Math 351

Workshop #1 **September 1, 2010**

The first problem sketchs a proof of a result you are probably familiar with: the binomial theorem.

#1(a) If $0 \leq i \leq n$ are integers, define

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

(Recall that $n! = (n)(n-1)...(3)(2)(1)$ if $n \geq 1$ and that $0! = 1$.) Also, define $\binom{n}{i} = 0$ whenever $i < 0$ or $i > n$. Prove that for $n \geq i \geq 0$ we have

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$$

and that $\binom{n}{i}$ is an integer.

(b) Let $a, b, n \in \mathbf{Z}, n \geq 0$. Prove that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

The next problem uses the binomial theorem to derive a celebrated result. We will see another way to prove this result later in the course (using material of Chapter 7).

#2(a) Let p be a prime and $0 < i < p$. Prove that p divides $\binom{p}{i}$.

(b) Let p be a prime and $a, b \in \mathbf{Z}$. Prove that p divides $(a+b)^p - a^p - b^p$.

(c) Prove:

Fermat's Little Theorem: Let p be a prime and $a \in \mathbf{Z}$. Then p divides $a^p - a$.

It is frequently interesting to invetigate whether all the hypotheses of a result are really needed and, if so, why. This explains the next problem.

#3 Does the Fermat's Little Theorem continue to hold if the hypothesis that p is prime is omitted? Justify your answer by giving a proof or an example.

Now some unrelated problems:

#4 Let n be an integer that is not divisible by 2 or 5. For any integer $m \geq 1$, let J_m denote the integer 11...1 where there are m $1's$. Thus $J_7 = 1,111,111 = 1 + 10 + (10)^2 + ... + (10)^6$, and, in general, $J_m = 1 + 10 + (10)^2 + ... + (10)^{m-1}$. Prove that n divides J_m for some m.

#5 Let $r, s, t \in \mathbf{Z}$ be such that the only positive integer dividing r, s, and t is 1. Prove that there are integers a, b, c such that $ar + bs + ct = 1$. (Note that this generalizes the result for two integers that was proved in class.)