Math 300-02 - REVIEW PROBLEMS FOR FINAL EXAM - APRIL 2009

REVIEW SESSION - WEDNESDAY, MAY 6, NOON - 3PM - ARC-110 (BUSCH)

#1 Suppose A and B are true while P and Q are false. State whether or not each of the following is true and justify your answer.
(a) $(A \Rightarrow P) \Rightarrow Q$;
(b) $(P \Rightarrow A) \Rightarrow Q$;
(c) $(P \Rightarrow A) \Rightarrow B$;

#2: Make truth tables for each of the following propositional forms:
(a) $(R \vee S) \Rightarrow (R \wedge S)$;
(b) $R \vee (S \wedge T)$.

#3 Prove that $(\sim R) \vee S$ is equivalent to $\sim (\sim S \wedge R)$.

#4 Is each of the following a tautology, a contadiction, or neither?
(a) $(P \Rightarrow Q) \vee (Q \Rightarrow P)$
(b) $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
(c) $(P \Rightarrow Q) \wedge (P \wedge \sim Q)$.

#5 Prove that $P \Rightarrow Q$ is equivalent to $\sim Q \Rightarrow \sim P$.

#6 Which of the following statements are true, where the universe is the power set of $\{1, 2, 3, 4, 5\}$? Why?
(a) $(\forall A)(\exists B)(A \subseteq B)$;
(b) $(\forall A)(\exists B)(A = B)$;
(c) $(\exists A)(\forall B)(A \subseteq B)$;
(d) $(\exists A)(\forall B)(A = B)$;

#7 Prove that if n is an integer, the $n^2 + 5n$ is an even integer.

$8 (a) Give a direct proof that that if n is a natural number then

$$\frac{n}{n+1} < \frac{n+1}{n+2}.$$

(b) Give a proof by contradiction to show that if n is a natural number then

$$\frac{n}{n+1} < \frac{n+1}{n+2}.$$

#9 Let $A = \{1,2,3,4,5,6,7,8,9\}, B = \{2,4,6,8,10\}, C = (1,5)$, and $D = (3,7]$. Find:
(a) $A - B$;
(b) $B - A$;
(c) $A \cap B$;
(d) $C \cap D$;
(e) $\sim C \cap D$.
(f) $\overline{\overline{B}}$.
(g) $\overline{\overline{\mathcal{P}(B)}}$ (where $\mathcal{P}(B)$ denotes the power set of B).
(h) $\overline{\overline{B - \emptyset}}$.
(i) $\overline{\overline{\mathcal{P}(B) - \emptyset}}$.

#10 Let A, B, C be sets. Prove that $A \cap (B - C) = (A \cap B) - C$.

#11 Use the principle of mathematical induction to prove that for any natural number n we have:
(a)

$$\sum_{k=1}^{n}(6n-2) = 3n^2 + n.$$

(b)

$$\sum_{k=1}^{n} n^3 = \frac{n^2(n+1)^2}{4}.$$

#12 Use the well-ordering principle to prove that for any natural number $n > 3$ there are integers x and y such that

$$n = 2x + 5y.$$

#13 Use the well-ordering principle to prove that any natural number $n > 1$ is a product of prime numbers (that is, there is some natural number k and there are some prime numbers $a_1, , , , .a_k$ such that $n = a_1 a_2 ... a_n$).

#14 Suppose $\overline{\overline{A}} = 27, \overline{\overline{B}} = 15$, and $\overline{\overline{A \cap B}} = 8$. Find $\overline{\overline{A \cup B}}, \overline{\overline{A - B}}$ and $\overline{\overline{B - A}}$

#15 (a) Suppose $\overline{\overline{A}} = 6$, and $\overline{\overline{B}} = 11$. How many functions from A to B are there? How many one-to-one functions from A to B are there?
(b) Suppose further that $A = A_1 \cup A_2$ with $\overline{\overline{A_1}} = 4$ and $\overline{\overline{A_2}} = 2$ and that $B = B_1 \cup B_2$ with $\overline{\overline{B_1}} = 5$ and $\overline{\overline{B_2}} = 6$. How many of the one-to-one functions from A to B satisfy $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$?

#16 State the definitions of: the converse of a conditional sentence, the contrapositive of a conditional sentence, a relation from A to B, the domain of the relation R, the range of the relation R, a function from A to B, a function from A onto B, a one-to-one function from A to B, a finite set, an infinite set,a denumerable set, a countable set, congruence modulo n, an equivalence relation, a partial order, a total order.

#17 Suppose a relation R from a set A to itself is both an equivalence relation and a partial order. What is R?

#18 Let $R = \{(n, n^2)|n \in \mathbf{Z}\}$ which is a relation from $\mathbf{Z}$ to $\mathbf{Z}$. What is the inverse relation?

#19 Prove that congruence modulo n is an equivalence relation on the set of integers and describe the corresponding partition.

#20 (a) Define a relation R on the integers by aRb if and only if $a^2 < b$. Is R a partial order? Why or why not? Is it a total order?

(b) Define a relation S on the integers by aSb if and only $a^2 \equiv_n b$ where n is a natural number. Suppose that S is an equivalence relation. What can you say about n.

#21 Let $f\{(x, x^3)|x \in \mathbf{R}\}$ and $g(x) = \{(x, |x| - 1)|x \in \mathbf{R}\}$. These are two functions from $\mathbf{R}$ to $\mathbf{R}$.

(a) Find the domain of $f \circ g$ and of $g \circ f$.

(b) Let h be a function from $\mathbf{R}$ to $\mathbf{R}$. Prove that h is one-to-one if and only if $f \circ h$ is one-to-one and is onto if and only if $f \circ h$ is onto.

#22 Let A and B be countable sets. Prove that $A \cup B$ and $A \times B$ are countable.

#23 Prove that $\mathbf{N}$ is not finite.

#24 (a) Let $(x_1, x_2, ...)$ be a sequence of real numbers and L be a real number. State the definition of $lim_{n\to\infty} x_n = L$.

(b) Find

$$lim_{n\to\infty} \frac{n^2+1}{n^2-1}$$

and prove your assertion using your definition from part (a). (If you want to, you may use (without proving it) the following fact: given any real number r there is some natural number K such that $K > r$.)

(c) Show that the sequence given by $x_n = cos(n\pi)$ diverges.

#25 Arrange the following cardinal number in order:

$$\overline{\{\pi, e, -1\}}, \overline{\mathbf{Q}}, \overline{\mathbf{N}}, \overline{\mathcal{P}(\{1,2,3\})}, \overline{\mathbf{R}}, \overline{\mathbf{N}\times\mathbf{N}}, \overline{\mathbf{Z}\times\mathbf{N}}, \mathbf{N}_3, \mathbf{N}_4.$$