MATH 350, Section 01 - Spring 2008 - Solutions to review Problems - corrected May 7

#1 Let $S = \{w_1, ..., w_k\}$ be an orthogonal set of nonzero vectors. Prove that S is linearly independent.

Solution: Suppose $0 = a_1 w_1 + ... + akw_k$ for some $a_1, ..., a_k \in F$. Then for each i, $1 \leq i \leq k$,

$$0 =< 0, w_i >=< a_1 w_1 + ... + akw_k, W_i >= a_1 < w_1, w_i > + ... + a_k < w_k, w_i > .$$

Since S is orthogonal, $< w_j, w_i >= 0$ for all $j \neq i$. Thus

$$0 = a_i < w_i, w_i > .$$

Since $w_i \neq 0$ we have $< w_i, w_i >\neq 0$ and so $a_i = 0$. Since this is true for all $i, 1 \leq i \leq k$, S is linearly independent.

#2 Let V be a finite-dimensional vector space and let U and W be subspaces of V. Prove that

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W).$$

Solution: Let $X = \{x_1, ..., x_l\}$ be a basis for $U \cap W$. Then we may extend X to a basis $\{x_1, ..., x_l, y_1, ..., y_m\}$ for U and we may also extend X to a basis $\{x_1, ..., x_l, z_1, ..., z_n\}$ for W.

We claim that $\{x_1, ..., x_l, y_1, ..., y_m, z_1, ..., z_m\}$ is a basis for $U + W$.

We will first show that this set is linearly independent. Suppose

$$a_1 x_1 + ... + a_l x_l + b_1 y_1 + ... + b_m y_m + c_1 z_1 + ... + c_n z_n = 0$$

for some $a_1, ..., a_l, b_1, ..., b_m, c_1, ..., c_n \in F$. Then

$$a_1 x_1 + ... + a_l x_l + b_1 y_1 + ... + b_m y_m = -(c_1 z_1 + ... + c_n z_n).$$

Now the vector on the right-hand side of this equation is in U and the vector on the left-hand side of the equation is in W. Since these vectors are equal we have

$$a_1 x_1 + ... + a_l x_l + b_1 y_1 + ... + b_m y_m \in U \cap W.$$

But X is a basis for $U \cap W$ and so

$$a_1 x_1 + ... + a_l x_l + b_1 y_1 + ... + b_m y_m = d_1 x_1 + ... + d_l x_l$$

for some $d_1, ..., d_l \in F$. Then

$$(a_1 - d_1)x_1 + ... + (a_l - d_l)x_l + b_1 y_1 + ... + b_m y_m = 0$$

and, since $\{x_1, ..., x_l, y_1, ..., y_m\}$ is linearly independent we have $b_1 = ... = b+m = 0$. Thus

$$a_1 x_1 + ... + a_l x_l + c_1 z_1 + ... + c_n z_n = 0$$

and, since $\{x_1, ..., x_l, z_1, ..., z_n\}$ is linearly independent, we have $a_1 = ... = a_l = c_1 = ... = c_n = 0$. This shows that $\{x_1, ..., x_l, y_1, ..., y_m, z_1, ..., z_m\}$ is linearly independent.

Now we show that $\{x_1, ..., x_l, y_1, ..., y_m, z_1, ..., z_m\}$ spans $U + W$. Let $v \in U + W$. Then $v = u + w, u \in U, w \in W$. Since $\{x_1, ..., x_l, y_1, ..., y_m\}$ is a basis for U, we have

$$u = a_1 x_1 + ... + a_l x_l + b_1 y_1 + ... + b + m y_m$$

for some $a_1, ..., a_l, b_1, ..., b_m \in F$. Similarly, since $\{x_1, ..., x_l, z_1, ..., z_n\}$ is a basis for W, we have

$$w = c_1 x_1 + ... + c_l x_l + d_1 z_1 + ... + d_n z_n$$

for some $c_1, ..., c_l, d_1, ..., d_n \in F$. Then

$$v = u + w = (a_1 + c_1)x_1 + ... + (a_l + c_l)x_l + b_1 y_1 + ... + b_m y_m + d_1 z_1 + ... + d_n z_n.$$

Thus $v \in Span\{x_1, ..., x_l, y_1, ..., y_m, z_1, ..., z_m\}$.

Now we can prove the dimension forumla. We have that $dim(U+W) = l+m+n, dim(U) = l + m, dim(W) = l + n$, and $dimU \cap W) = l$. Thus $dim(U) + dim(W) - dim(U \cap W) = l + m + l + n - l = l + m + n = dim(U + W)$ as required.

#3 Let

$$\beta = \{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\gamma = \{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}.$$

These are two ordered bases for $M_{2\times2}(\mathbf{R})$. Let

$$T : M_{2\times2}(\mathbf{R}) \to M_{2\times2}(\mathbf{R})$$

be the linear transformation defined by

$$T(A) = A + A^t.$$

(a) Find $[T]_\beta$.

(b) Find $[T]_\gamma$.

(c) Find the change of basis matrix from β to γ.

(d) Find the change of basis matrix from γ to β.

(e) Explain how your answers to (a) - (d) are related.

Solution: Write

$$w_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, w_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, w_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

With this notation we have:

(a) $T(w_1) = 2w_1, T(w_2) = 2w_2, T(w_3) = w_2 - w_4, T(w_4) = 2w_4$. Thus

$$[T]_\beta = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(b) $T(v_1) = 2v_1, T(v_2) = 2v_2, T(v_3) = v_3 + v_4, T(v_4) = v_3 + v_4$. Thus

$$[T]_\gamma = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(c) $w_1 = v_2+v_3+v_4, w_2 = (\frac{1}{2})v_1+(\frac{1}{2})v_2+v_3+v_4, w_3 = (\frac{1}{2})v_1+(\frac{1}{2})v_2+v_3, w_4 = (\frac{1}{2})v_1+(\frac{1}{2})v_2$. Thus

$$[I]_\beta^\gamma = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

(d) $v_1 = -w_1 + w_2 + w_4, v_2 = w_1 - w_2 + w_4, v_3 = w_3 - w_4, v_4 = w_2 - w_3$. Thus

$$[I]_\gamma^\beta = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

(e) $([I]_\gamma^\beta)^{-1} = [I]_\beta^\gamma$ and $T_\beta = [I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma$.

#4 (a) Is the set of vectors $\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix}\}$ in $\mathbf{R}^3$ linearly independent? Why or why not?

(b) Is the vector $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -2 \end{bmatrix}$ in $Span\{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -1 \\ 1 \end{bmatrix}\}$? Why or why not?

(c) Does the set of vectors $\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ 4 \end{bmatrix}\}$ span $\mathbf{R}^4$? Why or why not

Solution: (a) The matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ -1 & 0 & -1 \\ 2 & 3 & 8 \end{bmatrix}$$

has row echelon form

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence has rank 2. Thus the set of vectors is not linearly independent.

(b) The augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ -1 & -2 & -3 & -2 \\ -1 & 0 & -1 & 3 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

has row echelon form

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the last column does not contain an initial 1, the the vector $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -2 \end{bmatrix}$ is in $Span\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -1 \\ 1 \end{bmatrix}$

(c) The matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & 0 & 8 & 4 \end{bmatrix}$$

has row echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence has rank 3. Thus the given set does not span $\mathbf{R}^4$.

#5 Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & -1 \\ 1 & 2 & 0 & 1 & -1 \\ 2 & 5 & -1 & 0 & -2 \\ 2 & 3 & 1 & 4 & -1 \end{bmatrix}.$$

(a) Find the reduced row echelon form for A

(b) Find a basis for the null space $N(L_A)$

(c) Find a basis for $Col\ A$

(d) Find a basis for $Row\ A$

Solution: (a) The reduced row echelon form is

$$R = \begin{bmatrix} 1 & 0 & 2 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$, and $0 = Ax$ then

$$0 = Rx = \begin{bmatrix} x_1 + 2x_3 + 5x_4 \\ x_2 - x_3 - 2x_4 \\ x_5 \\ 0 \end{bmatrix}.$$

Thus

$$x_1 = -2x_3 - 5x_4,$$

$$x_2 = x_3 + 2x_4,$$

$$x_3 = x_3,$$

$$x_4 = x_4,$$

and

$$x_5 = 0.$$

Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and so

$$\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \}$$

is a basis for $N(L_A)$.

(c) The columns of R containing an initial 1 are the first,second and fifth columns. The corresponding coluns for A form a basis for $Col\ A$, so

$$\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \\ -1 \end{bmatrix} \}$$

is a basis for $Col\ A$.

(d) The nonzero rows for R form a basis for $Row\ A$. Thus

$$\{[1 \quad 0 \quad 2 \quad 5 \quad 0], [0 \quad 1 \quad -1 \quad -2 \quad 0], [0 \quad 0 \quad 0 \quad 0 \quad 1]\}$$

is a basis for $Row\ A$.

#6 Let $P = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Find P^{-1}.

Solution: Applying elementary row operations

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ -1 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & 1 & | & 1 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 2 & 1 & | & 1 & 1 & 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 0 & -1 & | & 1 & 1 & -2 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 3 & 2 & -4 \\ 0 & 1 & 0 & | & 1 & 1 & -1 \\ 0 & 0 & 1 & | & -1 & -1 & 2 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 & 1 & -3 \\ 0 & 1 & 0 & | & 1 & 1 & -1 \\ 0 & 0 & 1 & | & -1 & -1 & 2 \end{bmatrix}.$$

Thus

$$P^{-1} = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

#7 Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Find all eigenvalues for A and find a basis for each eigenspace.

(b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution: (a) $det(A - \lambda I) =$

$$det \begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = (3-\lambda)^2(1-\lambda) - 1 - 1 + (3-\lambda) + (3-\lambda) - (1-\lambda) =$$

$$(3-\lambda)^2(1-\lambda) + (3-\lambda) = (3-\lambda)((3-\lambda)(1-\lambda)+1) =$$

$$(3-\lambda)(\lambda^2 - 4\lambda + 4) = (3-\lambda)(2-\lambda)^2.$$

Thus the eigenvalues are 2 and 3. Now

$$E_2 = N(\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}) = N(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus E_2 has basis

$$\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}.$$

Also

$$E_3 = N(\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix} = N(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}).$$

thus E_3 has basis

$$\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}.$$

(b) We may take $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

#8 (a) Compute $det\ A$ if

$$A = \begin{bmatrix} 1 & -1 & -1 & -2 \\ 1 & -2 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix}$$

(b) Compute $det\ B$ if

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Solution: (a)

$$det(A) = det \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & -1 & 2 & 6 \\ 0 & 2 & 2 & 3 \\ 0 & 1 & 0 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & -1 & 2 & 6 \\ 0 & 0 & 6 & 15 \\ 0 & 0 & 2 & 11 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & -1 & 2 & 6 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix} = -36.$$

vskip 6 pt (b) Expanding along the first row gives

$$det(B) = 5det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 5.$$

#9 Suppose A is a 5 by 6 matrix over $\mathbf{R}$ and let R be the reduced row echelon form of A. Suppose that the columns of R form an orthogonal set. Prove that some column of A is 0.

Solution: If the columns of R are all nonzero, then the set of columns of R, being an orthogonal set of nonzero vectors, is a linearly independent set. But there are six columns of R and these columns are in the 5-dimensional space $\mathbf{R}^5$. Thus the set of columns of R cannot be linearly independent and so some column of R must be 0. But then the corresponding column of A must be 0.

The problem was originally stated as "Suppose A is a 5 by 6 matrix over $\mathbf{R}$ and let R be the reduced row echelon form of A. Suppose that the columns of R form an orthogonal set. Prove that some column of A is 0." This is actually easier since the argument given above for R can be applied directly to A.

#10 Let $W = Span(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix})$, a subspace of $\mathbf{R}^4$.

(a) Use the Gram-Schmidt procedure to find an orthogonal basis for W.

(b) Find an orthonormal basis β for W.

(c) Express $\begin{bmatrix} 9 \\ 2 \\ 2 \\ -2 \end{bmatrix}$ as a linear combination of the elements of β.

Solution: (a) Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$). Then applying the Gram-Schmidt procedure we get an orthogonal basis $\{w_1, w_2, w_3\}$ for W where

$$w_1 = v_1,$$

$$w_2 = v_2 - \frac{< v_2, w_1 >}{< w_1, w_1 >} w_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix},$$

$$w_3 = v_3 - \frac{< v_3, w_1 >}{< w_1, w_1 >} w_1 - \frac{< v_3, w_2 >}{< w_2, w_2 >} w_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}.$$

(b) Dividing each of the w_i by its length we get that

$$\{\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

is an orthonormal basis for W.

(c) If v is any vector in W, then $v = \frac{<v,w_1>}{<w_1,w_1>} w_1 + \frac{<v,w_2>}{<w_2,w_2>} w_2 + \frac{<v,w_3>}{<w_3,w_3>} w_3$. Applying this to the given vector we get

$$\begin{bmatrix} 9 \\ 2 \\ 2 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}.$$

#11 Let T be the linear operator on $P_3(\mathbf{R})$ defined by

$$T(f) = xf''.$$

(Here $f = f(x) \in P_2(\mathbf{R})$, f' denotes the derivative of f, and f'' denotes the second derivative of f.) Let W be the T-cyclic subspace of $P_3(\mathbf{R})$ generated by x^3.

(a) Find a basis for W.

(b) Find the characteristic polynomial of T_W, the restriction of T to W.

Solution: (a) $T(x^3) = x(6x) = 6x^2, T(6x^2) = x(12) = 12x, T(12x) = x(0) = 0$. Thus $\{x^3, 6x^2, 12x\}$ is a basis for T_W.

(b) The matrix of T_W with respect to the basis found in part (a) is $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Thus the characteristic polynomial of T_W is

$$det \begin{bmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}.$$

Since this matrix is lower triangular, its determinant is the product of the diagonal entries. Thus the characteristic polynomial of T_W is $-\lambda^3$.

#12 Let A be a 9 by 9 matrix with eigenvalues $1, 2$ and 3. Suppose

$$rank(A - I) = 7, rank(A - I)^2 = 6, rank(A - I)^3 = 5, rank(A - I)^4 = 5;$$

$$rank(A - 2I) = 8, rank(A - 2I)^2 = 8;$$

$$rank(A - 3I) = 7.$$

Find all possible Jordan canonical forms of A. (There is more than one.)

Solution: First consider the eigenvalue 1. We have

$$nullity(A - I) = 2, nullity(A - I)^2 = 3, nullity(A - I)^3 = 4, nullity(A - I)^4 = 4.$$

Thus

$$nullity(A - I) = 2, nullity(A - I)^2 - nullity(A - I) = 1,$$

$$nullity(A - I)^3 - nullity(A - I)^2 = 1, nullity(A - I)^4 - nullity(A - I)^3 = 0.$$

Thus the dot diagram for the eigenvalue 1 is

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \\ \bullet & \end{array}.$$

Thus there are blocks of size 3 and 1 with eigenvalue 1. Note that this means that $dim(K_1) = 4$.

Now consider the eigenvalue 2. We have

$$nullity(A - 2I) = 1, nullity(A - 2I)^2 = 1.$$

Thus

$$nullity(A - 2I) = 1, nullity(A - 2I)^2 - nullity(A - 2I) = 0.$$

Thus the dot diagram for the eigenvalue 2 is

$$\bullet\,.$$

Thus there is a single block of size 1 with the eigenbalue 2. Note that this means that $dim(K_2) = 1$.

Finally consider the eigenvalue 3. We have $dim(K_3) = 9 - dim(K_1) - dim(K_2) = 9 - 4 - 1 = 4$. Also $nullity(A - 3I) = 2$ and so the first row of the dot diagram must contain two dots. Now the number of dots in the diagram must be the dimension of K_3, i.e., it must be 4 Thus there are two possible dot diagrams:

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array}$$

and

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \\ \bullet & \end{array}.$$

.

There are then two possible Jordan canonical forms for A. The first has diagonal blocks

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, [1], [2], \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix},$$

and the second has diagonal blocks

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, [1], [2], \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, [3].$$

#13 Suppose A has reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let a_i denote the i-th column of A and suppose

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, a_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, a_5 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 2 \end{bmatrix}.$$

Find A.

Solution: Let R denote the reduced row echelon form and let r_i denote the i-th column of R. Then $r_2 = 2r_1, r_4 = r_1 + r_3, r_6 = -r_1 + 3r_3 - r_5$. Since the columns of A satisfy the same relations we have $a_2 = 2a_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 6 \end{bmatrix}, a_3 = a_4 - a_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, a_6 = -a_1 + 3a_3 - a_5 = \begin{bmatrix} -3 \\ 2 \\ 2 \\ -5 \end{bmatrix}$.
Thus

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 2 & -6 \\ -1 & -2 & 2 & 1 & 2 & 5 \\ 2 & 4 & -1 & 1 & -1 & -4 \\ 3 & 6 & -3 & 0 & 2 & -14 \end{bmatrix}.$$

#14 Find all values of a such that the following system of linear equations has a solution. Then, for each such a, find all of the solutions.

$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_1 + 3x_2 + x_3 + x_4 = 4$$

$$2x_2 + x_3 - x_4 = a$$

$$x_1 + 3x_2 + 2x_3 = 2a$$

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 1 & 1 & 4 \\ 0 & 2 & 1 & -1 & a \\ 1 & 3 & 2 & 0 & 2a \end{bmatrix}.$$

This has row echelon form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & a-2 \\ 0 & 0 & 0 & 0 & a-2 \end{bmatrix}.$$

Thus there is a solution if and only if $a = 2$. Setting $a = 2$ we see that the reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then x_4 is the only free variable and if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a solution we have

$$x_1 + 2x_4 = 1,$$

$$x_2 = 1,$$

$$x_3 - x_4 = 0.$$

Thus the set of solutions is

$$\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \| x_4 \in \mathbf{R} \}.$$

#15 Let A be an m by n matrix over a field F. Assume that, for any $b \in F^m$, the equation $Ax = b$ has a unique solution. Prove that $m = n$.

Solution: If $Ax = b$ has a solution, then $b \in Col(A)$. Thus if $Ax = b$ has a solution for every $b \in F^m$ we have $Col(A) = F^m$ and so $rank(A) = m$. If the solution of $Ax = b$ is unique, then $nullity(L_A) = 0$ and so $rank(A) = n$. Thus if $Ax = b$ has a unique solution for every $b \in F^m$ we have $m = ran(A) = n$.

#16 Let A be an 5 by 3 matrix over $\mathbf{R}$. Let b and c be two vectors in $\mathbf{R}^5$. Assume that $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ are solutions of $Ax = b$ and that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a solution of $Ax = c$. Find infinitely many solutions of $Ax = 2b + c$.

Solution: We have that

$$\begin{bmatrix}-1\\3\\1\end{bmatrix}-\begin{bmatrix}2\\1\\2\end{bmatrix}=\begin{bmatrix}-3\\2\\-1\end{bmatrix}$$

is a solution of $Ax=0$ and hence for any $a\in\mathbf{R}$ we have that

$$2\begin{bmatrix}-1\\3\\1\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}+a\begin{bmatrix}-3\\2\\-1\end{bmatrix}$$

is a solution of $Ax=2b+c$.

#17 Let

$$A=\begin{bmatrix}1&1&1&-1\\1&1&1&-1\\1&1&1&-1\\-1&-1&-1&1\end{bmatrix}.$$

Find an orthogonal matrix P and a diagonal matrix D such that such that

$$P^tAP=D.$$

Solution: By expanding along the first row and evaluating each of the resulting 3 by 3 determinants, we see that $det(A-\lambda I)=(1-\lambda)^3(4-\lambda)$. Thus the eigenvalues ar 1 and 4. Now

$$E_1=N(\begin{bmatrix}1&1&1&-1\\1&1&1&-1\\1&1&1&-1\\-1&-1&-1&1\end{bmatrix})=$$

$$N(\begin{bmatrix}1&1&1&-1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}).$$

Then we see that E_1 has basis $\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$. By applying the Gram-Schmidt procedure we see tht E_1 has orthogonal basis

$$\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}\frac{-1}{2}\\1\\0\\\frac{1}{2}\end{bmatrix},\begin{bmatrix}\frac{-1}{3}\\\frac{-1}{3}\\1\\\frac{1}{3}\end{bmatrix}\}$$

and hence has orthonormal basis

$$\{\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} \frac{-\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ 0 \\ \frac{\sqrt{6}}{6} \end{bmatrix}, \begin{bmatrix} \frac{-\sqrt{3}}{6} \\ \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{6} \end{bmatrix}\}.$$

We also see that

$$E_4 = N(\begin{bmatrix} -3 & 1 & 1 & -1 \\ 1 & -3 & 1 & -1 \\ 1 & 1 & -3 & -1 \\ -1 & -1 & -1 & -3 \end{bmatrix}) = N(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.)$$

Thus E_4 has basis $\{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\}$ and so has orthonormal basis

$$\{\begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix}\}.$$

Then we may take

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{6}}{6} & \frac{-\sqrt{3}}{6} & \frac{-1}{2} \\ 0 & \frac{\sqrt{6}}{3} & \frac{-\sqrt{3}}{6} & \frac{-1}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{6} & \frac{1}{2} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

#18 Let T be a self-adjoint linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^4$ with exactly 3 eigenvalues: $0, 1,$ and 2. Suppose that

$$T(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix},$$

$$T(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$T(\begin{bmatrix} -4 \\ -2 \\ 3 \\ 0 \end{bmatrix}) = 2\begin{bmatrix} -4 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

Suppose that

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector for T. What is the characteristic polynomial of T?

Solution: The eigenvector

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is not orthogonal to

$$\begin{bmatrix} -4 \\ -2 \\ 3 \\ 0 \end{bmatrix}$$

which is an eigenvector belonging to 2. Thus

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

must be an eigenvector belonging to 2, so $dim(E_2) = 2$. Hence the characteristic polynomial of T is $\lambda(1-\lambda)(2-\lambda)^2$.

#19 Let $V = P_2(\mathbf{C})$. Define

$$< f, g >= \int_0^1 f(t)g(\bar{t})dt.$$

Find an orthonormal basis for V.

Solution: $P_2(\mathbf{C})$ has basis $1, t, t^2$. We apply the Gram-Schmidt process to this to get a basis consisting of

$$1,$$

$$t - \frac{<1,t>}{<1,1>}1 = (t - \frac{1}{2}),$$

and

$$t^2 - \frac{<1,t^2>}{<1,1>}1 - \frac{<t-\frac{1}{2},t^2>}{<t-\frac{1}{2},t-\frac{1}{2}>} = t^2 - t + \frac{1}{6}.$$

Dividing each of these basis elements by its length we get the orthonormal basis

$$\{1, \sqrt{12}(t-\frac{1}{2}), \sqrt{180}(t^2 - t + \frac{1}{6})\}.$$

#20 State the definitions of: an inner product space, the orthogonal complement of a subspace, the projection of a vector u on the line through a vector v, the adjoint of a linear transformation, a self-adjoint matrix, an orthogonal matrix, an orthonormal set, the generalized eigenspace corresponding to an eigenvalue λ. You should also be able state definitions of any of the terms listed in the previous review sheets.

Solution:

These definitions are in the text.

#21 Let T be a linear transformation from a vector space V to V. let K_λ denote the generalized eigenspace of T corresponding to an eigenvalue λ.

(a) Show that K_λ is a T invariant subspace of V.

(b) Show that if $\mu \neq \lambda$ then the restriction of $T - \mu I$ to $K\lambda$ is invertible.

(c) If the distinct eigenvalues of T are $\lambda_1, ..., \lambda_k$ show that

$$V = K_{\lambda_1} \oplus ... \oplus K_{\lambda_k}.$$

Solution:

See the proofs of Theorems 7.1, 7.2 and 7.3 i the text.

#22 Let W denote the subspace of $\mathbf{R}^5$ spanned by

$$\{\begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\}.$$

Find a basis for $W^\perp$.

Solution:

$$W^\perp = N(\begin{bmatrix} 1 & 2 & 1 & -3 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 & 1 \end{bmatrix}) = N(\begin{bmatrix} 1 & 0 & 0 & 10 & -2 \\ 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 1 & -5 & 2 \end{bmatrix}).$$

Therefore $\{\begin{bmatrix} -10 \\ 4 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis for $W^\perp$.