MATH 250 **April 28, 2012**

REVIEW PROBLEMS FOR FINAL EXAMINATION

#1 Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Find each of the following if it is defined. If it is not defined, say so.

a) BA; b) AC; c) BC; d) $\mathbf{u}{\cdot}\mathbf{v}$; e) $\mathbf{uv}$; f) $(\mathbf{u}^T)\mathbf{v}$; g) $\mathbf{u}(\mathbf{v}^T)$; h) $AB - BA$; i) $BA - C$; j) $\|\mathbf{v}\|$;

#2 Find the inverse of $A = \begin{bmatrix} 1 & 1 & -8 & -2 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Then find the solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

of the equation

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

#3 Let $A = [a_{ij}]$ be a 5 by 5 matrix with $det\ A = 1$. Let $B = [ija_{ij}]$. (That is, the (i, j) entry of B is ij times the (i, j) entry of A.) What is $det\ B$?

#4 Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 0 & 3 & 2 \\ 1 & -1 & -1 & 1 & 1 \\ 3 & 0 & -1 & 4 & 3 \end{bmatrix}.$$

Find: a) a basis for the row space of A; b) a basis for the column space of A; c) a basis for the nullspace of A; d) the dimension of the row space of A;

e) the dimension of the column space of A; f) the dimension of the nullspace of A. State two general results relating these numbers.

#5 If the following system of linear equations is consistent, solve it for $\mathbf{x} = [x_1, x_2, x_3]$. If it is not consistent explain why.

$$2x_1 + 4x_2 + 6x_3 = 4$$

$$x_2 + 3x_3 = 2$$

$$3x_1 + 5x_2 + 6x_3 = 1$$

#6 Find all values of a such that the following system of linear equations has a solution, and then find all of the solutions.

$$x_1 + 2x_2 + 2x_3 = 6$$

$$x_1 - 2x_2 + 2x_3 = a$$

$$2x_2 + 4x_3 = 1$$

$$-x_2 + 2x_3 = 2$$

#7 You should be able to state the definitions of each of the following terms. (I give a page reference for each term.)

the dot product of vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^n$ (page 363);
two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if ... (page 363);
the norm (or length) of a vector $\mathbf{v}$ (page 361);
an orthogonal set of vectors $\{\mathbf{x}_1, ..., \mathbf{x}_k\}$ (page 374);
a unit vector (page 361);
an orthogonal basis for a subspace W of $\mathbf{R}^n$ (page 375);
an orthonormal basis for a subspace W of $\mathbf{R}^n$ (page 375);
the orthogonal complement of a set S of vectors in $\mathbf{R}^n$ (page 389);
the orthogonal projection of a vector $\mathbf{u}$ on the line through a vector $\mathbf{v}$ (page 366);
the orthogonal projection of a vector $\mathbf{u}$ on a subspace W of $\mathbf{R}^n$ (page 393);
the distance from a vector $\mathbf{v}$ to a subspace W of $\mathbf{R}^n$ (page 397);
a symmetric matrix A (page 12);
an orthogonal matrix Q (page 412).

You should also be able to state the definitions of terms listed in the previous review sheets. These are:

the transpose of a matrix (page 7);

a linear combination of a set of vectors (page 14);
the standard vectors $\mathbf{e}_1, ..., \mathbf{e}_n$ (page 17);
a consistent (or inconsistent) system of linear equations (page 29);
the augmented matrix of a system of linear equations (page 31);
the coefficient matrix of a system of linear equations (page 31);
a basic variable for a system of linear equations (page 35);
a free variable for a system of linear equations (page 35);
the rank of a matrix (page 47);
the nullity of a matrix (page 47);
the phrase "a set of vectors is linearly dependent" (page 75);
the phrase "a set of vectors is linearly independent" (page 75);
an upper triangular matrix (page 153);
a lower triangular matrix (page 153);
a subspace V of $\mathbf{R}^n$ (page 227);
the null space of a matrix A (page 232);
the column space of a matrix A (page 233);
the row space of a matrix (page 236);
a basis for a subspace V of $\mathbf{R}^n$ (page 241);
the dimension of a subspace V of $\mathbf{R}^n$ (page 246);
an eigenvector $\mathbf{v}$ of an n by n matrix A (page 294);
the eigenvalue λ of an n by n matrix A that corresponds to an eigenvector $\mathbf{v}$ (page 294);
the eigenspace of an n by n matrix A that corresponds to an eigenvalue λ (page 296);
the characteristic polynomial of an n by n matrix A (page 302);
the multiplicity of an eigenvalue λ of an n by n matrix A (page 305);
an n by n matrix A is diagonalizable if ... (page 315);

#8 Find an orthonormal basis for the null space of the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 2 & 4 \end{bmatrix}.$$

#9 Find the equation of the least-squares line for the data:

$$(-1, 1), (0, 0), (1, 3), (2, 4).$$

#10

Find the vector in $Span\ \{\begin{bmatrix}1\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\2\end{bmatrix}\}$ which is closest to the vector $\begin{bmatrix}-1\\-1\\2\\1\end{bmatrix}$.

#11 Let $A = \begin{bmatrix}-3 & 6 & 0\\0 & 3 & 0\\-3 & 2 & 0\end{bmatrix}$.

a) Find the eigenvalues of A and a basis for each eigenspace.

b) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (or, equivalently, $D = P^{-1}AP$.

c) Find the general solution of the system of differential equations

$$y_1' = -3y_1 + 6y_2$$
$$y_2' = 3y_2$$
$$y_3' = -3y_1 + 2y_2.$$

#12 Let $W = \ Span\ \{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}\}$, a subspace of $\mathbf{R}^4$.

a) Find the orthogonal complement $W^{\perp}$.

b) Find the orthogonal projection of $\mathbf{v} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ on W.

c) Find an orthonormal basis for W.

#13 Consider the real symmetric matrices $A = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 2\\0 & 2 & 4\end{bmatrix}$ and $B = \begin{bmatrix}1 & 2 & 2\\2 & 4 & 2\\1 & 2 & 1\end{bmatrix}$.

a) Find the eigenvalues of A and an orthonormal basis for each eigenspace. Find the eigenvalues of B and an orthonormal basis for each eigenspace.

b) Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$ (or, equivalently, $D = Q^{-1}AQ$). Find an orthogonal matrix

Q_1 and a diagonal matrix D_1 such that $B = Q_1 D_1 Q_1^{-1}$ (or, equivalently, $D_1 = Q_1^{-1} A_1 Q_1$).

c) Write down Q_1^{-1} explicitly.

#14 Compute *det* A if

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 7 & 1 \\ 3 & 3 & 3 & 6 \end{bmatrix}$$

#15 Compute *det* B if

$$B = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 3 & 1 \\ 5 & 2 & 5 \end{bmatrix}$$

#16 Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbf{R}^n$. Suppose

$$||\mathbf{u}|| = 2, ||\mathbf{v}|| = 5, ||\mathbf{w}|| = 3, \mathbf{u}.\mathbf{v} = -4, \mathbf{u}.\mathbf{w} = 2,$$

and

$$\mathbf{v}.\mathbf{w} = 6.$$

Find

$$||3\mathbf{u} - 2\mathbf{v} + \mathbf{w}||.$$

#17 Suppose R and S are two 3 by 3 matrices, that *det* $R = 3$ and *det* $S = 5$. Find a) $det(RS)$; b) *det* $(2S)$; c) *det* R^2.

#18 Suppose that P is a 2 by 3 matrix, that Q is a 3 by 2 matrix and that *det* $PQ = 7$. What is *det* QP? Why?

#19 (a) Prove that if A is an n by n symmetric matrix and $\mathbf{u}$ and $\mathbf{v}$ are column vectors in $\mathbf{R}^3$ then $(A\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(A\mathbf{v})$.

(b) Prove that if A is an n by n symmetric matrix and $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for A corresponding to different eigenvalues then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

#20 Let A be a 4 by 4 symmetric matrix. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}$.
Suppose $A\mathbf{u} = 3\mathbf{u}$ and $A\mathbf{v} = a\mathbf{v}$ for some real number a. What is a? Why?

#21 Find the LU decomposition of

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ -1 & -1 & 3 & 0 \\ 2 & 8 & 1 & 8 \end{bmatrix}.$$

#22 A is a 5 by 5 matrix. The equation $A\mathbf{x} = \mathbf{0}$ has free variables x_2, x_3, and x_5 and has general solution

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

(a) Find the reduced row echelon form of A.
(b) If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{bmatrix}$ and

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

and

$$\mathbf{a}_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

find $\mathbf{a}_2$ and $\mathbf{a}_3$ and $\mathbf{a}_5$.

#23 Let U, V and W be subspaces of $\mathbf{R}^5$. Assume

$$U \subseteq V \subseteq W^{\perp}$$

and that

$$dim\ U = 2, dim\ W = 3.$$

What is $dim\ V$?

#24 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a linearly independent set of vectors in $\mathbf{R}^6$ and let V be *Span* S.
(a) Show that $T = \{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_4, \mathbf{v}_3 + \mathbf{v}_4\}$ is a basis for V.
(b) Show that $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_4, \mathbf{v}_4 - \mathbf{v}_1\}$ is linearly dependent.