Math 350 - Solutions to review problems for Exam #1 - March 1, 2008

#1 Let V and W be finite dimensional vector spaces and let $T \in \mathcal{L}(V, W)$. Prove that

$$rank(T) + nullity(T) = dim(V).$$

Solution: This is in the text (Theorem 2.3, page 70). Here is a proof. Let $\beta = \{v_1, ..., v_k\}$ be a basis for $N(T)$. Then β can be extended to a basis $\gamma = \{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ of V. Then $dimV = n, nullity(T) = dim(N(T)) = k$, and we must show that $rank(T) = n - k$. Since $rank(T) = dim(R(T))$ we can do this by showing that $\alpha = \{T(v_{k+1}), ..., T(v_n)\}$ is a basis for $R(T)$. First we will show that α spans $R(T)$. Let $w \in R(T)$. Then $w = T(v)$ for some $v \in V$. Since $\{v_1, ..., v_n\}$ is a basis for V, we have $v = a_1v_1 + ... + a_nv_n$ for some scalars $a_1, ..., a_n \in F$. Then $w = T(v) = a_1T(v_1) + ... + a_nT(v_n)$. But $v_1, ..., v_k$ are in $N(T)$, so $T(v_1) = ... = T(v_k) = 0$. Thus $w = a_{k+1}T(v_{k+1}) + ... + a_nT(v_n) \in Span(\alpha)$. Now we show that α is linearly independent. Suppose $0 = b_{k+1}T(v_{k+1}) + ... + b_nT(v_n)$. Then $0 = T(b_{k+1}v_{k+1} + ... + b_nv_n)$ and so $b_{k+1}v_{k+1} + ... + b_nv_n \in N(T)$. Since β is a basis for $N(T)$ this means that

$$b_{k+1}v_{k+1} + ... + b_nv_n = c_1v_1 + ... + c_kv_k$$

for some scalars $c_1, ..., c_k \in F$. But then

$$-c_1v_1 - ... - c_kv_k + b_{k+1}v_{k+1} + ... + b_nv_n = 0.$$

Since γ is linearly independent, this means that $c_1 = ... = c_k = b_{k+1} = ... = b_n = 0$. Thus all the b_i are equal to 0. This shows that α is linearly independent and our proof is complete.

#2 Let V be a finite-dimensional vector space over F and let X and Y be subspaces of V. Recall that $X + Y$ denotes $\{x + y | x \in X, y \in Y\}$.

(a) Show that $X + Y$ is a subspace of V.

(b) Show that $X \cap Y$ is a subspace of V.

(c) Prove that

$$dim(X + Y) = dim(X) + dim(Y) - dim(X \cap Y).$$

Solution: (a) Since $0 \in X$ and $0 \in Y$ we have $0 = 0 + 0 \in X + Y$. Now let $u_1, u_2 \in X + Y, a \in F$. Then $u_1 = x_1 + y_1, u_2 = x_2 + y_2$ where $x_1, x_2 \in X, y_1, y_2 \in Y$. Then $u_1 + u_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2)$. Since $x_1 + x_2 \in X$ and $y_1 + y_2 \in Y$ we have $u_1 + u_2 \in X + Y$. Also $au_1 = a(x_1 + y_1) = (ax_1) + (ay_1)$. Since $ax_1 \in X$ and $ay_1 \in Y$ we have $au_1 \in X + Y$. Thus $X + Y$ is a subspace.

(b) Since $0 \in X$ and $0 \in Y$ we have $0 \in X \cap Y$. Now let $u_1, u_2 \in X \cap Y$ and $a \in F$. Then $u_1, u_2 \in X$ so $u_1 + u_2 \in X$. Also $u_1, u_2 \in Y$ so $u_1 + u_2 \in Y$. Thus $u_1 + u_2 \in X \cap Y$. Now $au_1 \in X$ and $au_1 \in Y$ so $au_1 \in X \cap Y$. Thus $X \cap Y$ is a subspace.

(c) Let $\alpha = \{u_1, ..., u_k\}$ be a basis for $X \cap Y$. Then we may extend α to a basis $\beta = \{u_1, ..., u_k, x_1, ..., x_l\}$ of X and we may extend α to a basis $\gamma = \{u_1, ..., u_k, y_1, ..., y_m\}$ of Y. Thus we have $dim(X) + dim(Y) - dim(X \cap Y) = (k+l) + (k+m) - k = k + l + m$, and so we need to prove that $dim(X+Y) = k + l + m$. We will verify this by showing that $\delta = \{u_1, ..., u_k, x_1, ..., x_l, y_1, ..., y_m\}$ is a basis for $X + Y$.

First we show that δ spans $X+Y$. Let $u \in X+Y$. Then $u = x+y, x \in X, y \in Y$. Then x is a linear combination of elements of β. Thus $x = a_1u_1 + ... + a_ku_k + b_1x_1 + ... + b_lx_l$ for some $a_1, ..., a_k, b_1, ..., b_l \in F$. Similarly y is a linear combination of elements of γ. Thus $y = c_1u_1 + ... + c_ku_k + d_1y_1 + ... + d_my_m$ for some $c_1, ..., c_k, d_1, ..., d_m \in F$. Then $x + y = (a_1 + c_1)u_1 + ... + (a_k + c_k)u_k + b_1x_1 + ... + b_lx_l + d_1y_1 + ... + d_my_m \in Span(\delta)$.

Now we show that δ is linearly independent. Suppose there exist elements

$$r_1, ..., r_k, s_1, ..., s_l, t_1, ..., t_m \in F$$

such that

$$r_1u_1 + ... + r_ku_k + s_1x_1 + ... + s_lx_l + t_1y_1 + ... + t_my_m = 0.$$

Then

$$r_1u_1 + ... + r_ku_k + s_1x_1 + ... + s_lx_l = -t_1y_1 - ... - t_my_m.$$

Dentote this element by E. Then $E = r_1u_1 + ... + r_ku_k + s_1x_1 + ... + s_lx_l \in X$ and $E = -t_1y_1 - ... - t_my_m \in Y$. Thus $E \in X \cap Y$ and so it is a linear combination of elements of α. Thus

$$z_1u_1 + ... + z_ku_k = -t_1y_1 - ... - t_my_m$$

and so

$$z_1u_1 + ... + z_ku_k + t_1y_1 + ... + t_my_m = 0$$

for some $z_1, ..., z_m \in F$. Since γ is linearly independent we have $z_1 = ... = z_k = y_1 = ... = y_m = 0$. But then $E = 0$ and so

$$r_1u_1 + ... + r_ku_k + s_1x_1 + ... + s_lx_l = 0.$$

Since β is linearly independent we have $r_1 = ... = r_k = x_1 = ... = x_l = 0$. This shows that δ is linearly independent and our proof is complete.

#3 Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, (x+1), (x+1)^2\}$. These are two ordered bases for $P_2(\mathbf{R})$. Let

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$$

be the linear transformation defined by

$$T(f) = xf'.$$

(Here $f = f(x) \in P_2(\mathbf{R})$ and f' denotes the derivative of f.)

(a) Find $[T]_\beta$.
(b) Find $[T]_\gamma$.
(c) Find the change of basis matrix from β to γ.
(d) Find the change of basis matrix from γ to β.
(e) Explain how your answers to (a) - (d) are related.
(f) Find $[T^t]_{\beta^*}$.

Solution: (a) We have $T(1) = 0, T(x) = x, T(x^2) = 2x^2$. Thus $[T]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) We have $T(1) = 0, T(x+1) = x = -1 + (x+1), T((x+1)^2) = 2x + 2x^2 = -2(x+1) + 2(x+1)^2$. Thus $[T]_\gamma = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

(c) This is $[I]_\beta^\gamma = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) This is $[I]_\gamma^\beta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(e) $([I]_\beta^\gamma)^{-1} = [I]_\gamma^\beta$ and

$$([I]_\beta^\gamma)^{-1}[T]_\gamma[I]_\beta^\gamma = [T]_\beta.$$

(f) $[T^t]_{\beta^*} = ([T]_\beta)^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

#4 (a) Is the set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbf{R}^3$ linearly independent? Why or why not?

(b) Is the vector $\begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix}$ in $Span\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix} \right\}$? Why or why not?

(c) Does the set of vectors $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ span $\mathbf{R}^4$? Why or why not?

Solution: (a) No. A set of 4 vectors in a 3-dimensional vector space cannot be linearly independent.

(b) No. Suppose $a_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix}$. Then

$$\begin{bmatrix} a_1 + a_2 + 3a_3 \\ a_1 - a_2 - 3a_3 \\ -a_1 + a_3 \\ a_1 + a_2 + a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix}.$$

Thus we have the system of equations

$$a_1 + a_2 + 3a_3 = 1$$

$$a_1 - a_2 - 3a_3 = -2$$

$$-a_1 + a_3 = 2$$

$$a_1 + a_2 + a_3 = 1.$$

Subtracting the first equation from the fourth gives $-2a_3 = 0$ and so $a_3 = 0$. Then we have

$$a_1 + a_2 = 1$$

$$a_1 - a_2 = -2$$

$$-a_1 = 2.$$

Then adding the first and third equations gives $a_2 = 3$ while adding the second and third equations gives $a_2 = 0$. Thus there is no solution, giving the result.

(c) Yes. This set of 4 vectors in the 4-dimensional space $\mathbf{R}^4$ spans $\mathbf{R}^4$ if and only if it is linearly independent. If

$$a_1 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + a_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

then

$$-a_1 + a_2 - a_3 + a_4 = 0$$
$$2a_1 - a_3 + a_4 = 0$$
$$a_1 + a_2 + a_3 + a_4 = 0$$
$$a_3 + 2a_4 = 0.$$

The adding twice the first equation to the second and adding the first equation to the third gives

$$-a_1 + a_2 - a_3 + a_4 = 0$$
$$2a_2 - 3a_3 + 3a_4 = 0$$
$$2a_2 + 2a_4 = 0$$
$$a_3 + 2a_4 = 0.$$

Subtracting the second equation from the third gives

$$-a_1 + a_2 - a_3 + a_4 = 0$$
$$2a_2 - 3a_3 + 3a_4 = 0$$
$$3a_3 - a_4 = 0$$
$$a_3 + 2a_4 = 0.$$

Finally, subtracting $\frac{1}{3}$ times the third equation from the fourth gives

$$-a_1 + a_2 - a_3 + a_4 = 0$$
$$2a_2 - 3a_3 + 3a_4 = 0$$
$$3a_3 - a_4 = 0$$
$$\frac{7}{3}a_4 = 0.$$

Thus $a_4 = a_3 = a_2 = a_1 = 0$ so the set is linearly independent.

#5 (a) Let $W_1 = \{f(x) \in \mathcal{P}_3 | f(1) = f(2)\}$. Is W_1 a subspace of $\mathcal{P}_3$? Why or why not?
(b) Let $W_2 = \{f(x) \in \mathcal{P}_3 | f(1) = 2\}$. Is W_2 a subspace of $\mathcal{P}_3$? Why or why not?

Solutions: (a) Clearly the zero function is in W_1. If $f, g \in W_1$, then $(f+g)(1) = f(1) + g(1) = f(2) + g(2) = (f+g)(2)$ so $f + g \in W_1$. Also, if $a \in F$ we have $(af)(1) = af(1) = af(2) = (af)(2)$ so $af \in W_1$. Thus W_1 is a subspace.
(b) The zero function is not in W_2 and so W_2 is not a subspace.

#6 Let V and W be vector spaces and $v_1, ..., v_n \in V$. State the definition of each of the following terms:

(a) The span of $\{v_1, ..., v_n\}$
(b) $\{v_1, ..., v_n\}$ is linearly independent
(c) A basis of V.
(d) The dimension of V
(e) A linear transformation from V to W.

Solution: (a) Let V be a vector space over F. $Span\{v_1, ..., v_n\} = \{a_1v_1 + ... + a_nv_n | a_1, ..., a_n \in F\}$.

(b) $\{v_1, ..., v_n\}$ is linearly independent if whenever $a_1v_1 + .. + a_nv_n = 0$ we must have $a_1 = ... = a_n = 0$.

(c) A subset S of a vector space V is a basis for V if it spans V and is linearly independent.

(d) The dimension of V is the number of elements in any basis for V.

(e) A function T from V to W (which are vector spaces over a field F) is a linear transformation if, for every $v_1, v_2 \in V$ and every $a \in F$ we have $T(v_1+v_2) = T(v_1)+T(v_2)$ and $T(av_1) = aT(V_1)$.

#7 (a) Is F^3 isomorphic to $M_{2\times2}(F)$? Why or why not?

(b) Is F^3 isomorphic to $\{A \in M_{2\times2}(F) | A = A^t\}$? Why or why not?
(c) Is F^2 isomorphic to $\{A \in M_{2\times2}(F) | A = A^t\}$? Why or why not?
(d) Is F^2 isomorphic to $\{A \in M_{2\times2}(F) | A = -A^t\}$? Why or why not?

Solution: (a) No, since $dim(F^3) = 3$ and $dim(M_{2\times2}(F)) = 4$.

(b) $\{A \in M_{2\times2}(F) | A = A^t\}$ has basis

$$\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

and hence has dimension 3. Thus the two spaces have the same dimension and so are isomorphic.

(c) No, as the two spaces have different dimensions.

(d) First assume that the characteristic of the field F is not 2, i.e., assume that $2 \neq 0$ in F. Then $\{A \in M_{2\times2}(F) | A = -A^t\}$ is 1-dimensional with basis $\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.\}$ Hence the two spaces have different dimensions and so are not isomorphic. If F has characteristic 2 then $-A^t = A^t$ so this is the same as part (b) (and the two spaces are not isomorphic).

#8 Let $V = \mathbf{R}^3$, let $\{e_1, e_2, e_3\}$ be the standard basis, and let $\beta = \{f_1, f_2, f_3\}$ be the dual basis. Define $g \in \mathcal{L}(V, \mathbf{R})$ by

$$g(\begin{bmatrix} r \\ s \\ t \end{bmatrix}) = r + 2s - 3t.$$

Express g as a linear combination of elements of β.

Solution: Write

$$g = a_1 f_1 + a_2 f_2 + a_3 f_3.$$

Recall that $f_i(e_j) = 0$ if $i \neq j$ and that $f_i(e_i) = 1$. Then we have

$$1 = g(e_1) = a_1 f_1(e_1) + a_2 f_2(e_1) + a_3 f_3(e_1) = a_1,$$

$$2 = g(e_2) = a_1 f_1(e_2) + a_2 f_2(e_2) + a_3 f_3(e_2) = a_2,$$

$$-3 = g(e_3) = a_1 f_1(e_3) + a_2 f_2(e_3) + a_3 f_3(e_3) = a_3.$$

Thus $g = f_1 + 2f_2 - 3f_3$.

#9 Let

$$A = \begin{bmatrix} 3 & -1 & 2 & 2 \\ 1 & 0 & 0 & 1 \\ -1 & 2 & 2 & 4 \end{bmatrix},$$

$$B = \begin{bmatrix} 4 & -1 & 2 & 3 \\ 1 & 0 & 0 & 1 \\ -1 & 2 & 2 & 4 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 3 & 5 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 4 \end{bmatrix}.$$

Find elementary matrices P and Q such that $PA = B$ and $AQ = C$.

Solution: The matrix B is obtained from A by adding the second row to the first row. We obtain P by performing this same row operation on the (3 by 3) identity matrix. Thus $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The matrix C is obtained from A by adding twice the first column to the second column. We obtain Q by performing this same column operation on the (4 by 4) identity matrix. Thus $Q = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

#10 Suppose $V_1, ..., V_6$ are vector spaces with

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subseteq V_5 \subseteq V_6$$

and $dim(V_6) = 4$. Prove that $V_i = V_{i+1}$ for some $i, 1 \leq i \leq 5$.

Solution: We have

$$0 \leq dim(V_1) \leq dim(V_2) \leq dim(V_3) \leq dim(V_4) \leq dim(V_5) \leq dim(V_6) = 4.$$

Since there are only five possibilities $(0, 1, 2, 3, 4)$ for the six integers $dim(V_1), ..., dim(V_6)$ we must have $dim(V_i) = dim(V_{i+1})$ for some $i, 1 \leq i \leq 5$. Since $V_i \subseteq V_{i+1}$ this implies $V_i = V_{i+1}$.

#11 Let $P = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Find P^{-1}. Show your work.

Solution:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 2 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Therefore

$$P^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$