Math 351

Review problems for Exam #1 **October 8, 2007**

Exam #1 will be given during the normal class period on Monday, October 15. It will cover material through Section 4.4. This set of review problems is about twice as long as the exam.

#1 Find the greatest common divisor of 561 and 1336 and write it in the form $561a+1336b$ where a and b are integers.

#2 Find the greatest common divisor of the polynomials $f(x) = x^5 + 2x^4 + 8x^3 + 16x^2 + 11x + 2$ and $g(x) = x^5 + 11x^3 + 2x^2 + 28x + 8$ and write it in the form $a(x)f(x) + b(x)g(x)$ where $a(x), b(x) \in \mathbf{R}[x]$.

#3 Let a, b, and n be integers. State the definition of $a \equiv b(mod\ n)$ and prove that if $a_1 \equiv b_1(mod\ n)$ and $a_2 \equiv b_2(mod\ n)$ then $a_1a_2 \equiv b_1b_2(mod\ n)$.

#4 Let $n > 1$ be an integer. State the definition of $\mathbf{Z}_n$. Using the fact that $\mathbf{Z}$ is a ring, prove that addition in $\mathbf{Z}_n$ is associative.

$5 Let R, S, T be rings, let f be a homomorphism from R to S and g be a homomorphism from S to T. Prove that the composition $g \circ f$ is a homomorphism from R to T.

#6 Let R be a ring, with addition $+$ and multiplication $\times_R$. Define a new multiplication $\times_{op}$ on R by $a \times_{op} b = b \times_R a$ for all $a, b \in R$. Then R with addition $+$ and multiplication $\times_{op}$ is a ring. (You don't have to verify this.) Show that $M(\mathbf{R})$ is isomorphic to $M(\mathbf{R})^{op}$. (Hint: Use the transpose map.)

#7 Let I be an ideal in a ring R and $a \in R$.

(a) State the definition of the coset $a + I$ and of the quotient ring R/I

(b) Prove that if $a_1 + I = b_1 + I$ and $a_2 + I = b_2 + I$, then $(a_1 + a_2) + I = (b_1 + b_2) + I$.

#8 (a) Is $\{3n|n \in \mathbf{Z}\}$ a subring of $\mathbf{Z}$? Why or why not?

(b) Is $\{3n + 1|n \in \mathbf{Z}\}$ a subring of $\mathbf{Z}$? Why or why not.

#9 Let U denote the set of upper triangular matrices in $M(\mathbf{R})$, D denote the set of diagonal matrices in $M(\mathbf{R})$, and N denote the set of strictly upper triangular matrices in $M(\mathbf{R})$.

(a) Verify that U and D are subrings of $M(\mathbf{R})$.

(b) Verify that N is an ideal in U.

(c) Show that the map $f : U \to D$ defined by $f(\begin{vmatrix} a & b \\ 0 & d \end{vmatrix}) = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}$ is a homomorphism of U onto D.

(d) Show that $D \cong U/N$.

#10 (a) Is the map $A \to\ tr(A)$ (where $tr(A)$ is the trace of the matrix A, i.e., the sum of its diagonal elememts) a homomorphism from $M(\mathbf{R})$ to $\mathbf{R}$? Why or why not?

(b) Is the map $A \to\ det(A)$ a homomorphsim from $M(\mathbf{R})$ to $\mathbf{R}$? Why or why not?

#11 Let S_1 and S_2 be subrings of a ring R.

(a) Is $S_1 + S_2$ (which, by definition, is $\{a+b|a \in S_1, b \in S_2\}$) necessarily a subring of R? Why or why not?

(b) Is S_1S_2 (which, by definition, is $\{ab|a \in S_1, b \in S_2\}$ necessaily a subring of R? Why or why not?

(c) Suppose S_2 is an ideal of R. Is $S_1 + S_2$ necessarily a subring of R? Why or why not?

(d) Show that $S_1 \cap S_2$ is a subring of R.

#12 Prove that if $f(x), g(x), h(x) \in F[x]$ (where F is a field), $(f(x), g(x)) = 1$, and $f(x)$ divides $g(x)h(x)$, then $f(x)$ divides $h(x)$.

#13 Prove that if $f(x) \in F[x]$ where F is a field, $a \in F$ and $f(a) = 0$ then $x - a$ divides $f(x)$.

$14 (a) Find all the irreducible polynomials of degree 3 over $\mathbf{Z}_2$.

(b) Find all the irreducible polynomials of degree 3 over $\mathbf{Z}_3$.

(c) Find all the irreducible polynomials of degree 4 over $\mathbf{Z}_2$.