Math 300 - Solutions to review problems for Exam #2 - April 12, 2009

#1 Use mathematical induction to prove that 8 divides $5^{2n} - 1$ for every integer $n \geq 1$.

Solution: Let

$$S = \{k \in \mathbf{N} | 8 \ \ divides \ \ 5^{2k} - 1\}.$$

Since 8 divides $5^2 - 1 = 25 - 1 = 24$ we have $1 \in S$. Next assume that $k \in S$. Now

$$5^{2(k+1)} - 1 = (5^2 - 1)5^{2k} + (5^{2k} - 1) = (24)5^{2k} + (5^{2k} - 1).$$

Since 8 divides 24, the first summand is divisible by 8, and since we are assuming that $k \in S$, the second summand is divisible by 8. Thus $5^{2(k+1)} - 1$ is divisible by 8 and so $k+1 \in S$. Thus, by the principle of mathematical inuction, $S = \mathbf{N}$ and so 8 divides $5^{2n} - 1$ for all $n \in \mathbf{N}$.

#2 Use the well ordering principle to show that if a, b are natural numbers then there exist integers $q, r \geq 0$ such that $a = qb + r$ and $0 \leq r < b$.

Solution: Let

$$A = \{a - q_1 b | a - q_1 b > 0, q_1 \geq 0, \ \ and \ \ q_1 \ \ is \ \ an \ \ integer\}.$$

Then A is a subset of the natrual numbers. Furthermore, $A \neq \emptyset$ because $a = a - 0(b) \in A$.. Thus by the well-ordering principle, A contains a smallest element. Call this element r_1. Then since $r_1 \in A$ we have $r_1 = a - q_1 b$ and so

$$a = q_1 b + r_1.$$

Thus if $r_1 < b$ we may take $q = q_1$, $r = r_1$ and we are done. Now suppose $r_1 > b$. Then $a - (q_1 + 1)b = (a - q_1 b) - b = r_1 - b > 0$ and so $r_1 - b \in A$. But this is impossible, since r_1 is the smallest element of A and $r_1 > r_1 - b$. Finally, suppose $r_1 = b$. Then $b = a - q_1 b$ and so $0 = a - (q_1 + 1)b$. Then taking $q = q_1 + 1$ and $r = 0$ gives the result.

#3 Suppose $\overline{\overline{A}} = 33, \overline{\overline{B}} = 17$, and $\overline{\overline{A \cap B}} = 12$. Find $\overline{\overline{A \cup B}}$.

Solution: Recall that, for any finite sets X, Y we have

$$\overline{\overline{X}} + \overline{\overline{Y}} = \overline{\overline{X \cup Y}} + \overline{\overline{X \cap Y}}.$$

Thus $33 + 17 = 12 + \overline{\overline{A \cup B}}$ and so $\overline{\overline{A \cup B}} = 38$.

#4 Suppose $\overline{\overline{A}} = 11$.

(a) Find $\overline{\overline{\mathcal{P}(A)}}$.

Solution: $\overline{\overline{\mathcal{P}(A)}} = 2^{11}$.

(b) How many subsets $B \subseteq A$ satisfy $\overline{\overline{B}} = 4$?

Solution: The number of such sets is $\binom{11}{4} = (11)(10)(9)(8)/4! = 330$.

#5 (a) State the definition of a relation from A to B.

Solution: A relation from A to B is a subset of $A \times B$. (See page 133.)

(b) Suppose $\overline{\overline{A}} = 7, \overline{\overline{B}} = 5$. How many relations from A to B are there?

Solution: Since $\overline{\overline{A \times B}} = 35$, the number of relations from A to B is

$$\overline{\overline{\mathcal{P}(A \times B)}} = 2^{35}.$$

#6 Let $A = \{1, 2, 3, 4, 5\}$. For each of the following relations from A to A state whether or not it is an equivalence relation. If it is an equivalence relation give the corresponding partition of A.

(a) $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4), (5, 5)\}$.

Solution: This is an equivalence relation and the corresponding partition of A is

$$\{\{1, 2, 3\}, \{4\}, \{5\}\}.$$

(b) $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4), (4, 5), (5, 5)\}$.

Solution: This is not an equivalence relation since $(4, 5) \in R$, but $(5, 4) \notin R$.

#7 (a) State the reflexive, symmetric, anti-symmetric, and transitive properties of a relation.

Solution: See page 145 for reflexive, symmetric and transitive and page 160 for antisymmetric.

(b) State the definition of an equivalence relation, of a partial order, of a total order, and of a partition.

Solution: See page 147 for equivalence relation, page 161 for partial order, page 165 for total order and page 154 for partition.

#8 For $0 \le n \le 6$, let $J_n = \{k \in \mathbf{Z} | 7 \ \ divides \ \ k - n\}$. Show that $\{J_0, ..., J_6\}$ is a partition of the integers and describe the corresponding equivalence relation.

Solution: If $0 \le n < 7$, then $n \in J_n$, so $J_n \ne \emptyset$. Suppose $J_m \cap J_n \ne \emptyset$. Then there is some integer $k \in J_m \cap J_n$ and so 7 divides $k - m$ and 7 divides $k - n$. Then 7 divides $(k - m) - (k - n) = n - m$. Since $0 \le m, n < 7$ this implies that $m = n$ and so $J_m = J_n$. Finally, let be any integer. Then we may write $k = 7q + r$ for some integers q and r with $0 \le r < 7$. (This is essentially the result proved in problem #2.) Then $k \in J_r$. Hence $\mathbf{Z} = \cup_{0 \le r < 7} J_r$. Let R be the corresponding equivalence relation. Then, for integers a and b , we have aRb if and only if there is some k such that $a, b \in J_k$. (That is, a and b belong to the same one of the J's.) Then 7 divides $a - k$ and 7 divides $b - k$ so 7 divides $a - b$. Thus $a \equiv_7 b$. On the other hand, if a and b are integers with $a \equiv_7 b$ and $a \in J_k$ then 7 divides $a - k$ and 7 divides $a - b$ so 7 divides $b - k = (a - k) - (a - b)$. Thus $b \in J_k$ and so aRb. This shows that R, the equivalence relation corresponding to the partition $\{J_0, ..., J_6\}$, is $\equiv_7$.

#9 (a) Define a relation R on the integers by aRb if and only if either $a = b$ or $a + 2 < b$. Is R a partial order? Is it a total order?

Solution: Since aRa for every integer a, R is reflexive. Suppose aRb and bRa. If $a \ne b$ we must have $a+2 < b$ and $b+2 < a$ so $a+2 < b < b+2 < a$ which is impossible. Thus $a = b$ and so R is antisymmetric. Finally, if we have aRb and bRc then we have either $a = b$ or $a + 2 < b$ and we also have either $b = c$ or $b + 2 < c$. Now if $a = b$ then either $a = b = c$ or $a + 2 = b + 2 < c$ so we have aRc. Also, if $b = c$ we have either $a = b = c$ or $a + 2 < b = c$ and so we have aRc. Finally, if $a + 2 < b$ and $b + 2 < c$, then $a + 2 < b < b + 2 < c$ so we have aRc. Thus R is transitive and so R is a partial order. It is not a total order since, for any integer a, neither $aR(a + 1)$ or $(a + 1)Ra$ holds.

(b) Define a relation S on the integers by aSb if and only a and b have the same parity (i.e., both are even or both are odd) and $a \le b$. Is S a partial order? Is it a total order?

Solution:

Clearly aSa holds for every integer a, so S is reflexive. If aSb and bSa then, $a \le b$ and $b \le a$ so $a = b$. Thus S is antisymmetric. Finally, if aSb and bSc hold, then a, b and c all have the same parity and $a \le b \le c$ so aSc holds. Thus S is transitive. S is not a total order since if m is even and n is odd, then neither mSn or nSm holds.

#10 State the definition of a function f from a set A to a set B. State the definition of the domain, codomain, and range of f.

Solution: See page 179 for function, domain and codomain, page 135 for range (as this is defined for all relations, not just for functions). Note that domain of any relation from A to B is defined (page 135) but that, for a function from A to B , the domain is A.

#11 Let $A = \{1, 2, 3, 4\}, B = \{x, y, z\}$. Let f be the function from A to B defined by $f(1) = z, f(2) = x, f(3) = y, f(4) = z$ and g be the function from B to A defined by

$g(x) = 4, g(y) = 3, g(z) = 2$. Find $f \circ g$ and $g \circ f$. Is either of the functions $f \circ g, g \circ f$ one-to-one? Onto?

Solution:

$$(f \circ g)(x) = f(g(x)) = f(4) = z, (f \circ g)(y) = f(g(y)) = f(3) = y,$$

$$(f \circ g)(z) = f(g(z)) = f(2) = x.$$

This function is one-to-oe and onto.

$$(g \circ f)(1) = g(f(1)) = g(z) = 2, (g \circ f)(2) = g(f(2)) = g(x) = 4,$$
$$(g \circ f)(3) = g(f(3)) = g(y) = 3, (g \circ f)(4) = g(f(4)) = g(z) = 2.$$

This function is not one-to-one (for $(g \circ f)(1) = (g \circ f)(4) = 2$) and is not onto (since 1 is not in its range).

#12 Suppose f is a function from A to C an g is a function from B to C. When will $f \cup g$ be a function? Why?

Solution: $f \sup g$ will be a function if and only if $f|_{A\cap B} = g|_{A\cap B}$. To see this, note that if $x \in A \cap B$ then the pairs $(x, f(x))$ and $(x, g(x))$ are both in $f \cup g$. But if $f \cup g$ is a function, there can only be one $b \in B$ such that (x, b) is in $f \cup g$. Thus, for $x \in A \cap B$ we must have $f(x) = g(x)$.

#13 State the definition of: $lim_{n\to\infty} x_n = L$.

Solution: See page 215.

#14 (a) Show that $lim_{n\to\infty} \frac{n+1}{1-2n} = \frac{-1}{2}$.

Solution: Given $\epsilon > 0$ we must find a natural number N so that whenever $n > N$ we have

$$|\frac{n+1}{1-2n} - \frac{-1}{2}| < \epsilon.$$

Now , adding the two fractions within the absolute value, we have

$$|\frac{n+1}{1-2n} - \frac{-1}{2}| = |\frac{(2n+2)+(1-2n)}{2(1-2n)}| = |\frac{3}{2(1-2n)}|.$$

Now if $n \geq 1$ we have $1 - 2n < 0$ and so $|\frac{3}{2(1-2n)}| = \frac{3}{4n-2}$. Furthermore, $\frac{3}{4n-2} < \epsilon$ if and only if $\frac{3}{\epsilon} < 4n - 2$. Now suppose N is an integer such that $N > \frac{3}{4\epsilon} + \frac{1}{2}$ and that $n > N$.

Then $4n-2 > 4N-2 > 4(\frac{3}{4\epsilon}+\frac{1}{2})-2 = \frac{3}{\epsilon}$. We have seen that this holds if and only if $|\frac{n+1}{1-2n} - \frac{-1}{2}| < \epsilon$ and so we are done.

(b) Show that $lim_{n\to\infty}\frac{2^n}{3^n} = 0$.

Solution: We must show that for any $\epsilon > 0$ there is some natural number n such that if n is a natural number and $n > N$ then $|\frac{2^n}{3^n}| < \epsilon$. This is the same as requiring that $\frac{1}{\epsilon} < (\frac{3}{2})^n$. Now one can show by induction tht $(\frac{3}{2})^n > n$ for all $n \geq 1$ and so it is sufficient to show that $\frac{1}{\epsilon} < n$. We can arrange for this to hold by taking N to be any integer greater than $\frac{1}{\epsilon}$.

(c) Show that the sequence given by $x_n = (-1)^n(1-\frac{1}{n})$ diverges.

Solution: If the sequence converges, then there is some L such that for any $\epsilon > 0$ there is a natural number N such that $|x_n - L| < \epsilon$ whenever n is a natural number larger than N. In particular there is some N such that $|x_n - L| < \frac{1}{3}$ whenever $n > N$. But if $n > N$ then we also have $n+1 > N$ and so we have

$$|x_n - x_{n+1}| \leq |x_n - L| + |x_{n+1} - L| < \frac{2}{3}.$$

Now

$$|x_n - x_{n+1}| = |(-1)^n(1-\frac{1}{n}) - (-1)^{n+1}(1-\frac{1}{n+1})| =$$

$$|(-1)^n||(1-\frac{1}{n}) + (1-\frac{1}{n+1})| = |2 - \frac{1}{n+1} - \frac{1}{n}|.$$

But if $n > 1$ we have $\frac{1}{n} + \frac{1}{n+1} < \frac{1}{2} + \frac{1}{3} < 1$ and so $|2 - \frac{1}{n+1} - \frac{1}{n}| = 2 - \frac{1}{n+1} - \frac{1}{n} > 1$. Thus the sequence cannot converge.