Math 350 - Review problems - February 25, 2008

#1 Let V and W be finite dimensional vector spaces and let $T \in \mathcal{L}(V, W)$. Prove that

$$rank(T) + nullity(T) = dim(V).$$

#2 Let V be a finite-dimensional vector space over F and let X and Y be subspaces of V. Recall that $X + Y$ denotes $\{x + y | x \in X, y \in Y\}$.

(a) Show that $X + Y$ is a subspace of V.
(b) Show that $X \cap Y$ is a subspace of V.
(c) Prove that

$$dim(X + Y) = dim(X) + dim(Y) - dim(X \cap Y).$$

#3 Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, (x+1), (x+1)^2\}$. These are two ordered bases for $P_2(\mathbf{R})$. Let

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$$

be the linear transformation defined by

$$T(f) = xf'.$$

(Here $f = f(x) \in P_2(\mathbf{R})$ and f' denotes the derivative of f.)

(a) Find $[T]_\beta$.
(b) Find $[T]_\gamma$.
(c) Find the change of basis matrix from β to γ.
(d) Find the change of basis matrix from γ to β.
(e) Explain how your answers to (a) - (d) are related.
(f) Find $[T^t]_{\beta^*}$.

#4 (a) Is the set of vectors $\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\}$ in $\mathbf{R}^3$ linearly independent? Why or why not?

(b) Is the vector $\begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix}$ in $Span\{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}\}$? Why or why not?

(c) Does the set of vectors $\{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}\}$ span $\mathbf{R}^4$? Why or why not?

#5 (a) Let $W_1 = \{f(x) \in \mathcal{P}_3 | f(1) = f(2)\}$. Is W_1 a subspace of $\mathcal{P}_3$? Why or why not?
(b) Let $W_2 = \{f(x) \in \mathcal{P}_3 | f(1) = 2\}$. Is W_2 a subspace of $\mathcal{P}_3$? Why or why not?

#6 Let V and W be vector spaces and $v_1, ..., v_n \in V$. State the definition of each of the following terms:
(a) The span of $\{v_1, ..., v_n\}$
(b) $\{v_1, ..., v_n\}$ is linearly independent
(c) A basis of V.
(d) The dimension of V
(e) A linear transformation from V to W.

#7 (a) Is F^3 isomorphic to $M_{2\times 2}(F)$? Why or why not?
(b) Is F^3 isomorphic to $\{A \in M_{2\times 2}(F) | A = A^t\}$? Why or why not?
(c) Is F^2 isomorphic to $\{A \in M_{2\times 2}(F) | A = A^t\}$? Why or why not?
(d) Is F^2 isomorphic to $\{A \in M_{2\times 2}(F) | A = -A^t\}$? Why or why not?

#8 Let $V = \mathbf{R}^3$, let $\{e_1, e_2, e_3\}$ be the standard basis, and let $\beta = \{f_1, f_2, f_3\}$ be the dual basis. Define $g \in \mathcal{L}(V, \mathbf{R})$ by

$$g(\begin{bmatrix} r \\ s \\ t \end{bmatrix}) = r + 2s - 3t.$$

Express g as a linear combination of elements of β.

#9 Let

$$A = \begin{bmatrix} 3 & -1 & 2 & 2 \\ 1 & 0 & 0 & 1 \\ -1 & 2 & 2 & 4 \end{bmatrix},$$

$$B = \begin{bmatrix} 4 & -1 & 2 & 3 \\ 1 & 0 & 0 & 1 \\ -1 & 2 & 2 & 4 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 3 & 5 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 4 \end{bmatrix}.$$

Find elementary matrices P and Q such that $PA = B$ and $AQ = C$.

#10 Suppose $V_1, ..., V_6$ are vector spaces with

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subseteq V_5 \subseteq V_6$$

and $dim(V_6) = 4$. Prove that $V_i = V_{i+1}$ for some $i, 1 \leq i \leq 5$.

#11 Let $P = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Find P^{-1}. Show your work.