Math 351

Workshop #4 **September 26, 2007**

The first problem asks you to find all the ideals in two important rings: $\mathbf{Z}$*, the ring of integers, and* $M(\mathbf{R})$ *the ring of two by two matrices over the real numbers.*

#1 (a) Let I be a ideal in $\mathbf{Z}$. Show that for any $n \in \mathbf{Z}, \{nk|k \in \mathbf{Z}\} = n\mathbf{Z}$ is an ideal in $\mathbf{Z}$. We will denote this ideal by (n).

(b) Let I be a ideal in $\mathbf{Z}$ and let $P = \{k \in I|k > 0\}$. Show that if $P = \emptyset$, then $I = (0)$ while if $P \neq \emptyset$ and n is the smallest element of P, then $I = (n)$.

(c) For $1 \leq i, j \leq 2$, let e_{ij} denote the matrix in $M(\mathbf{R})$ with 1 in the (i, j) position and 0 in all the other positions. Thus

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let I be an ideal in $M(\mathbf{R})$. Show that if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in I$ and $a_{ij} \neq 0$, then $e_{ij} \in I$.

(d) Let I be an ideal in $M(\mathbf{R})$. Show that if $e_{ij} \in I$ for some $1 \leq i, j \leq 2$, then $e_{11} \in I$.

(e) Let I be an ideal in $M(\mathbf{R})$. Show that if $e_{11} \in I$, then $I = M(\mathbf{R})$.

(f) Show that any ideal in $M(\mathbf{R})$ is either $\{0\}$ or $M(\mathbf{R})$.

The second problem asks you (in part (f) after several preliminary steps) to derive an importat result - the Second Isomorphism Theorem.

#2 Recall that if R is a ring and A, B are two subsets of R, then $A + B$ denotes $\{a + b|a \in A, b \in B\}$.

(a) Show that if R is a ring, S is a subring of R and I is an ideal in R, then $S + I$ is a subring of R.

(b) Give an example to show that if R is a ring, and S_1 and S_2 are subrings of R then $S_1 + S_2$ is not necessarily a subring of R. (Hint: Try looking as some subrings of $M(\mathbf{R})$.)

(c) Show that if R is a ring, and I and J are ideals in R, then $I + J$ is an ideal in R.

(d) Show that if R is a ring, and I and J are ideals in R, then $I \cap J$ is an ideal in R.

(e) Show that if R is a ring, S is a subring of R and I is an ideal in R, then $S \cap I$ is an ideal in S.

(f) Prove the Second Isomorphism Theorem: If R is a ring, S is a subring of R and I is an ideal in R, then $(S + I)/I$ is isomorphic to $S/(S \cap I)$. (Hint: Show that if $s_1, s_2 \in S, x_1, x_2 \in I$ and $s_1 + x_1 = s_2 + x_2$ then $s_1 + (S \cap I) = s_2 + (S \cap I)$. Thus we may define a map $f : (S + I) \to S/(S \cap I)$ by $f(s + x) = s + (S \cap I)$ where $s \in S, x \in I$. Show that f is a homomorphism of $(S + I)$ onto $S/(S \cap I)$ with kernel I and then apply Theorem 6.13.)