 = ... = \sigma p_k + cp_k$. Consequently, $p_1 = ... = p_k = 0$. Furthermore, the induction assumption implies that $h_1 \in F[b_1 + b_2, b_2 b_1, a_1, ..., a_k]$ and so, replacing f by $f - b_2 b_1 h_1$, we see that we may assume $h_1 = 0$ and so $4f = 4b_2^2 h_2 = z(s-1)z(s-1)h_2 = zsz(s-1)h_2 - z^2(s-1)h_2 = zszq - z^2 q$ where $q = (s-1)h_2 \in B$. Then

$$0 = 4(\sigma f - cf) = -zsz(\sigma q + cq) - z^2(\sigma q - cq).$$

As $\{a_1, ..., a_k, s, z\}$ is algebraically independent, it follows that $\sigma q + cq = 0$ and $\sigma q - cq = 0$. Thus $q = 0$ and so $h_2 = 0$, proving the proposition.

Let $y_1, ..., y_n$ be algebraically independent elements of an associative algebra A over a field F. Let

$$Y = F[y_1, ..., y_n]$$

be the subalgebra of A generated by $y_1, ..., y_n$. Note that, by the algebraic independence of the y_i, we have

$$Y = \oplus_{i=1}^{n} y_i Y.$$

Write

$$Y_i = \sum_{l=i}^{n} y_l Y$$

for $1 \leq i \leq n$ and set $Y_{n+1} = (0)$. For $1 \leq i < n$, set

$$Y^{[i]} = F[y_1, ..., y_{i-1}, y_i + y_{i+1}, y_{i+1} y_i, y_{i+2}, ..., y_n].$$

For $1 \leq j < i \leq n$, define

$$\Lambda_{i,j} = 0$$

and for $1 \leq j \leq n$, define

$$\Lambda_{0,j} = 1.$$

For $1 \leq i \leq j \leq n$, define

$$\Lambda_{i,j} = \sum_{j \geq l_1 > ... > l_i \geq 1} y_{l_1} y_{l_2} ... y_{l_i}.$$

For $1 \leq j \leq n$, set

$$\Lambda_j = F[\Lambda_{1,j}, ..., \Lambda_{j,j}, y_{j+1}, ..., y_n].$$

Lemma 3.2. *If* $1 \leq j \leq n-1$*, then* $\Lambda_j \cap Y^{[j]} = \Lambda_{j+1}$.

Proof: Note that for $1 \leq j \leq n-1$ and $1 \leq i \leq n$, we have

$$\Lambda_{i,j+1} = y_{j+1}\Lambda_{i-1,j} + \Lambda_{i,j}$$

and consequently

$$\Lambda_{j+1} \subseteq \Lambda_j.$$

Furthermore, we also see that

$$\Lambda_{i,j+1} = y_{j+1}y_j\Lambda_{i-2,j-1} + (y_{j+1} + y_j)\Lambda_{i-1,j-1} + \Lambda_{i,j-1}$$

for $2 \leq i \leq n, 1 \leq j \leq n-1$, and so

$$\Lambda_{j+1} \subseteq Y^{[j]}.$$

Thus

$$\Lambda_{j+1} \subseteq \Lambda_j \cap Y^{[j]}.$$

Hence we need to show that if $f \in \Lambda_j \cap Y^{[j]}$, then $f \in \Lambda_{j+1}$. Without loss of generality, we may assume that f is homogeneous of degree $t \geq 0$ in $\{y_1, ..., y_n\}$. The assertion is clearly true if $t = 0$. We now proceed by induction on t, assuming that the assertion is true for homogeneous polynomials of degree $< t$.

Suppose that for $1 \leq i \leq n$, we have $f \in Y_i \cap \Lambda_j \cap Y^{[j]}$. Then we may write

$$f = y_i f_i + ... + y_n f_n$$

where $f_i, ..., f_n \in Y$,

$$f = \Lambda_{1,j} g_1 + ... + \Lambda_{j,j} g_j + y_{j+1} g_{j+1} + ... + y_n g_n$$

where $g_1, ..., g_n \in \Lambda_j$, and $f =$

$$y_1 h_1 + ... + y_{j-1} h_{j-1} + (y_j + y_{j+1}) h_j + y_{j+1} y_j h_{j+1} + y_{j+2} h_{j+2} + ... + y_n h_n,$$

where $h_1, ..., h_n \in Y^{[j]}$. We will show that

$$f \in Y_i \cap \Lambda_{j+1} + Y_{i+1} \cap \Lambda_j \cap Y^{[j]}.$$

Note that, since $Y_1 = Y$ and $Y_{n+1} = 0$, iterating this result proves the lemma.

To prove our assertion, first suppose that $i \leq j$. Then $g_1 = ... = g_{i-1} = h_1 = ... = h_{i-1} = 0$ and $y_i f_i = y_i y_{i-1} ... y_1 g_i = y_i h_i$. Thus $y_{i-1} ... y_1 g_i \in Y^{[j]}$, and so $g_i \in Y^{[j]}$. But then $g_i \in \Lambda_j \cap Y^{[j]}$ and so, by the induction assumption, $g_i \in \Lambda_{j+1}$. But then $\Lambda_{i,j+1} g_i \in Y_i \cap \Lambda_{j+1}$ and so, since $f - \Lambda_{i,j+1} g_i = f - (y_{j+1} \Lambda_{i-1,j} + \Lambda_{i,j}) g_i \in Y_{i+1}$, we have $f - \Lambda_{i,j+1} g_i \in Y_i \cap \Lambda_{j+1}$, proving our assertion.

Next suppose that $i = j + 1$. Then we have $y_{j+1} g_{j+1} = y_{j+1} y_j h_{j+1}$, and so $g_{j+1} = y_j h_{j+1} \in \Lambda_j$. Then $g_{j+1} = \Lambda_{j,j} h'_{j+1}$ with $h'_{j+1} \in \Lambda_j$ and so $h_{j+1} = y_{j-1} ... y_1 h'_{j+1} \in Y^{[j]}$. It follows that $h'_{j+1} \in \Lambda_{j+1}$, so by the induction assumption $h'_{j+1} \in \Lambda_{j+1}$. Then $y_{j+1} g_{j+1} = y_{j+1} \Lambda_{j,j} h'_{j+1} = \Lambda_{i+1,j+1} h'_{j+1} \in \Lambda_{j+1}$. Since $f - y_{j+1} g_{j+1} \in Y_{i+1}$, our assertion is proved in this case.

Finally, suppose $i > j+1$. Then $y_i g_i = y_i h_i$ so $g_i = h_i\ in \Lambda_j \cap Y^{[j]}$ and, by the induction assumption $g_i \in \Lambda_{j+1}$. Therefore $y_i g_i \in \Lambda_{j+1}$. Since $f - y_i g_i \in Y_{i+1}$, our assertion is proved in this case as well, completing the proof of the lemma.

Noting that $Y_1 = \Lambda_2$ we obtain the following immediate consequence of Lemma 3.2:

Proposition 3.3. *For $2 \leq j \leq n$, we have $\cap_{i=1}^{j-1} Y^{[i]} = \Lambda_j$.*

4. Proof of the Gelfand-Retakh conjecture

Theorem 4.1. *Let $f \in F[y_1, ..., y_n]$ and suppose $\sigma f = f$ for all $\sigma \in S_n$. Then $f \in F[\Lambda_{1,n}, ..., \Lambda_{n,n}]$.*

Proof: We have $\|y_i - x_i\|_1 < 0$ and $\|y_i - \sigma_i(y_i)\| = 0$. Thus $y_i - \sigma_i(y_i) \neq 0$ and so $(y_i - \sigma_i(y_i))^{-1}$ exists.

It is clear from the definition that

$$\sigma_i(y_j) = y_j$$

whenever $j < i$. By the Gelfand-Retakh Theorem we have $y_{i+1}y_iy_{i-1}...y_1 = \Lambda_{i+1,i+1} = \sigma_i(\Lambda_{i+1,i+1}) = \sigma_i(y_{i+1})\sigma_i(y_i)(y_{i-1}...y_1)$ and so

$$\sigma_i(y_{i+1})\sigma_i(y_i) = y_{i+1}y_i.$$

Similarly, we have $\Lambda_{i,i+1} = (y_{i+1} + y_i)\Lambda_{i-1,i-1} + y_{i+1}y_i\Lambda_{i-2,i-1} = \sigma_i(\Lambda_{i,i+1}) = (y_{i+1}+y_i)\Lambda_{i-1,i-1}+y_{i+1}y_i\Lambda_{i-2,i-1} = \sigma_i(y_{i+1}+y_i)\Lambda_{i-1,i-1}+ y_{i+1}y_i\Lambda_{i-2,i-1}$. Therefore

$$\sigma_i(y_{i+1} + y_i) = y_{i+1} + y_i.$$

Also, for $j > i + 1$ we have $\Lambda_{j,j} = y_j\Lambda_{j-1,j-1} = \sigma_iy_j(\Lambda_{j-1,j-1}) = \sigma_i(y_j)\Lambda_{j-1,j-1}$. Thus

$$\sigma_i(y_j) = y_j$$

whenever $j > i + 1$.

Let $u_i = (y_i + \sigma_i(y_i))/2, v_i = (y_{i+1} + \sigma_i(y_{i+1}))/2$, and $z_i = (y_i - \sigma_i(y_i))/2$. Since $\sigma_i(y_{i+1} + y_i) = y_{i+1} + y_i$, we also have $z_i = -(y_{i+1} - \sigma_i(y_{i+1}))/2$. We have noted that z_i^{-1} exists; set $s_i = u_iz_i^{-1}$. By Lemma 2.4 we have that the set $\{y_1, ..., y_{i-1}, s_i, z_i, y_{i+2}, ..., y_n\}$ is algebraically independent. Since $\sigma_i(y_{i+1})\sigma_i(y_i) = y_{i+1}y_i$, we have that $z_iu_i = (\sigma_i(y_{i+1})y_i - y_{i+1}\sigma(y_i))/4 = v_iz_i$. Then we have

$$y_i = u_i + z_i = s_iz_i + z_i,$$

$$y_{i+1} = v_i - z_i = z_is_i - z_i.$$

Note that $\sigma_i(s_i) = -s_i$ and $\sigma_i(z_i) = -z_i$. Therefore the hypotheses of Proposition 3.1 are satisfied (with $k = n-2, a_j = y_j$ for $1 \leq j \leq i-1, s = s_i, z = z_i, a_j = y_{j+2}$ for $i \leq j \leq n-2$ and $\sigma = \sigma_i$). Therefore $f \in F[y_1, ..., y_{i-1}, y_i + y_{i+1}, y_{i+1}y_i, y_{i+2}, ..., y_n]$ or, in the notation of Proposition 3.3, $f \in Y^{[i]}$. Since this holds for all $i, 1 \leq i \leq n-1$, Proposition 3.3 shows that $f \in \Lambda_n$, proving the theorem.

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