Math 351

Review problems for Exam #2 **November 12, 2007**

Exam #2 will be given during the normal class period on Monday, November 19. It will cover material from Sections 4.5, 4.6, 5.1 - 5.3, 6.3, 9.1, 9.4, 7.1 - 7.4. This set of review problems is about twice as long as the exam. As usual, **Z** denotes the ring of integers, **Q** denotes the field of rational numbers, and **C** denotes the field of complex numbers.

#1 Let $f(x) \in \mathbf{R}[x]$ have degree 7. Prove that $f(x)$ is a reducible polynomial in $\mathbf{R}[x]$. You may want to use the fact that every irreducible polynomial in $\mathbf{C}[x]$ has degree 1.

#2 Let $f(x)$ and $g(x)$ be polynomials in $\mathbf{Z}[x]$. Let p be a prime integer. Prove that if p divides every coefficient of $f(x)g(x)$ then either p divides every coefficient of $f(x)$ or p divides every comefficient of $g(x)$.

#3 Let $f(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbf{Z}[x]$ and suppose that $\frac{r}{s} \neq 0$ is a root of $f(x)$ where $r, s \in \mathbf{Z}$ and r and s are relatively prime. Prove that $r|a_0$ and $s|a_n$.

#4 Let $f(x) \in \mathbf{Z}[x]$ and assume that $f(x)$ is an irreducible polynomial in $\mathbf{Z}[x]$. Prove that $f(x)$ is an irreducible polynomial in $\mathbf{Q}[x]$. You may want to use the results of problems #2 and #3.

#5 Show (by constructing an example) that there is a field with 8 elements.

#6 Let F be a field and $f(x) \in F[x]$. Let $p(x) \in F[x]$ be a polynomial of degree ≥ 1. Prove that $f(x) + (p(x))$ is a unit in $F[x]/(p(x))$ if and only if $f(x)$ and $p(x)$ are relatively prime.

#7 (a) State the definition a prime ideal in a ring R.

(b) Prove that an ideal I in a commutative ring with identity R is a prime ideal if and only if R/I is an integral domain.

(c) State the definition of a maximal ideal in a ring R.

(d) Prove that if R is a commutative ring with identity, then an ideal I in R is maximal if and only if R/I is a field.

#8 Show that $\mathbf{Z}[\sqrt{-2}]$ is a Euclidean domain with $\delta(a + b\sqrt{-2}) = a^2 + 2b^2$.

#9 Let R be an integral domain. Define $S = \{(a, b)|a, b \in R, b \neq 0\}$. Define $(a, b) \sim (c, d)$ if $ad = bc$. Show that $\sim$ is an equivalence relation.

#10 Let $R = \{a + b\sqrt{3}|a, b \in \mathbf{Z}\}$. Then R is an integal domain (why?) and so R has a quotient field F. What is F?

#11 Let G be a group, $g, h, k \in G$ and $gh = gk$. Prove that $h = k$. Conclude that the multiplicative inverse of g is unique.

#12 Compute the product

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 1 & 2 & 3 & 7 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 \end{pmatrix}$$

in the symmetric group on 7 elements.

#13 Let $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 6 & 4 & 1 & 3 \end{pmatrix}$ in the symmetric group on 7 elements.

(a) Find g^{-1}.

(b) Find the order of g.

#14 Let G be a group with identity element e. Suppose $g^2 = e$ for all $g \in G$. Prove that G is commutative.

#15 Let G be a commutative group with identity element e and let $n \in \mathbf{Z}, n \geq 1$. Let $H = \{g \in G | g^n = e\}$. Prove that H is a subgroup of G.