MATH 350-01 - Solutions to review problems for Exam #2

The solution to #2 has been corrected and some minor typos have been fixed as of 6PM on Sunday, 4/13.

#1 Suppose that A is a 5 by 5 matrix and

$$B = A + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $det(A) = 1$ and $det(B) = 3$, what is $det(2A + B)$. Why?

Solution: Let a_i denote the i-th row of A and b_i denote the i-th row of B. Thus $b_1 = a_1, b_2 = a_2 + [1, -1, 2, 0, 1], b_3 = a_3, b_4 = a_4, b_5 = a_5$, and we may write

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, B = \begin{bmatrix} a_1 \\ b_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Let $C = (\frac{2}{3})A + (\frac{1}{3})B$. Thus

$$C = \begin{bmatrix} a_1 \\ (\frac{2}{3})a_2 + (\frac{1}{3})b_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Then

$$detC = (\frac{2}{3})det \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} + (\frac{1}{3}) \begin{bmatrix} a_1 \\ b_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = (\frac{2}{3})det(A) + (\frac{1}{3})det(B) = (\frac{2}{3}) + (\frac{1}{3})3 = \frac{5}{3}.$$

Now $2A + B = 3C = (3I)C$ and so

$$det(2A + B) = det(3I)det(C) = 3^5(\frac{5}{3}) = 3^4(5) = 405.$$

#2 Let the 4 by 7 matrix A have columns $a_1, ..., a_7$. Suppose the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Suppose further that $a_2 = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 6 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, and $a_5 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ -3 \end{bmatrix}$. Find A.

Solution: Let R denote the reduced row echelon form of A and let r_i denote the i-th column of R. Then we know that if $b_1, ..., b_7 \in F$ we have $b_1a_1 + ... + b_7a_7 = 0$ if and only if $b_1r_1 + ... + b_7r_7 = 0$. Now $r_2 = 2r_1, r_5 = -r_1 + 2r_3 + r_4, r_6 = r_4$, and $r_7 = 3r_1 + r_3 + 3r_4$. Hence $a_2 = 2a_1$ and so

$$a_1 = (\frac{1}{2})a_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}.$$

Also $a_5 = -a_1 + 2a_3 + a_4$ and so

$$a_4 = a_1 - 2a_3 + a_5 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} 12 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -1 \\ -4 \end{bmatrix}.$$

Finally,

$$a_6 = a_4 = \begin{bmatrix} -2 \\ -2 \\ -1 \\ -4 \end{bmatrix},$$

and

$$a_7 = 3a_1 + a_3 + 3a_4 = 3 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ -2 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -11 \\ -2 \\ -1 \end{bmatrix}.$$

Thus

$$A = \begin{bmatrix} 1 & 2 & 1 & -2 & -1 & -2 & -2 \\ -2 & -4 & 1 & -2 & 2 & -2 & -11 \\ 0 & 0 & 1 & -1 & 1 & -1 & -2 \\ 3 & 6 & 2 & -4 & -3 & -4 & -1 \end{bmatrix}.$$

#3 A 9 by 9 diagonalizable matrix A has three eigenvalues: $1, 2$ and 3. If

$$rank(A - I) = 7$$

and

$$rank(A - 2I) = 5,$$

what is the multiplicity of the eigenvalue 3? Why?

Solution: Since the matrix is diagonalizable, the sum of the dimensions of the eigenspaces must equal 9. Now the 1-eigenspace, E_1, is equal to $N(A - I)$ and so its dimension is the nullity of $A - I$ which is equal to $9 - rank(A - I) = 9 - 7 = 2$. Similarly, the dimension of E_2 is $9 - rank(A - 2I) = 9 - 5 = 4$. Then $2 + 4 + dim(E_3) = 9$ and so $dim(E_3) = 3$. This is the (geometric) multiplicity of the eigenvalue 3.

#4 Let A be an m by n matrix. Write $A = [\, a_1 \quad a_2 \quad ... \quad a_n \,]$ where A_i denotes the i-th column of A. Let $A_k = [\, a_1 \quad ... \quad a_k \,]$, i.e., the matrix consisting of the first k columns of A. Set $s_i(A) = rank(A_i)$ for $1 \leq i \leq n$, and let $s(A)$ denote the n-tuple $[\, s_1(A) \quad ,..., \quad s_n(A) \,]$.

(a) Let P be an invertible m by m matrix. Prove that $s(A) = s(PA)$.

(b) Let R be the reduced row echelon form of A. Prove that $s(R) = s(A)$.

(c) Say that a column of A is a basic column if the corresponding column of R contains the initial nonzero entry of some row. Show how to determine the basic columns from the n-tuple $s(A)$.

(d) Show that the column a_i of A is a linear combination of the columns a_j such that $j \leq i$ and a_j is basic.

(e) Explain why a matrix A has only one reduced row echelon form.

Solution:

(a) We know from the definition of matrix multiplication that the i-th column of PA is Pa_i. Therefore $(PA)_k = P(A_k)$ and so, $s_k(PA) = rank((PA)_k) = rank(P(A_k)) = rank(A_k) = A_k$.

(b) Since $R = PA$ for some invertible matrix P, this follows from part (a).

(c) The k-th column of R is basic if and only if it is not contained in the span of the first $k - 1$ columns. This occurs if and only if either $k = 1$ and $s_1(R) \neq 0$ or if $k > 1$ and $s_k(R) > s_{k-1}(R)$. In view of part (b), this means that the k-th column is basic if and only if either $k = 1$ and $s_1(A) \neq 0$ or if $k > 1$ and $S_k(A) > s_{k-1}(A)$.

(d) We know that for scalars $b_1, ..., b_n$ we have $b_1 a_1 + ... + b_n a_n = 0$ if and only if $b_1 r_1 + ... + b_n r_n$. Since r_i is a linear combination of the columns r_j such that $j \leq i$ and r_j is basic, the same result holds for the a_i.

(e) Suppose A has reduced row echelon forms

$$R = [\, r_1 \quad r_2 \quad ... \quad r_n \,]$$

and

$$T = [\, t_1 \quad t_2 \quad \ldots \quad t_n \,].$$

Then by (c) the basic columns of R are the same as the basic columns of T. Furthermore, any column of A is a linear combination of basic columns of A. Therefore the corresponding column of R is the same linear combination of the basic columns of R and the corresponding column of T is the same linear combination of the basic columns of T. Thus every column of R is equal to the corresponding column of T and so the two matrices are equal.

#5 Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & -1 \\ 1 & 2 & 0 & 1 & -1 \\ 2 & 5 & -1 & 0 & -2 \\ 2 & 3 & 1 & 4 & -1 \end{bmatrix}.$$

(a) Find the reduced row echelon form for A
(b) Find a basis for the null space $N(L_A)$
(c) Find a basis for the row space of A
(d) Find a basis for the column space of A.

Solution:

(a) $R = \begin{bmatrix} 1 & 0 & 2 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the reduced row echelon form.

(b) The free variables are x_3 and x_4. Suppose $R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$. Then

$$\begin{bmatrix} x_1 + 2x_3 + 5x_4 \\ x_2 - x_3 - 2x_4, x_5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and so

$$x_1 = -2x_3 - 5x_4$$

$$x_2 = x_3 + 2x_4$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$x_5 = 0.$$

Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 - 5x_4 \\ x_3 + 2x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus

$$\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \}$$

is a basis for $N(L_A)$.

(c) The set of nonzero rows of the reduced row echelon form of A is (one) basis for the row space of A. Thus $\{[1 \quad 0 \quad 2 \quad 5 \quad 0], [0 \quad 1 \quad -1 \quad -2 \quad 0], [0 \quad 0 \quad 0 \quad 0 \quad 1]\}$ is a basis for the row space of A.

(d) The set of basic columns of A (that is, those columns corresponding to the columns of R containing the initial nonzero element of some row) is one basis for the column space of A. Thus $\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \\ -1 \end{bmatrix} \}$ is a basis for the column space of A.

#6 Let $A = \begin{bmatrix} -3 & 0 & -5 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find all eigenvalues for A and find a basis for each eigenspace.

(b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution:

(a) $det \begin{bmatrix} -3-\lambda & 0 & -5 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{bmatrix} =$

$(2-\lambda)det \begin{bmatrix} -3-\lambda & -5 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda)(\lambda^2 - 9 + 5) = (2-\lambda)((\lambda^2 - 4) = -(\lambda - 2)^2(\lambda + 2).$

Thus the eigenvalues are 2 and -2. Now $E_2 = N(A - 2I) = N(\begin{bmatrix} -5 & 0 & -5 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix})$.

Thus $\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$ is a basis for E_2. Also $E_{-2} = N(A - (-2)I) = N(A + 2I) =$ $N(\begin{bmatrix} -1 & 0 & -5 \\ 0 & 4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$. Thus $\{\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$ is a basis for E_{-2}.

(b) $P = \begin{bmatrix} -1 & 0 & -5 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is one choice for O and D.

#7

(a) Compute *det* A if

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & -1 & -4 \end{bmatrix}$$

(b) Compute *det* B if

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 \\ 0 & 3 & 7 & 3 & 0 \\ 0 & 0 & 4 & 13 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix}$$

(c) Let $a_1, ..., a_n \in F$. Compute

$$det \begin{bmatrix} a_1^{(n-1)} & a_2^{(n-1)} & \ldots & a_n^{(n-1)} \\ a_1^{(n-2)} & a_2^{(n-2)} & \ldots & a_n^{(n-2)} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ a_1 & a_2 & \ldots & a_n \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

(d) Let $a_0, ..., a_{n-1} \in F$. Find the characteristic polynomial of

$$\begin{bmatrix} 0 & 0 & 0 & ... & 0 & a_0 \\ 1 & 0 & 0 & ... & 0 & a_1 \\ 0 & 1 & 0 & ... & 0 & a_2 \\ 0 & 0 & 1 & ... & 0 & a_3 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & a_{n-1} \end{bmatrix}.$$

Solution:

(a) $$detA = det \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & 2 & 3 \\ 0 & 2 & 0 & -2 \end{bmatrix} =$$

$$-det \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 3 \\ 0 & 2 & 2 & 6 \\ 0 & 2 & 0 & -2 \end{bmatrix} = -det \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & 4 & 4 \end{bmatrix} = -det \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & 0 & -4 \end{bmatrix} = -24.$$

(b) $$detB = det \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 7 & 3 & 0 \\ 0 & 0 & 4 & 13 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 4 & 13 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & -15 \end{bmatrix} = -15.$$

(c) Subtract a_1 times the second row from the first row. Then subtract a_1 times the third row from the second row. Continue in this way, finally subtracting a_1 times the n-th row from the $n-1$st row to get

$$det \begin{bmatrix} a_1^{(n-1)} & a_2^{(n-1)} & ... & a_n^{(n-1)} \\ a_1^{(n-2)} & a_2^{(n-2)} & ... & a_n^{(n-2)} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_1 & a_2 & ... & a_n \\ 1 & 1 & ... & 1 \end{bmatrix} =$$

$$det \begin{bmatrix} 0 & (a_2-a_1)a_2^{(n-2)} & ... & (a_n-a_1)a_n^{(n-2)} \\ 0 & (a_2-a_1)a_2^{(n-3)} & ... & (a_n-a_1)a_n^{(n-3)} \\ . & . & ... & . \\ . & . & ... & . \\ 0 & a_2-a_1 & ... & a_n-a_1 \\ 1 & 1 & ... & 1 \end{bmatrix}.$$

Expanding along the first column shows that this is

$$\begin{bmatrix} (a_2-a_1)a_2^{(n-2)} & ... & (a_n-a_1)a_n^{(n-2)} \\ (a_2-a_1)a_2^{(n-3)} & ... & (a_n-a_1)a_n^{(n-3)} \\ . & ... & . \\ . & ... & . \\ a_2-a_1 & ... & a_n-a_1 \end{bmatrix}.$$

Factoring out the common factors from each column gives

$$(-1)^{n+1}(a_2-a_1)(a_3-a_1)...(a_n-a_1)det \begin{bmatrix} a_2^{(n-2)} & a_3^{(n-2)} & ... & a_n^{(n-2)} \\ a_2^{(n-3)} & a_3^{(n-3)} & ... & a_n^{(n-3)} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_2 & a_3 & ... & a_n \\ 1 & 1 & ... & 1 \end{bmatrix} =$$

$$(a_1-a_2)(a_1-a_3)...(a_1-a_n)det \begin{bmatrix} a_2^{(n-2)} & a_3^{(n-2)} & ... & a_n^{(n-2)} \\ a_2^{(n-3)} & a_3^{(n-3)} & ... & a_n^{(n-3)} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_2 & a_3 & ... & a_n \\ 1 & 1 & ... & 1 \end{bmatrix}.$$

Continuing in this way gives

$$det \begin{bmatrix} a_1^{(n-1)} & a_2^{(n-1)} & ... & a_n^{(n-1)} \\ a_1^{(n-2)} & a_2^{(n-2)} & ... & a_n^{(n-2)} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_1 & a_2 & ... & a_n \\ 1 & 1 & ... & 1 \end{bmatrix} = (a_1-a_2)...(a_1-a_n)(a_2-a_3)...(a_2-a_n)...(a_{n-1}-a_n).$$

(d) Expanding along the first row gives

$$det \begin{bmatrix} -\lambda & 0 & 0 & ... & 0 & a_0 \\ 1 & -\lambda & 0 & ... & 0 & a_1 \\ 0 & 1 & -\lambda & ... & 0 & a_2 \\ 0 & 0 & 1 & ... & 0 & a_3 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & a_{n-1} - \lambda \end{bmatrix} =$$

$$(-\lambda det \begin{bmatrix} -\lambda & 0 & 0 & ... & 0 & a_1 \\ 1 & -\lambda & 0 & ... & 0 & a_2 \\ 0 & 1 & -\lambda & ... & 0 & a_3 \\ 0 & 0 & 1 & ... & 0 & a_4 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & a_{n-1} - \lambda \end{bmatrix} + (-1)^{1+n} det \begin{bmatrix} 1 & -\lambda & 0 & ... & 0 \\ 0 & 1 & -\lambda & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}.$$

Since the matrix in the second summand is upper triangular with diagonal entries 1, its determinant is 1. Thus the characteristic polynomial of the given matrix is

$$(-\lambda det \begin{bmatrix} -\lambda & 0 & 0 & ... & 0 & a_1 \\ 1 & -\lambda & 0 & ... & 0 & a_2 \\ 0 & 1 & -\lambda & ... & 0 & a_3 \\ 0 & 0 & 1 & ... & 0 & a_4 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & a_{n-1} - \lambda \end{bmatrix} + (-1)^{1-n} a_0.$$

Continuing in this way shows that the characteristic polyomial is

$$(-1)^n(\lambda^n - a_{n-1}\lambda^{n-1} - ... - a_1\lambda - a_0).$$

#8 Let A be an m by n matrix over $\mathbf{R}$ and let R be the reduced row echelon form of A. Suppose that the columns of A are $a_1, ..., a_n$ and that the columns of R are $r_1, ..., r_n$. Let $k_1, ..., k_n \in \mathbf{R}$. Prove that

$$k_1 a_1 + ... + k_n a_n = 0$$

if and only if

$$k_1 r_1 + ... + k_n r_n = 0.$$

Solution: Write $k = \begin{bmatrix} k_1 \\ k_2 \\ . \\ . \\ . \\ k_n \end{bmatrix}$. Then $k_1a_1 + ... + k_na_n = Ak$ and $k_1r_1 + ... + k_nr_n = Rk$.
But $R = PA$ for some invertible n by m matrix P. Now if $Ak = 0$ then $Rk = (PA)k = P(Ak) = 0$ and if $Rk = 0$ then $Ak = (P^{-1}R)k = P^{-1}(Rk) = 0$.

#9 Let T be the linear operator on $P_3(\mathbf{R})$ defined by

$$T(f) = 3f - xf' + f''.$$

(Here $f = f(x) \in P_3(\mathbf{R})$, f' denotes the derivative of f, and f'' denotes the second derivative of f.) Let W be the T-cyclic subspace of $P_3(\mathbf{R})$ generated by x^3.

(a) Find a basis for W.
(b) Find the characteristic polynomial of T_W, the restriction of T to W.

Solution:
(a) $T(x^3) = 3x^3 - x(3x^2) + 6x = 6x$ and so $T^2(x^3) = T(6x) = 18x - x(6) + 0 = 12x$. Thus $T^2(x^3) \in span\{x^3, T(x^3)\}$. Since $\{x^3, T(x^3)\} = \{x^3, 6x\}$ is linearly independent it is a basis for W.
(b) $T^2(x^3) = 2T(x^3)$ and therefore $t^2 - 2t$ is the characteristic polynomial of T_W.

#10 State the definitions of the following terms.
(a) An eigenvalue (respectively eigenvector, eigenspace) of a linear transformation from V to V.
(b) An eigenvalue (respectively eigenvector, eigenspace) of an n by n matrix A.
(c) The direct sum of subspaces $V_1, ..., V_k$ of a vector space V.
(d) The determinant of an n by n matrix A.
(e) The characteristic polynomial of an n by n matrix A.
(f) Similar

Solution:
(a) A scalar $\alpha \in F$ such that $T(v) = \alpha v$ for some nonzero $v \in V$ is called an eigenvalue for T and such a v is called an eigenvector belonging to α. The α-eigenspace, denoted E_α is $\{v \in V | T(v) = \alpha v\}$.
(b) A scalar $\alpha \in F$ such that $Av = \alpha v$ for some nonzero column vecgtor $v \in F^n$ is called an eigenvalue for A and such a v is called an eigenvector belonging to α. The α-eigenspace, denoted E_α is $\{v \in F^n | Av = \alpha v\}$.

(c) The sum, $V_1 + ... + V_k$ of the subspaces $V_1, ..., V_k$ is

$$\{v_1 + ... + v_k | v_1 \in V_1, ..., v_k \in V_k\}.$$

The sum $V_1 + ... + V_k$ is said to be a direct sum (and written $V_1 \oplus ... \oplus V_k$) if $V_i \cap (V_1 + ... + v_{i-1} + V_{i+1} + ... + V_k) = \{0\}$ for all $i, 1 \le i \le k$.

(d) The determinant of the 1 by 1 matrix $[a]$ is a. Assume that determinants of $n-1$ by $n-1$ matrices have been defined and that $A = [a_{ij}]$ is an n by n matrix. Then

$$det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} det \bar{A^{1j}}$$

where $\bar{A^{1j}}$ is the matrix obtained from A by deleting the first row and the j-th column.

(e) The characteristic polynomial of the n by n matrix A is $det(A - \lambda I)$ where I denotes the n by n identity matrix.

(f) Two n by n matrices A and B are similar if there is an invertible n by n matrix P such that $B = PAP^{-1}$.

#11 Prove that similar matrices have the same characteristic polynomials and (hence) the same eigenvalues. Give an example to show that they do not necessarily have the same eigenvectors.

Solution:

Suppose $B = PAP^{-1}$ where P is invertible. Then

$$det(B - \lambda I) = det(PAP^{-1} - \lambda I) = det(P(A - \lambda I)P^{-1}) = det(P)det(A - \lambda I)det(P^{-1}) =$$

$$det(P)det(A - \lambda I)det(P)^{-1} = det(P)det(P)^{-1}det(A - \lambda I) = det(A - \lambda I).$$

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then A and B are similar since $B = PAP^{-1}$ where $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. However, the 0-eigenspace of A is $N(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = F \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the 0-eigenspace of B is $N(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = F \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

#12 Let A be an m by n matrix and B be an n by p matrix.

(a) Is the row space of AB contained in the row space of A? Why or why not?

(b) Is the row space of AB contained in the row space of B? Why or why not?

(c) Is the column space of AB contained in the column space of A? Why or why not?

(d) Is the column space of AB contained in the column space of B? Why or why not?

(e) Prove that $rank(AB) \leq rank(A)$ and $rank(AB) \leq rank(B)$.

Solution:

(a) No. In fact, the row space of A consists of (row) vectors in F^n and the row space of AB consists of vectors in F^p, so if $n \neq p$ an inclusion is impossible. Even if $n = p$ the inclusion does not hold. For example, if $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then the row space of AB is $F\begin{bmatrix} 0 & 1 \end{bmatrix}$ while the row space of A is $F\begin{bmatrix} 1 & 0 \end{bmatrix}$.

(b) Yes. Let E_{ij} denote the matrix with entry 1 in the (i, j) position and 0 in every other position. Then the ith row of $E_{ij}B$ is equal to the jth row of B and all other rows of $E_{ij}B$ are 0. Thus the row space of $E_{ij}B$ is contained in the row space of B. Since A is a linear combination of the E_{ij} it follows tht the row space of AB is contained in the row space of B.

(c) The column space of AB is the row space of $(AB)^t = B^tA^t$. Now the row space of B^tA^t is contained in the ros space of A^t which is the column space of A. Thus the column space of AB is contained in the column space of A.

(d) The example of (a) shows that the answer is no.

(e) We know that the rank of A is equal to the dimension of the row space. Thus (b) gives $rank(AB) \leq rank(B)$. We also know that the rank of A is equal to the dimension of the column space. Thus (c) gives $rank(AB) \leq rank(A)$.

#13 Suppose A is a 5 by 7 matrix and B is a 7 by 5 matrix. Suppose further that $det(AB) = 3$. What is $det(BA)$? Why?

Solution: We have $rankA \leq 5$ (since A has only 5 rows). Thus by (e) of the previous problem, $rank(BA) \leq 5$. But BA is a 7 by 7 matrix. Hence BA is not invertible and so its determinant is equal to 0.

#14 Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues for A and for each eigenvalue find a basis for the corresponding eigenspace.

(b)Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (This is equivalent to $P^{-1}AP = D$.)

(c) Using your answer to (b), find the general solution of the following system of linear differential equations:

$$y_1' = y_1 + y_2 - y_3$$

$$y_2' = 2y_2 + y_3$$
$$y_3' = 3y_3$$

Solution: (a) The eigenvalues are $1, 2, 3$. The 1-eigenspace has basis $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$; the 2-eigenspace has basis $\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\}$; the 3-eigenspace has basis $\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}$.

(b) We may take $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(c) Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be the general solution to the system and let $x = P^{-1}y$. Then $Ay = y'$ and $Dx = P^{-1}APx = P^{-1}APP^{-1}y = P^{-1}Ay = P^{-1}y' = (P^{-1}y)' = x'$. Thus

$$x = \begin{bmatrix} C_1e^t \\ C_2e^{2t} \\ C_3e^{3t} \end{bmatrix}$$

and

$$y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1e^t \\ C_2e^{2t} \\ C_3e^{3t} \end{bmatrix}.$$

#15 A 3 by 3 matrix A has eigenvalues $1, 2$, and 3. What are the eigenvalues of the matrix $B = A^2 - I$? Why?

Solution: Suppose v is an eigenvector for the matrix A corresponding to the eigenvalue i. Then

$$A^2v = A(Av) = A(iv) = i(Av) = i(iv) = i^2v$$

and

$$(A^2 - I)v = a^2v - v = i^2v - v = (i^2 - 1)v.$$

Thus the eigenvalues of $A^2 - I$ are $1^1 - 1 = 0, 2^2 - 1 = 3$, and $3^2 - 1 = 8$.

#16 In each part state whether or not the given matrix is diagonalizable and give your reason.

(a) $R = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(b) $P = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $Q = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: (a) The characteristic polynomial is $(1-\lambda)(2-\lambda)(4-\lambda)$. Since there are three distinct roots (and hence 3 eigenvalues), the matrix is diagonalizable.

(b) The characteristic polynomial is $(2-\lambda)^2(3-\lambda)$ and $E_2 = N(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix})$. Thus E_2 has dimension 1, so the geometric multiplicity of the eigenvalue 2 is not equal to its algebaic multiplicity. Hence P is not diagonalizable.

(c) The characteristic polynomial is $(2-\lambda)^2(3-\lambda)$ and $E_2 = N(\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$. Thus E_2 has dimension 2, so the geometric multiplicity of the eigenvalue 2 is equal to its algebaic multiplicity. Hence Q is diagonalizable.