MATH 350, Section 01 - Spring 2008 - Review Problems (corrected as of May 7)

Problems 9,16,18 and 19 have been corrected since the original posting.

#1 Let $S = \{w_1, ..., w_k\}$ be an orthogonal set of nonzero vectors. Prove that S is linearly independent.

#2 Let V be a finite-dimensional vector space and let U and W be subspaces of V. Prove that

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W).$$

#3 Let

$$\beta = \{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}.$$

These are two ordered bases for $M_{2\times 2}(\mathbf{R})$. Let

$$T : M_{2\times 2}(\mathbf{R}) \to M_{2\times 2}(\mathbf{R})$$

be the linear transformation defined by

$$T(A) = A + A^t.$$

(a) Find $[T]_\beta$.

(b) Find $[T]_\gamma$.

(c) Find the change of basis matrix from β to γ.

(d) Find the change of basis matrix from γ to β.

(e) Explain how your answers to (a) - (d) are related.

#4 (a) Is the set of vectors $\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix}\}$ in $\mathbf{R}^3$ linearly independent? Why or why not?

(b) Is the vector $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -2 \end{bmatrix}$ in $Span\{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -1 \\ 1 \end{bmatrix}\}$? Why or why not?

(c) Does the set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ 4 \end{bmatrix} \right\}$ span $\mathbf{R}^4$? Why or why not

#5 Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & -1 \\ 1 & 2 & 0 & 1 & -1 \\ 2 & 5 & -1 & 0 & -2 \\ 2 & 3 & 1 & 4 & -1 \end{bmatrix}.$$

(a) Find the reduced row echelon form for A
(b) Find a basis for the null space $N(L_A)$
(c) Find a basis for $Col\ A$
(d) Find a basis for $Row\ A$

#6 Let $P = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Find P^{-1}.

#7 Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Find all eigenvalues for A and find a basis for each eigenspace.
(b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
#8 (a) Compute $det\ A$ if

$$A = \begin{bmatrix} 1 & -1 & -1 & -2 \\ 1 & -2 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix}$$

(b) Compute $det\ B$ if

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

#9 Suppose A is a 5 by 6 matrix over $\mathbf{R}$ and let R be the reduced row echelon form of A. Suppose that the columns of R form an orthogonal set. Prove that some column of A is 0.

#10 Let $W = Span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right)$, a subspace of $\mathbf{R}^4$.

a) Use the Gram-Schmidt procedure to find an orthogonal basis for W.

b) Find an orthonormal basis β for W.

c) Express $\begin{bmatrix} 9 \\ 2 \\ 2 \\ -2 \end{bmatrix}$ as a linear combination of the elements of β.

#11 Let T be the linear operator on $P_3(\mathbf{R})$ defined by

$$T(f) = xf''.$$

(Here $f = f(x) \in P_2(\mathbf{R})$, f' denotes the derivative of f, and f'' denotes the second derivative of f.) Let W be the T-cyclic subspace of $P_3(\mathbf{R})$ generated by x^3.

(a) Find a basis for W.

(b) Find the characteristic polynomial of T_W, the restriction of T to W.

#12 Let A be a 9 by 9 matrix with eigenvalues $1, 2$ and 3. Suppose

$$rank(A - I) = 7, rank(A - I)^2 = 6, rank(A - I)^3 = 5, rank(A - I)^4 = 5;$$

$$rank(A - 2I) = 8, rank(A - 2I)^2 = 8;$$

$$rank(A - 3I) = 7.$$

Find all possible Jordan canonical forms of A. (There is more than one.)

#13 Suppose A has reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let a_i denote the i-th column of A and suppose

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, a_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, a_5 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 2 \end{bmatrix}.$$

Find A. #14 Find all values of a such that the following system of linear equations has a solution. Then, for each such a, find all of the solutions.

$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_1 + 3x_2 + x_3 + x_4 = 4$$

$$2x_2 + x_3 - x_4 = a$$

$$x_1 + 3x_2 + 2x_3 = 2a$$

#15 Let A be an m by n matrix over a field F. Assume that, for any $b \in F^m$, the equation $Ax = b$ has a unique solution. Prove that $m = n$.

#16 Let A be an 5 by 3 matrix over $\mathbf{R}$. Let b and c be two vectors in $\mathbf{R}^5$. Assume that $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ are solutions of $Ax = b$ and that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a solution of $Ax = c$. Find infinitely many solutions of $Ax = 2b + c$.

#17 Let

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Find an orthogonal matrix P and a diagonal matrix D such that such that

$$P^t AP = D.$$

#18 Let T be a self-adjoint linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^4$ with exactly 3 eigenvalues: $0, 1,$ and 2. Suppose that

$$T(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix},$$

$$T(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$T(\begin{bmatrix} -4 \\ -2 \\ 3 \\ 0 \end{bmatrix}) = 2\begin{bmatrix} -4 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

Suppose that

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector for T. What is the characteristic polynomial of T?

#19 Let $V = P_2(\mathbf{C})$. Define

$$< f, g >= \int_0^1 f(t)g(\bar{t})dt.$$

Find an orthonormal basis for V.

#20 State the definitions of: an inner product space, the orthogonal complement of a subspace, the projection of a vector u on the line through a vector v, the adjoint of a linear transformation, a self-adjoint matrix, an orthogonal matrix, an orthonormal set, the generalized eigenspace corresponding to an eigenvalue λ. You should also be able state definitions of any of the terms listed in the previous review sheets.

#21 Let T be a linear transformation from a vector space V to V. let K_λ denote the generalized eigenspace of T corresponding to an eigenvalue λ.
(a) Show that K_λ is a T invariant subspace of V.
(b) Show that if $\mu \neq \lambda$ then the restriction of $T - \mu I$ to $K\lambda$ is invertible.
(c) If the distinct eigenvalues of T are $\lambda_1, ..., \lambda_k$ show that

$$V = K_{\lambda_1} \oplus ... \oplus K_{\lambda_k}.$$

#22 Let W denote the subspace of $\mathbf{R}^5$ spanned by

$$\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \}.$$

Find a basis for $W^\perp$.