MATH 350-01 - Review problems for Exam #2

This is the complete set of review problems. Problems #4, #7c,d, and #10 - #16 have been added since the original set was posted on 4/7. In addition, some typos have been corrected.

These problems will be worked at a review session on Sunday, 4/13, from 2:00 - 5:00 PM. The location of the review session will be posted on the door of Hill-340. (It will probably be a 4th floor classroom in Hill Center.)

#1 Suppose that A is a 5 by 5 matrix and

$$B = A + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $det(A) = 1$ and $det(B) = 3$, what is $det(2A + B)$. Why?

#2 Let the 4 by 7 matrix A have columns $a_1, ..., a_7$. Suppose the reduece row echelon form of A is

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Suppose further that $a_2 = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 6 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, and $a_5 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ -3 \end{bmatrix}$. Find A.

#3 A 9 by 9 diagonalizable matrix B has three eigenvalues: $1, 2$ and 3.

$$rank(A - I) = 7$$

and

$$rank(A - 2I) = 5,$$

what is the multiplicity of the eigenvalue 3? Why?

#4 Let A be an m by n matrix. Write $A = [\, a_1 \quad a_2 \quad ... \quad a_n \,]$ where A_i denotes the i-th column of A. Let $A_k = [\, a_1 \quad ... \quad a_k \,]$, i.e., the matrix consisting of the first k columns of A. Set $s_i(A) = rank(A_i)$ for $1 \leq i \leq n$, and let $s(A)$ denote the n-tuple $[\, s_1(A) \quad , ..., \quad s_n(A) \,]$.

(a) Let P be an invertible m by m matrix. Prove that $s(A) = s(PA)$.

(b) Let R be the reduced row echelon form of A. Prove that $s(R) = s(A)$.

(c) Say that a column of A is a basic column if the corresponding column of R contains the initial nonzero entry of some row. Show how to determine the basic columns from the n-tuple $s(A)$.

(d) Show that the column a_i of A is a linear combination of the columns a_j such that $j \leq i$ and a_j is basic.

(e) Explain why a matrix A has only one reduced row echelon form.

#5 Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & -1 \\ 1 & 2 & 0 & 1 & -1 \\ 2 & 5 & -1 & 0 & -2 \\ 2 & 3 & 1 & 4 & -1 \end{bmatrix}.$$

(a) Find the reduced row echelon form for A

(b) Find a basis for the null space $N(L_A)$

(c) Find a basis for the row space of A

(d) Find a basis for the column space of A.

#6 Let $A = \begin{bmatrix} -3 & 0 & -5 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find all eigenvalues for A and find a basis for each eigenspace.

(b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

#7

(a) Compute *det* A if

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & -1 & -4 \end{bmatrix}$$

(b) Compute *det* B if

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 \\ 0 & 3 & 7 & 3 & 0 \\ 0 & 0 & 4 & 13 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix}$$

(c) Let $a_1, ..., a_n \in F$. Compute

$$det \begin{bmatrix} a_1^{(n-1)} & a_2^{(n-1)} & ... & a_n^{(n-1)} \\ a_1^{(n-2)} & a_2^{(n-2)} & ... & a_n^{(n-2)} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_1 & a_2 & ... & a_n \\ 1 & 1 & ... & 1 \end{bmatrix}.$$

(d) Let $a_0, ..., a_{n-1} \in F$. Find the characteristic polynomial of

$$\begin{bmatrix} 0 & 0 & 0 & ... & 0 & a_0 \\ 1 & 0 & 0 & ... & 0 & a_1 \\ 0 & 1 & 0 & ... & 0 & a_2 \\ 0 & 0 & 1 & ... & 0 & a_3 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & a_{n-1} \end{bmatrix}.$$

#8 Let A be an m by n matrix over $\mathbf{R}$ and let R be the reduced row echelon form of A. Suppose that the columns of A are $a_1, ..., a_n$ and that the columns of R are $r_1, ..., r_n$. Let $k_1, ..., k_n \in \mathbf{R}$. Prove that

$$k_1 a_1 + ... + k_n a_n = 0$$

if and only if

$$k_1 r_1 + ... + k_n r_n = 0.$$

#9 Let T be the linear operator on $P_3(\mathbf{R}$ defined by

$$T(f) = 3f - xf' + f''.$$

(Here $f = f(x) \in P_2(\mathbf{R})$, f' denotes the derivative of f, and f'' denotes the second derivative of f.) Let W be the T-cyclic subspace of $P_3(\mathbf{R})$ generated by x^3.

(a) Find a basis for W.
(b) Find the characteristic polynomial of T_W, the restriction of T to W.

#10 State the definitions of the following terms.

(a) An eigenvalue (respectively eigenvector, eigenspace) of a linear transformation from V to V.
(b) An eigenvalue (respectively eigenvector, eigenspace) of an n by n matrix A.
(c) The direct sum of subspaces $V_1, ..., V_k$ of a vector space V.
(d) The determinant of an n by n matrix A.
(e) The characteristic polynomial of an n by n matrix A.
(f) Similar

#11 Prove that similar matrices have the same characteristic polynomials and (hence) the same eigenvalues. Give an example to show that they do not necessarily have the same eigenvectors.

#12 Let A be an m by n matrix and B be an n by p matrix.
(a) Is the row space of AB contained in the row space of A? Why or why not?
(b) Is the row space of AB contained in the row space of B? Why or why not?
(c) Is the column space of AB contained in the column space of A? Why or why not?
(d) Is the column space of AB contained in the column space of B? Why or why not?
(e) Prove that $rank(AB) \leq rank(A)$ and $rank(AB) \leq rank(B)$.

#13 Suppose A is a 5 by 7 matrix and B is a 7 by 5 matrix. Suppose further that $det(AB) = 3$. What is $det(BA)$? Why?

#14 Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues for A and for each eigenvalue find a basis for the corresponding eigenspace.

(b)Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (This is equivalent to $P^{-1}AP = D$.)

(c) Using your answer to (b), find the general solution of the following system of linear differential equations:

$$y_1' = y_1 + y_2 - y_3$$

$$y_2' = 2y_2 + y_3$$

$$y_3' = 3y_3$$

#15 A 3 by 3 matrix A has eigenvalues $1, 2,$ and 3. What are the eigenvalues of the matrix $B = A^2 - I$? Why?

#16 In each part state whether or not the given matrix is diagonalizable and give your reason.

(a) $R = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(b) $P = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $Q = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$