Math 300-02 - SOLUTIONS TO REVIEW PROBLEMS FOR FINAL EXAM - MAY 10, 2009

Note that there are were misprints in the statements of both parts of problem #11.
The solutions given here are to the corrected versions.

#1 Suppose A and B are true while P and Q are false. State whether or not each of the following is true and justify your answer.
(a) $(A \Rightarrow P) \Rightarrow Q$;

SOLUTION: Since A is true and P is false, $A \Rightarrow P$ is false. Then $(A \Rightarrow P) \Rightarrow Q$ is true.

(b) $(P \Rightarrow A) \Rightarrow Q$;

SOLUTION: Since A is true, $P \Rightarrow A$ is true. Since Q is false, we have that $(P \Rightarrow A) \Rightarrow Q$ is false.

(c) $(P \Rightarrow A) \Rightarrow B$;

SOLUTION: Since B is true, $(P \Rightarrow A) \Rightarrow B$ is true.

#2: Make truth tables for each of the following propositional forms:
(a) $(R \vee S) \Rightarrow (R \wedge S)$;

SOLUTION:

$$\begin{bmatrix} R & S & R \vee S & R \wedge S & (R \vee S) \Rightarrow (R \wedge S) \\ T & T & T & T & T \\ T & F & T & F & F \\ F & T & T & F & F \\ F & F & F & F & T \end{bmatrix}.$$

(b) $R \vee (S \wedge T)$.

SOLUTION:

$$\begin{bmatrix} R & S & T & S \wedge T & R \vee (S \wedge T) \\ T & T & T & T & T \\ T & T & F & F & T \\ T & F & T & F & T \\ T & F & F & F & T \\ F & T & T & T & T \\ F & T & F & F & F \\ F & F & T & F & F \\ F & F & F & F & F \end{bmatrix}.$$

#3 Prove that $(\sim R) \vee S$ is equivalent to $\sim(\sim S \wedge R)$.

SOLUTION: We will show that the truth tables are the same.

$$\begin{bmatrix} R & S & \sim R & \sim S & (\sim R) \vee S & (\sim S \wedge R) & \sim(\sim S \wedge R) \\ T & T & F & F & T & F & T \\ T & F & F & T & F & T & F \\ F & T & T & F & T & F & T \\ F & F & T & T & T & F & T \end{bmatrix}.$$

Since the columns headed $(\sim R)\vee S$ and $\sim(\sim S\wedge R)$ are the same, the two propositional forms are equivalent.

#4 Is each of the following a tautology, a contadiction, or neither?
(a) $(P \Rightarrow Q) \vee (Q \Rightarrow P)$

SOLUTION: The truth table for $(P \Rightarrow Q) \vee (Q \Rightarrow P)$ is

$$\begin{bmatrix} P & Q & P \Rightarrow Q & Q \rightarrow P & (P \Rightarrow Q) \vee (Q \Rightarrow P) \\ T & T & T & T & T \\ T & F & F & T & T \\ F & T & T & F & T \\ F & F & T & T & T \end{bmatrix}.$$

This shows that $(P \Rightarrow Q) \vee (Q \Rightarrow P)$ is always true, i.e., it is a tautology.

(b) $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

SOLUTION: If P and Q are both true, this is true. If P is false and Q is true then $P \Rightarrow Q$ is true and $Q \Rightarrow P$ is false, so that $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is false. Thus the propositional form is neither a tautology nor a contradiction.

(c) $(P \Rightarrow Q) \wedge (P \wedge \sim Q)$

SOLUTION: If P is false or if Q is true, then $P \wedge \sim Q$ is false and so $(P \Rightarrow Q) \wedge (P \wedge \sim Q)$ is false. Also, if P is true and Q is false, then $P \Rightarrow Q$ is false and so $(P \Rightarrow Q) \wedge (P \wedge \sim Q)$ is false. Thus, in any case, $(P \Rightarrow Q) \wedge (P \wedge \sim Q)$ is false, and so it is a contradiction.

#5 Prove that $P \Rightarrow Q$ is equivalent to $\sim Q \Rightarrow \sim P$.

SOLUTION: We will show that the truth tables are the same.

$$\begin{bmatrix} P & Q & P \Rightarrow Q & \sim Q & \sim P & \sim Q \Rightarrow \sim P \\ T & T & T & F & F & T \\ T & F & F & T & F & F \\ F & T & T & F & T & T \\ F & F & T & T & T & T \end{bmatrix}.$$

Since the coulmns headed $P \Rightarrow Q$ and $\sim Q \Rightarrow \sim P$ are the same, the two propositional forms are equivalent.

#6 Which of the following statements are true, where the universe is the power set of $\{1, 2, 3, 4, 5\}$? Why?
(a) $(\forall A)(\exists B)(A \subseteq B)$;

SOLUTION: This is true. For example, we can take $B = A$.

(b) $(\forall A)(\exists B)(A = B)$;

SOLUTION: This is true. We must take $B = A$.

(c) $(\exists A)(\forall B)(A \subseteq B)$;

SOLUTION: This is true. We must take $A = \emptyset$.

(d) $(\exists A)(\forall B)(A = B)$;

SOLUTION: This is false, for if were true, taking $B = \{1\}$ would imply $A = \{1\}$ while taking $B = \{2\}$ would imply $A = \{2\}$.

#7 Prove that if n is an integer, the $n^2 + 5n$ is an even integer.

SOLUTION: Since n is either even or odd, we can divide the proof into two cases.
Case I: n is even. Then $n = 2k$ for some integer k and $n^2+5n = n(n+5) = (2k)(n+5) = 2(k(n + 5))$. Since $k(n + 5)$ is an integer, this shows that $n^2 + 5n$ is even.
Case II: n is odd. Then $n = 2k + 1$ for some integer k and so

$$n^2 + 5n = n(n + 5) = n(2k + 1 + 5) = n(2(k + 3)) = 2n(k + 3).$$

Since $n(k + 3)4$ is an integer, $n^2 + 5n$ is even.

#8 (a) Give a direct proof that that if n is a natural number then

$$\frac{n}{n+1} < \frac{n+1}{n+2}.$$

SOLUTION: We know that $0 < 1$. Then $n(n+2) = n^2+2n = n^2+2n+0 < n^2+2n+1 = (n+1)^2$. Since n is a natural number, $(n+1)(n+2)$ is a positive number. Thus

$$\frac{n}{n+1} = \frac{n(n+2)}{(n+1)(n+2)} < \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

(b) Give a proof by contradiction to show that if n is a natural number then

$$\frac{n}{n+1} < \frac{n+1}{n+2}.$$

SOLUTION: Suppose the assertion is not true. Then

$$\frac{n}{n+1} > \frac{n+1}{n+2}.$$

Multiplying both sides by $(n+1)(n+2)$ (which is positive since n is a natural number), we see that

$$n(n+2) > (n+1)^2.$$

Thus

$$n^2+2n > n^2+2n+1$$

and so $0 > 1$, a contradiction.

#9 Let $A = \{1,2,3,4,5,6,7,8,9\}, B = \{2,4,6,8,10\}, C = (1,5)$, and $D = (3,7]$. Find:

(a) $A - B$;

SOLUTION: $\{1,3,5,7,9\}$

(b) $B - A$;

SOLUTION: $\{10\}$

(c) $A \cap B$;

SOLUTION: $\{2,4,6,8\}$

(d) $C \cap D$;

SOLUTION: $(3,5)$

(e) $\sim C \cap D$.

SOLUTION: $[5, 7]$

(f) $\overline{\overline{B}}$.

SOLUTION: 5

(g) $\overline{\overline{\mathcal{P}(B)}}$ (where $\mathcal{P}(B)$ denotes the power set of B).

SOLUTION: $2^5 = 32$

(h) $\overline{\overline{B - \emptyset}}$.

SOLUTION: 5

(i) $\overline{\overline{\mathcal{P}(B) - \emptyset}}$.

SOLUTION: $2^5 - 1 = 31$

#10 Let A, B, C be sets. Prove that $A \cap (B - C) = (A \cap B) - C$.

SOLUTION: Let $x \in A \cap (B - C)$. Then $x \in A$ and $x \in B - C$. Thus $x \in A, x \in B$ and $x \notin C$. Therefore $x \in A \cap B$ and, since $x \notin C$, we have $x \in (A \cap B) - C$. This shows that $A \cap (B - C) \subseteq (A \cap B) - C$.

Now let $x \in (A \cap B) - C$. Then $x \in A \cap B$ and $x \notin C$. Thus $x \in A, x \in B$, and $x \notin C$. Thus $x \in B - C$ and, since $x \in A$ we have $x \in A \cap (B - C)$. This shows that $(A \cap B) - C \subseteq A \cap (B - C)$.

Since we have proved both $A \cap (B - C) \subseteq (A \cap B) - C$ and $(A \cap B) - C \subseteq A \cap (B - C)$, we have $A \cap (B - C) = (A \cap B) - C$.

#11 Use the principle of mathematical induction to prove that for any natural number n we have:

(a)

$$\sum_{k=1}^{n} (6k - 2) = 3n^2 + n.$$

SOLUTION: Let $S = \{n | \sum_{k=1}^{n} (6k - 2) = 3n^2 + n\}$. We want to show that $S = \mathbf{N}$. The principle of mathematical induction says that if

(i) $1 \in S$ and

(ii) $(l \in S) \Rightarrow (l + 1 \in S)$ then $S = \mathbf{N}$. We will verify (i) and (ii).

First we consider (i). Since $\sum_{k=1}^{1} (6k - 2) = 4$ and $3(1^2) - 1 = 4$, we see that (i) holds.

Now assue that $l \in S$. Then $\sum_{k=1}^{l}(6k-2) = 3l^2 + l$. Hence

$$\sum_{k=1}^{l+1}(6k-2) = (\sum_{k=1}^{l}(6k-2)) + (6(l+1)-2) = 3l^2 + l + 6l + 4 =$$

$$3(l^2 + 2l + 1) + 1 = 3(l+1)^2 + 1.$$

This shows that $l+1 \in S$. Thus the principle of mathematical induction shows that $S = \mathbf{N}$ and so we have $\sum_{k=1}^{n}(6k-2) = 3n^2 + n$ for all natural numbers n.

(b)

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

SOLUTION: Let $S = \{n | \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. We want to show that $S = \mathbf{N}$. The principle of mathematical induction says that if

(i) $1 \in S$ and

(ii) $(l \in S) \Rightarrow (l+1 \in S)$ then $S = \mathbf{N}$. We will verify (i) and (ii).

First we consider (i). Since $\sum_{k=1}^{1} k^3 = 1$ and $\frac{1^1(1+1)^2}{4} = \frac{4}{4} = 1$, we see that (i) holds.

Now assue that $l \in S$. Then $\sum_{k=1}^{l} k^3 = \frac{l^2(l+1)^2}{4}$. Hence

$$\sum_{k=1}^{l+1} k^3 = (\sum_{k=1}^{l} k^3) + (l+1)^3 =$$

$$\frac{l^2(l+1)^2}{4} + (l+1)^3 = (l+1)^2(\frac{l^2}{4} + (l+1)) =$$

$$(l+1)^2(\frac{l^2+4l+4}{4}) = \frac{(l+1)^2(l+2)^2}{4}.$$

This shows that $l+1 \in S$. Thus the principle of mathematical induction shows that $S = \mathbf{N}$ and so we have $\sum_{k=1}^{n} n^3 = \frac{n^2(n+1)^2}{4}$ for all natural numbers n.

#12 Use the well-ordering principle to prove that for any natural number $n > 3$ there are integers x and y such that

$$n = 2x + 5y.$$

SOLUTION: Let

$$S = \{n \in \mathbf{N} | n > 3, n \neq 2x + 5y \ \ for \ \ any \ \ integers \ \ x, y\}.$$

We want to show that $S = \emptyset$. We will do this by contradiction. Thus we assume that $S \neq \emptyset$. Then by the well-ordering principle, S contains a smallest element, say m. Now $m \neq 4$, since $4 = 2(2) + 5(0)$ and 2 and 0 are integers. Also $m \neq 5$, since $5 = 2(0) + 5(1)$ and 0 and 5 are integers. Thus $m > 5$ so $m-2 > 3$. But $m-2 \notin S$ since m is the smallest element in S. Thus $m-2 = 2x+5y$ for some integers x and y. But then $m = 2(x+1)+5y$. Since $x+1$ and y are integers, this shows that $m \notin S$. This contradiction shows that $S \neq \emptyset$ is impossible. Thus $S = \emptyset$ and the result is proved.

#13 Use the well-ordering principle to prove that any natural number $n > 1$ is a product of prime numbers (that is, there is some natural number k and there are some prime numbers $a_1, , , , .a_k$ such that $n = a_1 a_2 ... a_k$).

SOLUTION: Let S denote the set of all natural numbers $n > 1$ such that n is not a product of primes. We want to show that $S = \emptyset$. We will do this by contradiction. Thus we assume that $S \neq \emptyset$. Then by the well-ordering principle, S contains a smallest element, say m. Then m cannot be a prime (for if it is, $m = a_1$ where a_1 is a prime so $m \notin S$). Since m is not prime, we can find natural numbers r and s such that $r, s \neq 1$ and $r, s \neq m$ with $m = rs$. Then $r = \frac{m}{s} < m$ and $s = \frac{m}{r} < m$. Thus, since m is the smallest element of S, $r, s \notin S$. But $r, s \neq 1$ so $r, s > 1$. Thus r must be a product of primes, say $r = a_1 ... a_k$ for some natural number k and some primes $a_1, ..., a_k$ Also s must be a product of primes, say $s = b_1 ... b_t$ for some natural number t and some primes $b_1, ..., b_t$. Then $m = a_1 ... a_k b_1 ... b_t$ is a product of primes, so $m \notin S$, a contradiction.

#14 Suppose $\overline{\overline{A}} = 27, \overline{\overline{B}} = 15$, and $\overline{\overline{A \cap B}} = 8$. Find $\overline{\overline{A \cup B}}, \overline{\overline{A - B}}$ and $\overline{\overline{B - A}}$

SOLUTION:

$$\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}} = 27 + 15 - 8 = 34.$$

$$\overline{\overline{A - B}} = \overline{\overline{A}} - \overline{\overline{A \cap B}} = 27 - 8 = 19.$$

$$\overline{\overline{B - A}} = \overline{\overline{B}} - \overline{\overline{A \cap B}} = 15 - 8 = 7.$$

#15 (a) Suppose $\overline{\overline{A}} = 6$, and $\overline{\overline{B}} = 11$. How many functions from A to B are there? How many one-to-one functions from A to B are there?

SOLUTION: There are 11^6 functions from A to B and (11)(10)(9)(8)(7)(6) of these functions are one-to-one.

(b) Suppose further that $A = A_1 \cup A_2$ with $\overline{\overline{A_1}} = 4$ and $\overline{\overline{A_2}} = 2$ and that $B = B_1 \cup B_2$ with $\overline{\overline{B_1}} = 5$ and $\overline{\overline{B_2}} = 6$. How many of the one-to-one functions from A to B satisfy $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$?

SOLUTION: ((5)(4)(3)(2))((6)(5))

#16 State the definitions of: the converse of a conditional sentence, the contrapositive of a conditional sentence, a relation from A to B, the domain of the relation R, the range of the relation R, a function from A to B, a function from A onto B, a one-to-one function from A to B, a finite set, an infinite set,a denumerable set, a countable set, congruence modulo n, an equivalence relation, a partial order, a total order.

SOLUTION: Here are page references to the text:

the converse of a conditional sentence and the contrapositive of a conditional sentence: page 12

a relation from A to B: page 133,

the domain of the relation R and the range of the relation R: page 135

a function from A to B: page 179

a function from A onto B page 198

a one-to-one function from A to B: page 201

a finite set and an infinite set: page 224

a denumerable set: page 230

a countable set: page 232

congruence modulo n page 248

an equivalence relation: page 147, though this uses terms defined on page 145

a partial order: page 161

a total order: page 165

#17 Suppose a relation R from a set A to itself is both an equivalence relation and a partial order. What is R?

SOLUTION: Since R is reflexive, we have aRa for every $a \in A$. Suppose $a, b \in A$ and aRb. Then, since R is symmetric, we have bRa and, since R is antisymmetric we have $a = b$. Thus $R = \{(a, a)|a \in A\}$.

#18 Let $R = \{(n, n^2)|n \in \mathbf{Z}\}$ which is a relation from $\mathbf{Z}$ to $\mathbf{Z}$. What is the inverse relation?

SOLUTION: The inverse relation is $\{(n^2, n)|n \in \mathbf{Z}\}$. Note that this is not a function.

#19 Prove that congruence modulo n is an equivalence relation on the set of integers and describe the corresponding partition.

SOLUTION: For any integer a we have $a - a = 0 = n0$. Thus n divides $a - a$ and so $a \equiv_n a$. Thus $\equiv_n$ is reflexive. If $a \equiv_n b$ then n divides $a - b$ so $a - b = nk$ for some integer k. Then $b - a = n(-k)$ and, since $-k$ is an integer, we have $b \equiv_n a$. Thus $\equiv_n$ is symmetric. Finally, suppose $a \equiv_n b$ and $b \equiv_n c$. Then n divides $a - b$ and $b - c$ so there are integers k and l such that $a - b = nk$ and $b - c = nl$. Then $a - c = (a - b) + (b - c) = nk + nl = n(k + l)$. Since $k + l$ is an integer, n divides $a - c$ so $a \equiv_n c$. Thus $\equiv_n$ is transitive and so it is an equivalence relation.

Now let $a \in \mathbf{Z}$. Then there are integers q and r with $0 \leq r < n$ such that $a = nq + r$. Then $a - r = nq$ so n divides $a - r$. Thus $a \equiv_n r$. Thus every integer is congruent modulo n to one of the integers $0, 1, ..., n-1$. Now the equivalence class of r is $r/\equiv_n = \{r + nq | q \in \mathbf{Z}\}$. Note that if $0 \leq r < s < n$ then $0 < s - r < n$ and so n does not divide $s - r$ which means that r and s are not congruent modulo n. Thus $(r/\equiv_n) \cap (s/\equiv_n) = \emptyset$. Hence the partition of $\mathbf{Z}$ corresponding to $\equiv_n$ is

$$\{nq | q \in \mathbf{Z}\} \cup \{1 + nq | q \in \mathbf{Z}\} \cup ... \cup \{(n-1) + nq | q \in \mathbf{Z}\}.$$

#20 (a) Define a relation R on the integers by aRb if and only if $a^2 < b$. Is R a partial order? Why or why not? Is it a total order?

SOLUTION: Note that $(-1)^2 > -1$ so $(-1, -1) \notin R$. Thus R is not reflexive so it is not a partial order (and so can't be a total order).

(b) Define a relation S on the integers by aSb if and only $a^2 \equiv_n b$ where n is a natural number. Suppose that S is an equivalence relation. What can you say about n.

SOLUTION: Since S is an equivalence relation, it must be reflexive. Thus, in particular, (-1)S(-1) and so n divides $(-1)^2 - (-1) = 2$. Thus $n = 1$ or $n = 2$. Note that in either of these cases S is an eqiuvalence relation, for if $n = 1$ then aSb for all integers a and b (so S is reflexive, symmetric and transitive) while if $n = 2$ we have aSb if and only if a and b have the same parity so S is just congruence modulo 2.

#21 Let $f\{(x, x^3) | x \in \mathbf{R}\}$ and $g(x) = \{(x, |x| - 1) | x \in \mathbf{R}\}$. These are two functions from $\mathbf{R}$ to $\mathbf{R}$.

(a) Find the domain of $f \circ g$ and of $g \circ f$.

SOLUTION: The domain of f and the domain of g are both $\mathbf{R}$ so the domain of $f \circ g$ and the domain of $g \circ f$ are both $\mathbf{R}$.

(b) Let h be a function from $\mathbf{R}$ to $\mathbf{R}$. Prove that h is one-to-one if and only if $f \circ h$ is one-to-one and is onto if and only if $f \circ h$ is onto.

SOLUTION: Note that f is one-to-one and onto. Let $a, b \in \mathbf{R}$. Then, since f is one-to-one, $h(a) = h(b)$ if and only if $f(h(a)) = f(h(b))$. But $f(h(a)) = f \circ h(a)$ and $f(h(b)) =$

$f \circ h(b)$. Thus $h(a) = h(b)$ if and only if $f \circ h(a) = f \circ h(b)$, showing that h is one-to-one if and only if $f \circ h$ is one-to-one.

Since f is onto, for any $a \in \mathbf{R}$ there is some $b \in \mathbf{R}$ such that $f(b) = a$. Suppose h is onto. Then there is some $c \in \mathbf{R}$ such that $h(c) = b$. But then $a = f(b) = f(h(c)) = f \circ h(c)$ so $f \circ h$ is onto. Conversely, suppose $f \circ h$ is onto. Then for any $a \in \mathbf{R}$ there is some $d \in \mathbf{R}$ such that $a = f \circ h(d)$. Then $a = f(h(d)$, so f is onto.

#22 Let A and B be countable sets. Prove that $A \cup B$ and $A \times B$ are countable.

SOLUTION: See the proof of Theorem 5.26 for $A \cup B$. Now for $b \in B$ note that $A \times \{b\}$ is a subset of $A \times B$ which is equivalent to A (for the function g from A to $A \times \{b\}$ defined by $g(a) = (a, b)$ is one-to-one and onto). Thus $A \times B = \cup_{b \in B} A \times \{b\}$ is a countable union of countable sets. If B is finite then the argument of Theorem 5.26 shows that $A \times B$ is countable. If A is finite, interchange the roles of A and B and use the arguent of Theorem 5.26 again. If A and B are both denumberable, use the arguent of Theorem 5.28.

#23 Prove that $\mathbf{N}$ is not finite.

SOLUTION: Suppose $\mathbf{N}$ is finite. Then it is equivalent to either $\emptyset$ or to $\mathbf{N}_k$ for some natural number k. Since $\mathbf{N}$ is not empty, it cannot be equivalent to $\emptyset$. Thus for some natural number k there must be an one-to-one onto function f from $\mathbf{N}_k$ to $\mathbf{N}$. Then $f(1) + f(2) + ... + f(k) + 1$ is an natural number, so it must equal $f(j)$ for some $j, 1 \leq j \leq k$. Then $f(1) + ... + f(j-1) + f(j+1) + ... + f(k) + 1 = 0$. But this number is greater than or equal to 1, a contradiction. Thus $\mathbf{N}$ cannot be finite.

#24 (a) Let $(x_1, x_2, ...)$ be a sequence of real numbers and L be a real number. State the definition of $lim_{n \to \infty} x_n = L$.

SOLUTION: See page 215.

(b) Find

$$lim_{n \to \infty} \frac{n^2 + 1}{n^2 - 1}$$

and prove your assertion using your definition from part (a). (If you want to, you may use (without proving it) the following fact: given any real number r there is some natural number K such that $K > r$.)

SOLUTION: We will show that

$$lim_{n \to \infty} \frac{n^2 + 1}{n^2 - 1} = 1.$$

Let $\epsilon > 0$ be given. Now

$$|\frac{n^2+1}{n^2-1} - 1| = |\frac{n^2+1}{n^2-1} - \frac{n^2-1}{n^2-1}| = |\frac{2}{n^2-1}|.$$

Now whenever $n > 1$ we have $n^2 - 1 > n$. Then if N is a natural number with $N > \frac{2}{\epsilon}$ and $n > N$ we have

$$n^2 - 1 > n > N > \frac{2}{\epsilon}$$

and so

$$\epsilon > \frac{2}{n^2-1} = |\frac{2}{n^2-1}| = |\frac{n^2+1}{n^2-1} - 1|$$

proving that

$$lim_{n\to\infty} \frac{n^2+1}{n^2-1} = 1.$$

(c) Show that the sequence given by $x_n = cos(n\pi)$ diverges.

SOLUTION: Note that $cos(n\pi) = (-1)^n$. Suppose $lim_{n\to\infty}(-1)^n = L$. Then we can find some integer natural number N so that for any natural number $n > N$ we have

$$|(-1)^n - L| < \frac{1}{2}.$$

But there are both even and odd natural numbers greater than N, so we have both $|1-L| < \frac{1}{2}$ (which implies $L > 0$) and $|-1-L| < \frac{1}{2}$ (which implies $L < 0$). Thus the assumption that the sequence converges leads to a contradiction, so the series must diverge.

#25 Arrange the following cardinal number in order:

$$\overline{\{\pi, e, -1\}}, \overline{\mathbf{Q}}, \overline{\mathbf{N}}, \overline{\mathcal{P}(\{1,2,3\})}, \overline{\mathbf{R}}, \overline{\mathbf{N}\times\mathbf{N}}, \overline{\mathbf{Z}\times\mathbf{N}}, \mathbf{N}_3, \mathbf{N}_4.$$

SOLUTION:

$$3 = \overline{\overline{\{\pi, e, -1\}}} = \overline{\overline{\mathbf{N}_3}} < 4 = \overline{\overline{\mathbf{N}_4}} < 8 = \overline{\overline{\mathcal{P}(\{1,2,3\})}} < \overline{\overline{\mathbf{Q}}} =$$

$$\overline{\overline{\mathbf{N}}} = \overline{\overline{\mathbf{N}\times\mathbf{N}}} = \overline{\overline{\mathbf{Z}\times\mathbf{N}}} < \overline{\overline{\mathbf{R}}}.$$