MATH 354-03 **February 19, 2005**

Solutions to practice questions for exam #1

#1 A furniture manufacturer wishes to determine how many tables, chairs, desks, or bookcases he should make in order to optimize the use of his available resources. These products utilize two different types of lumber: pine and oak. The manufacturer has on hand 1,500 board feet of pine and 1,000 board feet of oak. He has 800 hours of his employees' time available for the entire job. His sales forecast plus his back orders require him to make at least 40 tables, 130 chairs, 30 desks and no more than 10 bookcases.

Each table requires 5 board feet of pine, 2 board feet of oak, and 3 hours of labor.

Each chair requires 1 board feet of pine, 3 board feet of oak, and 2 hours of labor.

Each desk requires 9 board feet of pine, 4 board feet of oak, and 5 hours of labor.

Each bookcase requires 12 board feet of pine, 1 board feet of oak, and 10 hours of labor.

The manufacturer makes a profit of $12 on a table, $5 on a chair, $15 on a desk, and $10 on a bookcase.

Set up a linear programming model of this situation. State explicitly what each of your variables (for example, $x_1, x_2, ...$) represents). DO NOT attempt to solve the resulting linear programming problem.

SOLUTION: Let x_1 denote the number of tables to be produced, x_2 the number of chairs, x_3 the number of desks and x_4 the number of bookcases.

maximize: $12x_1 + 5x_2 + 15x_3 + 10x_4$
subject to:

$5x_1 + x_2 + 9x_3 + 12x_4 \leq 1500$
$2x_1 + 3x_2 + 4x_3 + 1x_4 \leq 1000$
$3x_1 + 2x_2 + 5x_3 + 10x_4 \leq 800$
$x_1 \geq 40, x_2 \geq 130, x_3 \geq 30$
$0 \leq x_4 \leq 10.$

#2 A manufacturer has distribution centers located in Atlanta (A), Chicago (C), and New York (NY). These centers have available 40, 20, and 40 units of his product, respectively. His retail outlets require the following number of units: Cleveland (CL)- 25; Louisville (L) - 10; Memphis (M)- 20; Pittsburgh (P)- 30; and Richmond (R)- 15. The shipping cost per unit in dollars between each center and outlet is given in the following table:

	CL	L	M	P	R
A	55	30	40	50	40
C	35	30	100	45	60
NY	40	60	95	35	30

Set up a linear programming model of this situation. State explicitly what each of your variables (for example, $x_1, x_2, ...$) represents. DO NOT attempt to solve the resulting linear programming problem.

SOLUTION:

There are fifteen variables denoted $x_{I,J}$ where I stands for A, C, or NY (designating a distribution center) and J stands for CL, L, M, P, or R (designating a retail outlet). The variable $x_{I,J}$ denotes the number of units to be shipped from distribution center I to retail outlet J.

minimize: $55x_{A,CL} + 30x_{A,L} + 40x_{A,M} + 50x_{A,P} + 40x_{A,R} + 35x_{C,CL} + 30x_{C,L} + 100x_{C,M} + 45x_{C,P} + 60x_{C,R} + 40x_{NY,CL} + 60x_{NY,L} + 95x_{NY,M} + 35x_{NY,P} + 30x_{NY,R}$

subject to:

$x_{A,CL} + x_{A,L} + x_{A,M} + x_{A,P} + x_{A,R} \leq 40$

$x_{C,CL} + x_{C,L} + x_{C,M} + x_{C,P} + x_{C,R} \leq 20$

$x_{NY,CL} + x_{NY,L} + x_{NY,M} + x_{NY,P} + x_{NY,R} \leq 40$

$x_{A,CL} + x_{C,CL} + x_{NY,CL} \geq 25$

$x_{A,L} + x_{C,L} + x_{NY,L} \geq 10$

$x_{A,M} + x_{C,M} + x_{NY,M} \geq 20$

$x_{A,P} + x_{C,P} + 35x_{NY,P} \geq 30$

$x_{A,R} + x_{C,R} + x_{NY,R} \geq 15$

$x_{I,J} \geq 0$ for all I,J.

#3 Convert the following linear programming problem into (a) standard form, and (b) canonical form.

minimize: $-3x_1 + 2x_2 + x_4$
subject to:
$x_1 + x_2 \geq 5 - x_1 - x_3 + 2x_4$
$x_2 + 3x_4 = 5$
$x_1, x_2 \geq 0, x_3 \leq 0, x_4$ unconstrained.

SOLUTION:
(a)
Replace x_3 with $-x_5$ and x_4 with $x_6 - x_7$. Then the problem becomes:

maximize: $3x_1 - 2x_2 - x_6 + x_7$
subject to:
$-2x_1 - x_2 + x_5 + 2x_6 - 2x_7 \leq -5$
$x_2 + 3x_6 - 3x_7 \leq 5$
$-x_2 - 3x_6 + 3x_7 \leq -5$
$x_1, x_2, x_5, x_6, x_7 \geq 0.$
(b)
Replace x_3 with $-x_5$ and x_4 with $x_6 - x_7$ and introduce a slack variable, x_8, to change the first consraint in (a) to an equality. Then the problem becomes:

maximize $3x_1 - 2x_2 - x_6 + x_7$
subject to:
$-2x_1 - x_2 + x_5 + 2x_6 - 2x_7 + x_8 = -5$
$x_2 + 3x_6 - 3x_7 = 5$
$x_1, x_2, x_5, x_6, x_7, x_8 \geq 0.$

#4 Consider the linear programming problem
Maximize: $x + y$
Subject to:
$-x + y \leq 2$
$2x + y \leq 6$

$x + 2y \leq 6$
$x, y \geq 0.$

Sketch the feasible region and the lines with equations $x+y = 2, x+y = 3, x + y = 4, x + y = 5, x + y = 6$. Find the optimal solution and explain how you find it.

SOLUTION: The feasible region is the pentagon with vertices $(0, 0), (0, 2)$, $(\frac{2}{3}, \frac{8}{3})$, $(2, 2)$, and $(3, 0)$. The line with equation $x + y = k$ has x-intercept $(k, 0)$ and y-intercept $(0, k)$. The maximum value of the objective funcetion $x+y$ is the largest value of k such that the graph of the equation $x+y = k$ intersects the feasible region. This occurs for $k = 4$. Thus the optimal value of the objective function is 4 at the point $(2, 2)$.

#5 Consider the linear programming problem
Maximize: $x + y$
Subject to:
$-2x + y \leq 2$
$x - 2y \leq 2$
$3x + 5y \geq 15$
$x, y \geq 0.$

Sketch the feasible region. Does this problem have an optimal solution? Why or why not?

SOLUTION: The feasible region is that portion of the plane between the graphs of the equations $-2x+y = 2$ (a line with y-intercept $(0, 2)$ and slope 2) and $x - 2y = 2$ (a line with x-intercept $(2, 0)$ and slope $\frac{1}{2}$) and above the graph of the equation $3x + 5y = 15$ (a line with x-intercept $(5, 0)$ and y-intercept $(0, 3)$). This region is unbounded. Note, for example, that it contains the point (k, k) whenever $k \geq \frac{15}{8}$. Since the value of the objective function at this point is $2k$, we see that there is no optimal solution.

#6 Consider the linear programming problem
Maximize: $x + y$
Subject to:

$2x + y \leq 6$
$x + 2y \leq 6$
$2x + 3y \geq 24$
$x, y \geq 0$. Sketch the feasible region. Does this problem have an optimal solution? Why or why not?

SOLUTION: The feasible region is empty, so there is no optimal solution.

#7 State the definitions of the following terms:
(a) feasible region of a linear programming problem
(b) convex set in $\mathbf{R}^n$
(c) extreme point of a convex set

SOLUTION:
(a) The feasible region is the set of all points satisfying the constraints.
(b) A subset C of $\mathbf{R}^n$ is convex if whenever it contains points $\mathbf{x}$ and $\mathbf{y}$ it contains all points on the line segment joining $\mathbf{x}$ and $\mathbf{y}$.
(c) A point $\mathbf{x}$ in a convex set C is an extereme point if $\mathbf{x}$ is not in the interior of the line segment joining any two points $\mathbf{y}$ and $\mathbf{z}$ in C.

#8 Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbf{R}^n$. Describe the line segment joining $\mathbf{x}$ and $\mathbf{y}$.

SOLUTIONS: The line segment joining $\mathbf{x}$ and $\mathbf{y}$ consists of all $\lambda\mathbf{x}+(1-\lambda)\mathbf{y}$ where $0 \leq \lambda \leq 1$.

#9 Prove that the feasible region of a linear programming problem is convex.

SOLUTION: Suppose the problem is in standard form, so that the constraints are $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Let $\mathbf{y}$ and $\mathbf{z}$ be two feasible solutions. We must show that $\lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ satisfies the constraints whenever $0 \leq \lambda \leq 1$..

Now, since $0 \leq \lambda \leq 1$ we have $\lambda \geq 0$ and $1 - \lambda \geq 0$. Therefore

$$A(\lambda \mathbf{y} + (1-\lambda)\mathbf{z}) =$$

$$A(\lambda \mathbf{y}) + A((1-\lambda)\mathbf{z}) =$$

$$\lambda A\mathbf{y} + ((1-\lambda))\mathbf{z} \leq$$

$$\lambda \mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}$$

and

$$(\lambda \mathbf{y} + (1-\lambda)\mathbf{z}) \geq$$

$$\lambda \mathbf{0} + (1-\lambda)\mathbf{0} = \mathbf{0}.$$

#10 (a) Find an optimal solution (if there is one) to the following linear programming problem using the simplex method.

Maximize: $4x_1 + 3x_2 + 6x_3$
Subject to:
$3x_1 - 4x_2 - 6x_3 \leq 18$
$-2x_1 - x_2 + 2x_3 \leq 12$
$x_1 + 3x_2 + 2x_3 \leq 1$
$x_1, x_2, x_3 \geq 0.$

(b) Find an optimal solution (if there is one) to the following linear programming problem using the simplex method. (Note that only one coefficient has been changed from the problem in (a).)

Maximize: $4x_1 + 3x_2 + 6x_3$
Subject to:
$3x_1 - 4x_2 - 6x_3 \leq 18$
$-2x_1 - x_2 + 2x_3 \leq 12$
$-x_1 + 3x_2 + 2x_3 \leq 1$
$x_1, x_2, x_3 \geq 0.$

SOLUTION:

(a) Introduce slack variables x_4, x_5, and x_6. Then the initial tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	3	−4	−6	1	0	0	18
x_5	−2	−1	2	0	1	0	12
x_6	1	3	2	0	0	1	1
	−4	−3	−6	0	0	0	0

The negative entry of largest value in the objective row is in the third column. Th θ-ratios are $\frac{12}{2} = 6$ and $\frac{1}{2}$. Thus we pivot on the entry in the third row, third column. The new tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	6	5	0	1	0	3	21
x_5	−3	−4	0	0	−1	−1	11
x_3	$\frac{1}{2}$	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	−1	6	0	0	0	3	3

The only negative entry in thye objective row is in the first column. The θ-ratios are $\frac{21}{6}$ and $\frac{\frac{1}{2}}{\frac{1}{2}} = 1$. Thus we pivot on the entry in the third row, first column. The new tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	−13	−12	1	0	−3	15
x_5	0	5	6	0	1	2	14
x_1	1	3	2	0	0	1	1
	0	9	2	0	0	4	4

All entries are optimal, so the resulting solution ($z = 4$ at the point $x_1 = 1, x_2 = 0, x_3 = 0$) is optimal.

(b) The initial tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	3	−4	−6	1	0	0	18
x_5	−2	−1	2	0	1	0	12
x_6	−1	3	2	0	0	1	1
	−4	−3	−6	0	0	0	0

As in part (a) we pivot on the third row, third column, getting

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	5	0	1	0	3	21
x_5	$\frac{-7}{3}$	-4	0	0	-1	-1	11
x_3	$\frac{-1}{2}$	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	-7	6	0	0	0	3	3

Now the entry in the first column of the objective row is negative and every entry above it is ≤ 0. Thus there is no optimal solution (because the objective function may take arbitrarily large values).

#11 In each part, find an optimal solution (if there is one) to the following linear programming problem using the two-phase simplex method (or the big M method).

(a) Maximize: $x_1 + x_2$
Subject to:
$-x_1 - x_2 + x_3 + x_4 = 2$
$-4x_1 - x_2 + 2x_3 + 3x_4 = 5$
$x_1, x_2, x_3, x_4 \geq 0$

(b) Maximize: $x_1 + x_2$
Subject to:
$2x_1 + x_2 + x_4 = 6$
$3x_2 + x_3 + x_5 = 8$
$3x_1 + 6x_2 + 2x_3 + x_4 + x_5 = 20$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

SOLUTION:

(a) Introduce artificial variables y_1 and y_2 so that the constriants become

$$-x_1 - x_2 + x_3 + x_4 + y_1 = 2$$

$$-4x_1 - x_2 + 2x_3 + 3x_4 + y_2 = 5.$$

We want to minimize $y_1 + y_2$ or, equivalently, maximize $-y_1 - y_2$.

Now $-y_1 = -2 - x_1 - x_2 + x_3 + x_4$ and $-y_2 = -5 - 4x_1 - x_2 + 2x_3 + 3x_4$. Hence we want to maximize $-y_1 - y_2 = -7 - 5x_1 - 2x_2 + 3x_3 + 4x_4$. This is the same as maximizing $-5x_1 - 2x_2 + 3x_3 + 4x_4$. Thus the initial tableau for phase one is:

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	-1	-1	1	1	1	0	2
y_2	-4	-1	2	3	0	1	5
	5	2	-3	-4	0	0	

The negative entry of largest absolute value in the objective row is in the fourth column. The θ-ratios are 2 and $\frac{5}{3}$. Thus we pivot on the second row, fourth column. The new tableau is

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	$\frac{1}{3}$	$\frac{-2}{3}$	$\frac{1}{3}$	0	1	$\frac{-1}{3}$	$\frac{1}{3}$
x_4	$\frac{-4}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	$\frac{5}{3}$
	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-1}{3}$	0	0	$\frac{4}{3}$	

.

We may now pivot on the first row, first column. The new tableau is

	x_1	x_2	x_3	x_4	y_1	y_2	
x_1	1	-2	1	0	3	-1	1
x_4	0	-3	2	1	4	1	3
	0	0	0	0	1	1	

This shows that $x_1 = 1, x_2 = x_3 = 0, x_4 = 3$ is a feasible solution to the original problem.

The initial tableau for the second phase is obtained by deleting the columns corresponding to y_1 and y_2 from the previous tableau and replacing the objective row by the row obtained from the objective function for the original problem modified by row operations to make the entries in basic columns 0.

Thus, in this problem, we start with the row

$$-1 \quad -1 \quad 0 \quad 0$$

and add the row

$$1 \quad -2 \quad 1 \quad 0$$

to get

$$0 \quad -3 \quad 1 \quad 0\,.$$

Thus our initial tableau for phase two is

$$\begin{array}{cccccc} & x_1 & x_2 & x_3 & x_4 & \\ x_1 & 1 & -2 & 1 & 0 & 1 \\ x_4 & 0 & -3 & 2 & 1 & 3 \\ & 0 & -3 & 1 & 0 & \end{array}.$$

But now the entry in the second column of the objective row is negative and every entry above it is ≤ 0. This means that there is no optimal value of the objective function, i.e., the objective function can take arbitrarily large values.

(b) Introduce artificial variables y_1, y_2 and y_3 to get constraints
$2x_1 + x_2 + x_4 + y_1 = 6$
$3x_2 + x_3 + x_5 + y_2 = 8$
$3x_1 + 6x_2 + 2x_3 + x_4 + x_5 + y_3 = 20.$

We want to minimize $y_1 + y_2 + y_3$ or, equivalently, to maximize $-y_1 - y_2 - y_3$. Now $-y_1 - y_2 - y_3 = -6 + 2x_1 + x_2 + x_4 - 8 + 3x_2 + x_3 + x_5 - 20 + 3x_1 + 6x_2 + 2x_3 + x_4 + x_5 = -34 + 5x_1 + 10x_2 + 3x_3 + 2x_4 + 2x_5$. Thus the initial tableau for phase one is

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	2	1	0	1	0	1	0	0	6
y_2	0	3	1	0	1	0	1	0	8
y_3	3	6	2	1	1	0	0	1	20
	-5	-10	-3	-2	-2	0	0	0	

The negative entry of largest absolute value in the objective row is -10 in the second column. The smallest θ-ratio in the second column is $\frac{8}{3}$ which occurs in the second row. Thus we pivot on the second row, second column and obtain the new tableau

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	2	0	$\frac{-1}{3}$	1	$\frac{-1}{3}$	1	$\frac{-1}{3}$	0	$\frac{10}{3}$
x_2	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{8}{3}$
y_3	3	0	0	1	-1	0	-2	1	4
	-5	0	$\frac{1}{3}$	-2	$\frac{4}{3}$	0	$\frac{10}{3}$	0	

Now the entry of largest absolute value in the objective row is -5 in the first column. The smallest θ-ratio in the first column is $\frac{4}{3}$ which occurs in the third row. Thus we pivot on the third row, first column and obtain the new tableau

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
y_1	0	0	$\frac{-1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{-2}{3}$	$\frac{2}{3}$
x_2	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{8}{3}$
x_1	1	0	0	$\frac{1}{3}$	$\frac{-1}{3}$	0	$\frac{-2}{3}$	$\frac{1}{3}$	$\frac{4}{3}$
	0	0	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{-1}{3}$	0	0	$\frac{5}{3}$	

Now one of the entries of largest absolute value in the objective row is $\frac{-1}{3}$ in the fourth column. The smallest θ-ratio in the fourth column is 2 which occurs in the first row. Thus we pivot on the first row, fourth column and obtain the new tableau

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
x_4	0	0	-1	1	1	3	3	-2	2
x_2	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{8}{3}$
x_1	1	0	$\frac{1}{3}$	0	$\frac{-2}{3}$	-1	$\frac{-5}{3}$	1	$\frac{2}{3}$
	0	0	0	0	0	1	1	1	

This shows that $x_1 = \frac{2}{3}, x_2 = \frac{8}{3}, x_4 = 2, x_3 = x_5 = 0$ is a feasible solution to the original problem.

The initial tableau for the second phase is obtained by deleting the columns corresponding to y_1, y_2 and y_3 from the previous tableau and replacing the objective row by the row obtained from the objective function for the original problem modified by row operations to make the entries in basic columns 0. Thus, in this problem, we start with the row

$$-1 \quad -1 \quad 0 \quad 0 \quad 0$$

add the row

$$1 \quad 0 \quad \tfrac{1}{3} \quad 0 \quad \tfrac{-2}{3}$$

to get

$$0 \quad -1 \quad \tfrac{1}{3} \quad 0 \quad \tfrac{-2}{3}$$

then add the row

$$0 \quad 1 \quad \tfrac{1}{3} \quad 0 \quad \tfrac{1}{3}$$

to get

$$0 \quad 0 \quad \tfrac{2}{3} \quad 0 \quad \tfrac{-1}{3}\,.$$

Since the entries in the positions corresponding to basic columns are all 0, this will be our objective row. Thus our initial tableau for phase two is

	x_1	x_2	x_3	x_4	x_5	
x_4	0	0	-1	1	1	2
x_2	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{8}{3}$
x_1	1	0	$\frac{1}{3}$	0	$\frac{-2}{3}$	$\frac{2}{3}$
	0	0	$\frac{2}{3}$	0	$\frac{-1}{3}$	

The only negative entry in the objective row is in the fifth column. The smallest θ-ratio in this column is 2 which occurs in the first row. Thus we pivot on the first row, fifth column and get the new tableau

	x_1	x_2	x_3	x_4	x_5	
x_5	0	0	-1	1	1	2
x_2	0	1	$\frac{2}{3}$	$\frac{-1}{3}$	0	2
x_1	1	0	$\frac{-1}{3}$	$\frac{2}{3}$	0	2
	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	

This satisfies the optimality condition, so the optimal solution is $x_1 = x_2 = x_5 = 2, x_3 = x_4 = 0$, giving the value of 4 for the objective function.