

Solutions to practice questions for Final Exam

#1 A lawn products company has 80 tons of nitrate and 50 tons of phosphate to use in producing three types of fertilizer. "Regular lawn" fertilizer requires 4 tons of nitrate and 2 tons of phosphate per 1000 bags. "Super lawn" fertilizer requires 4 tons of nitrate and 3 tons of phosphate per 1000 bags. "Garden" fertilizer requires 2 tons of nitrate and 2 tons of phosphate per 1000 bags. The profit per 1000 bags of fertilizer is \$300 for "regular lawn" fertilizer, \$500 for "super lawn" fertilizer, and \$400 for "garden" fertilizer.

(a) Set up a linear programming model of this situation. State explicitly what each of your variables (for example, x_1, x_2, \dots) represents).

(b) Use the simplex method to find an optimal solution to this problem.

Solution: (a) Let x_1 represent the number of thousands of bags of "regular lawn", x_2 represent the number of thousands of bags of "super lawn", and x_3 represent the number of thousands of bags of "garden". Then the problem may be stated as:

Maximize: $300x_1 + 500x_2 + 400x_3$

Subject to:

$$4x_1 + 4x_2 + 2x_3 \leq 80$$

$$2x_1 + 3x_2 + 2x_3 \leq 50$$

$$x_1, x_2, x_3 \geq 0.$$

(b) After adding slack variables x_4 and x_5 , the initial tableau is

$$\left[\begin{array}{cccccc} & 300 & 500 & 400 & 0 & 0 \\ 4 & 4 & 2 & 1 & 0 & 80 \\ 2 & 3 & 2 & 0 & 1 & 50 \\ -300 & -500 & -400 & 0 & 0 & 0 \end{array} \right].$$

Pivot first on the (2, 2) position to get

$$\left[\begin{array}{cccccc} & 300 & 500 & 400 & 0 & 0 \\ \frac{4}{3} & 0 & \frac{-2}{3} & 1 & \frac{-4}{3} & \frac{40}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{50}{3} \\ \frac{100}{3} & 0 & \frac{-200}{3} & 0 & \frac{500}{3} & \frac{25000}{3} \end{array} \right].$$

Now pivot on the (2, 3) position to get

$$\left[\begin{array}{cccccc} & 300 & 500 & 400 & 0 & 0 \\ 2 & 1 & 0 & 1 & -1 & 30 \\ 1 & \frac{3}{2} & 1 & 0 & \frac{1}{2} & 25 \\ 100 & 200 & 0 & 0 & 200 & 10000 \end{array} \right].$$

As all entries in the objective row are positive, this tableau gives the optimal solution: $x_1 = x_2 = 0, x_3 = 25$.

#2 A manufacturer has distribution centers located in Atlanta (A), Chicago (C), and New York (NY). These centers have available 40, 20, and 40 units of his product, respectively. His retail outlets require the following number of units: Cleveland (CL)- 25; Louisville (L) - 10; Memphis (M)- 20; Pittsburgh (P)- 30; and Richmond (R)- 15. The shipping cost per unit in dollars between each center and outlet is given in the following table:

	<i>C</i>	<i>L</i>	<i>M</i>	<i>P</i>	<i>R</i>
<i>A</i>	55	30	40	50	40
<i>C</i>	35	30	100	45	60
<i>NY</i>	40	60	95	35	30

(a) Set up a linear programming model of this situation. State explicitly what each of your variables (for example, x_1, x_2, \dots) represents.

(b) Use the algorithm for the transportation problem to find an optimal solution.

Solution: (a) Designate the distribution centers by the numbers 1 for Atlanta, 2 for Chciago and 3 for New York. Designate the retail outlets

by the numbers 1 for Cleveland, 2 for Louisville, 3 for Memphis, 4 for Pittsburgh, and 5 for Richmond.

There are fifteen variables denoted $x_{i,j}$ where $1 \leq i \leq 3, 1 \leq j \leq 5$. The variable $x_{i,j}$ denotes the number of units to be shipped from distribution center i to retail outlet j .

minimize: $55x_{1,1} + 30x_{1,2} + 40x_{1,3} + 50x_{1,4} + 40x_{1,5} + 35x_{2,1} + 30x_{2,2} + 100x_{2,3} + 45x_{2,4} + 60x_{2,5} + 40x_{3,1} + 60x_{3,2} + 95x_{3,3} + 35x_{3,4} + 30x_{3,5}$
subject to:

$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} \leq 40$$

$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{2,5} \leq 20$$

$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} + x_{3,5} \leq 40$$

$$x_{1,1} + x_{2,1} + x_{3,1} \geq 25$$

$$x_{1,2} + x_{2,2} + x_{3,2} \geq 10$$

$$x_{1,3} + x_{2,3} + x_{3,3} \geq 20$$

$$x_{1,4} + x_{2,4} + 35x_{3,4} \geq 30$$

$$x_{1,5} + x_{2,5} + x_{3,5} \geq 15$$

$$x_{i,j} \geq 0 \text{ and integral for all } i,j.$$

Note that since the total demand is equal to the total supply (both are 100) we may replace the inequalities in the supply and demand constraints by equalities.

(b) Using the minimum cost rule produces an initial feasible solution:

$$x_{1,2} = 10, x_{1,3} = 20, x_{1,5} = 10;$$

$$x_{2,1} = 20;$$

$$x_{3,1} = 5, x_{3,4} = 30, x_{3,5} = 5.$$

Setting $v_i + w_j = c_{i,j}$ whenever $x_{i,j}$ is a basic variable gives:

$$v_1 + w_2 = 30, v_1 + w_3 = 40, v_1 + w_5 = 40;$$

$$v_2 + w_1 = 35;$$

$$v_3 + w_1 = 40, v_3 + w_4 = 35, v_3 + w_5 = 30.$$

Setting $v_1 = 0$ and solving for the remaining v_i and w_j gives:

$$v_1 = 0, v_2 = -15, v_3 = -10$$

$$w_1 = 50, w_2 = 30, w_3 = 40, w_4 = 45, w_5 = 40.$$

Then, for each non-basic variable $x_{i,j}$ we have $z_{i,j} = v_i + w_j$. Thus we compute

$$z_{1,1} - c_{1,1} = 0 + 50 - 55 = -5$$

$$z_{1,4} - c_{1,4} = 0 + 45 - 50 = -5$$

$$z_{2,2} - c_{2,2} = -15 + 30 - 30 = -15$$

$$z_{2,3} - c_{2,3} = -15 + 40 - 100 = -75$$

$$z_{2,4} - c_{2,4} = -15 + 45 - 45 = -15$$

$$z_{2,5} - c_{2,5} = -15 + 40 - 60 = -25$$

$$z_{3,2} - c_{3,2} = -10 + 30 - 60 = -40$$

$$z_{3,3} - c_{3,3} = -10 + 40 - 95 = -65.$$

Since all the $z_{i,j} - c_{i,j} \leq 0$, the solution is optimal. (If some $z_{i,j} - c_{i,j}$ had been > 0 we would proceed as described beginning on page 308 of the text.

#3 Problem #1 of Section 4.1.

Solution: Let x denote the number of machines of type A purchased and y denote the number of machines of type B purchased. Then the problem may be modeled as:

Minimize: $22000x + 48000y$

Subject to:

$$100x + 200y \geq 600$$

$$50x + 140y \leq 350$$

$$x, y \geq 0 \text{ and integral.}$$

#4 In each part, sketch the feasible region, find all extreme points of the feasible region, and sketch lines with equation $z = k$ for several values of k . Explain, on the basis of your sketch, whether the problem is feasible or not and whether or not it has an optimal solution. If it has an optimal solution, find it.

(a) Maximize $z = x_1 + 3x_2$

subject to

$$x_1 + 2x_2 \leq 3$$

$$2x_1 + x_2 \leq 3$$

$$4x_1 + 3x_2 \geq 12$$

$$x_1, x_2 \geq 0.$$

(b) Maximize $z = x_1 + 3x_2$
 subject to
 $-x_1 + x_2 \leq 1$
 $2x_1 - x_2 \geq 2$
 $x_1, x_2 \geq 0.$

(c) Maximize $z = x_1 + 3x_2$
 subject to
 $-x_1 + x_2 \leq 1$
 $2x_1 - x_2 \leq 2$
 $x_1, x_2 \geq 0.$

Solution: (a) The feasible region is empty, so there are no extreme points and there is no optimal solution.

(b) There are two extreme points, $(1, 0)$ and $(3, 4)$. The lines $-x_1 + x_2 = 1$

$2x_1 - x_2 = 2$ intersect at $(3, 4)$ and the feasible region is the region between these lines above and to the right of $(3, 4)$. The equations $z = k$ have slope $-\frac{1}{3}$ and x_2 -intercept $\frac{k}{3}$. These lines intersect the feasible region for all $k \geq 1$, so there is no optimal solution (as the problem is unbounded).

(c) The extreme points are $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(3, 4)$. The feasible region is the quadrilateral whose vertices are the extreme points. The line $z = k$ are the same as in part (b), and the largest value of k for which this intersects the feasible region is 15. Thus the optimal solution is $z = 15$ at $x_1 = 3, x_2 = 4$.

#5 Use the two-phase simplex method to solve each of the following linear programming problems.

(a) Maximize $z = x_1 + x_2$
 subject to
 $2x_1 - 3x_2 \geq 6$
 $-x_1 + 2x_2 \geq 4$
 $x_1, x_2 \geq 0.$

(b) Maximize $z = -x_1 - x_2$

subject to

$$2x_1 - 3x_2 \geq 6$$

$$-x_1 + 2x_2 \geq 4$$

$$x_1, x_2 \geq 0.$$

(c) Maximize $z = 2x_1 + x_2 + x_3$

subject to

$$x_1 + x_2 + x_3 = 3$$

$$x_1 + 2x_2 - 2x_3 = 5$$

$$x_1, x_2, x_3 \geq 0.$$

Solution: (a) After introducing slack variables x_3, x_4 and artificial variables y_1, y_2 we see that the problem for Phase I is

Minimize $y_1 + y_2$

subject to

$$2x_1 - 3x_2 - x_3 + y_1 = 6$$

$$-x_1 + 2x_2 - x_4 + y_2 = 4$$

$$x_1, x_2, x_3, x_4, y_1, y_2 \geq 0.$$

Now minimizing $y_1 + y_2$ is the same as maximizing $-y_1 - y_2$ and $-y_1 - y_2 = (2x_1 - 3x_2 - x_3 - 6) + (-x_1 + 2x_2 - x_4 - 4) = x_1 - x_2 - x_3 - x_4 - 10$. Thus the initial tableau for Phase I is:

$$\begin{bmatrix} 1 & -1 & -1 & -1 & & & \\ 2 & -3 & -1 & 0 & 1 & 0 & 6 \\ -1 & 2 & 0 & -1 & 0 & 1 & 4 \\ -1 & 1 & 1 & 1 & 0 & 0 & -10 \end{bmatrix}.$$

Applying the simplex method, we pivot on the (1, 1) position and obtain the new tableau

$$\begin{bmatrix} 1 & -1 & -1 & -1 & & & \\ 1 & \frac{-3}{2} & \frac{-1}{2} & 0 & \frac{1}{2} & 0 & 3 \\ 0 & \frac{1}{2} & \frac{-1}{2} & -1 & \frac{1}{2} & 1 & 7 \\ 0 & \frac{-1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & -7 \end{bmatrix}.$$

Applying the simplex method again, we pivot on the (2,2) position and obtain

$$\begin{bmatrix} 1 & -1 & -1 & -1 & & & \\ 1 & 0 & -2 & -3 & 2 & 3 & 24 \\ 0 & 1 & -1 & -2 & 1 & 2 & 14 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The basic variables are now x_1 and x_2 so we have eliminated the artificial variables from our solution, completing Phase I. The initial tableau for Phase II is obtained by deleting the columns corresponding to the artificial variables from the final tableau for Phase I and entering the values for the objective function given in the statement of the problem and computing the corresponding entries ($z_j - c_j$) in the objective row. Thus it is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & & & \\ 1 & 0 & -2 & -3 & 24 & & \\ 0 & 1 & -1 & -2 & 14 & & \\ 0 & 0 & -3 & -5 & 0 & & \end{bmatrix}.$$

Note that the objective row contains the negative entry -5 and that all entries above this are negative. Thus there is no optimal solution (the problem is unbounded).

(b) Note that the constraints in (b) are the same as in (a). Thus Phase I of the simplex method for (b) will be the same as Phase I for (a) and so (using the new values for the objective function) the initial tableau for Phase II will be

$$\begin{bmatrix} -1 & -1 & 0 & 0 & & & \\ 1 & 0 & -2 & -3 & 24 & & \\ 0 & 1 & -1 & -2 & 14 & & \\ 0 & 0 & 3 & 5 & 0 & & \end{bmatrix}.$$

Since all the values in the objective row are ≥ 0 , the current solution ($x_1 = 24, x_2 = 14$) is optimal.

(c) After adding artificial variables y_1 and y_2 we obtain the following problem for Phase I:

Minimize $y_1 + y_2$
 subject to
 $x_1 + x_2 + x_3 + y_1 = 3$
 $x_1 + 2x_2 - 2x_3 + y_2 = 5$
 $x_1, x_2, x_3, y_1, y_2 \geq 0.$

Since minimizing $y_1 + y_2$ is the same as maximizing $-y_1 - y_2$ and $-y_1 - y_2 = (x_1x_2 + x_3 - 3) + (x_1 + 2x_2 - 2x_3 - 5) = 2x_1 + 3x_2 - x_3 - 8$ we see that the initial tableau for Phase I is

$$\begin{bmatrix} 2 & 3 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & -2 & 0 & 1 & 5 \\ -2 & -3 & 1 & 0 & 0 & -8 \end{bmatrix}.$$

Applying the simplex method, we pivot on the (2, 2) position and obtain

$$\begin{bmatrix} 2 & 3 & -1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 2 & 1 & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & -1 & 0 & \frac{1}{2} & \frac{5}{2} \\ \frac{-1}{2} & 0 & -2 & 0 & \frac{3}{2} & \frac{-1}{2} \end{bmatrix}.$$

Applying the simplex method again, we pivot on the (1, 3) position and obtain

$$\begin{bmatrix} 2 & 3 & -1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 1 & \frac{1}{2} & \frac{-1}{4} & \frac{1}{4} \\ \frac{3}{4} & 1 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The basic variables are now x_2 and x_3 so we have eliminated the artificial variables from our solution, completing Phase I. The initial tableau for Phase II is obtained by deleting the columns corresponding to the artificial variables from the final tableau for Phase I and entering the values for the objective function given in the statement of the problem and computing the corresponding entries ($z_j - c_j$) in the objective row. Thus it is

$$\begin{bmatrix} 2 & 1 & 1 & \\ \frac{1}{4} & 0 & 1 & \frac{1}{4} \\ \frac{3}{4} & 1 & 0 & \frac{11}{4} \\ -1 & 0 & 0 & 3 \end{bmatrix}.$$

Applying the simplex method, we pivot on the (1, 1) position and obtain

$$\begin{bmatrix} 2 & 1 & 1 & \\ 1 & 0 & 4 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 4 & 4 \end{bmatrix}.$$

All the entries in the objective row are positive, so this tableau gives an optimal solution: $x_1 = 1, x_2 = 2, x_3 = 0$.

#6 In the matrix below, the (i, j) entry represents the capacity of the (directed) arc from node i to node j in a network. Use the labeling algorithm to find the maximal flow from the source (node 1) to the sink (node 10). Also find a minimal cut.

$$\begin{bmatrix} 0 & 7 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: We will use the labeling algorithm. I suggest you draw diagrams (which I do not know how to do for a web-site posting).

We start with the trivial initial feasible solution: $x_{i,j} = 0$ for all i, j .

Then the excess capacities are:

$$d_{1,2} = 7, d_{2,1} = 0 \quad d_{1,3} = 6, d_{3,1} = 0 \quad d_{1,4} = 7, d_{4,1} = 0 \quad d_{2,5} = 5, d_{5,2} = 0 \\ d_{2,6} = 5, d_{6,5} = 0 \quad d_{3,7} = 4, d_{7,3} = 0 \quad d_{4,5} = 4, d_{5,4} = 0 \quad d_{4,9} = 3, d_{9,4} = 0 \\ d_{5,8} = 6, d_{8,5} = 0 \quad d_{6,8} = 7, d_{8,6} = 0 \quad d_{7,9} = 6, d_{9,7} = 0 \quad d_{7,10} = 8, d_{10,7} = 0 \\ d_{8,10} = 6, d_{10,8} = 0 \quad d_{9,10} = 12, d_{10,9} = 0$$

We now start labeling nodes. We may label a node when it is not already labeled and it is connected to a previously labeled node (say node a) by an arc with positive excess capacity. (There are frequently several nodes we could label; we pick one arbitrarily.) We label the node with a pair (u, a) where u the the extra amount we could send to the node through node a (that is, the minimum of the amount node a is labeled with and the excess capacity of the arc from node a). In this problem, we may label node 2 with the pair $(7, 1)$ (indicating that we can send 7 more units to node 2 from node 1. This allow us to label node 6 with $(5, 2)$, then to label node 8 with $(5, 6)$ and to label node 10 (the sink) with $(5, 8)$. Since the sink is now labeled, we can increase the flow by following the indicated path (sending 5 units from node 1 to node 2 to node 6 to node 8 to node 10. We now update our list of excess capacities, getting

$$d_{1,2} = 2, d_{2,1} = 5 \quad d_{1,3} = 6, d_{3,1} = 0 \quad d_{1,4} = 7, d_{4,1} = 0 \quad d_{2,5} = 5, d_{5,2} = 0 \\ d_{2,6} = 0, d_{6,5} = 5 \quad d_{3,7} = 4, d_{7,3} = 0 \quad d_{4,5} = 4, d_{5,4} = 0 \quad d_{4,9} = 3, d_{9,4} = 0 \\ d_{5,8} = 6, d_{8,5} = 0 \quad d_{6,8} = 2, d_{8,6} = 5 \quad d_{7,9} = 6, d_{9,7} = 0 \quad d_{7,10} = 8, d_{10,7} = 0 \\ d_{8,10} = 1, d_{10,8} = 5 \quad d_{9,10} = 12, d_{10,9} = 0$$

Now we may label node 4 with $(7, 1)$, node 9 with $(3, 4)$ and node 10 with $(3, 9)$. Thus we can increase the flow by sending 3 units from node 1 to node 4 to node 9 to node 10. We again update our list of excess capacities, getting

$$d_{1,2} = 2, d_{2,1} = 5 \quad d_{1,3} = 6, d_{3,1} = 0 \quad d_{1,4} = 4, d_{4,1} = 3 \quad d_{2,5} = 5, d_{5,2} = 0 \\ d_{2,6} = 0, d_{6,5} = 5 \quad d_{3,7} = 4, d_{7,3} = 0 \quad d_{4,5} = 4, d_{5,4} = 0 \quad d_{4,9} = 0, d_{9,4} = 3 \\ d_{5,8} = 6, d_{8,5} = 0 \quad d_{6,8} = 2, d_{8,6} = 5 \quad d_{7,9} = 6, d_{9,7} = 0 \quad d_{7,10} = 8, d_{10,7} = 0 \\ d_{8,10} = 1, d_{10,8} = 5 \quad d_{9,10} = 9, d_{10,9} = 3$$

We may label node 3 with $(6, 1)$, node 7 with $(4, 3)$ and node 10 with $(4, 7)$. Thus we can increase the flow by sending 4 units from node 1 to node 3 to node 7 to node 10. We again update our list of excess capacities, getting

$$d_{1,2} = 2, d_{2,1} = 5 \quad d_{1,3} = 2, d_{3,1} = 4 \quad d_{1,4} = 4, d_{4,1} = 3 \quad d_{2,5} = 5, d_{5,2} = 0 \\ d_{2,6} = 0, d_{6,5} = 5 \quad d_{3,7} = 0, d_{7,3} = 4 \quad d_{4,5} = 4, d_{5,4} = 0 \quad d_{4,9} = 0, d_{9,4} = 3 \\ d_{5,8} = 6, d_{8,5} = 0 \quad d_{6,8} = 2, d_{8,6} = 5 \quad d_{7,9} = 6, d_{9,7} = 0 \quad d_{7,10} = 4, d_{10,7} = 4 \\ d_{8,10} = 1, d_{10,8} = 5 \quad d_{9,10} = 9, d_{10,9} = 3$$

We may label node 2 with (2, 1), node 5 with (2, 2), node 8 with (2, 6) and node 10 with (1, 8). Thus we can increase the flow by sending 1 unit from node 1 to node 2 to node 6 to node 8 to node 10. We again update our list of excess capacities, getting

$$d_{1,2} = 1, d_{2,1} = 6 \quad d_{1,3} = 2, d_{3,1} = 4 \quad d_{1,4} = 4, d_{4,1} = 3 \quad d_{2,5} = 4, d_{5,2} = 1 \\ d_{2,6} = 0, d_{6,5} = 5 \quad d_{3,7} = 0, d_{7,3} = 4 \quad d_{4,5} = 4, d_{5,4} = 0 \quad d_{4,9} = 0, d_{9,4} = 3 \\ d_{5,8} = 5, d_{8,5} = 1 \quad d_{6,8} = 2, d_{8,6} = 5 \quad d_{7,9} = 6, d_{9,7} = 0 \quad d_{7,10} = 4, d_{10,7} = 4 \\ d_{8,10} = 0, d_{10,8} = 6 \quad d_{9,10} = 9, d_{10,9} = 3$$

We can now label node 2 with (1, 1), node 3 with (2, 1), node 4 with (4, 1), node 5 with (4, 4), node 6 with (1, 2) and node 8 with (1, 6). We cannot label nodes 7, 9, 10. We now list all the arcs connecting a labeled node to an unlabeled node. These are the arcs from 3 to 7, from 4 to 9, and from 8 to 10. These arcs form a cut of value $4 + 3 + 6 = 13$. This is also the value of our flow, so the flow

$$x_{1,2} = 6 \quad x_{1,3} = 4 \quad x_{1,4} = 3 \quad x_{2,5} = 1 \quad x_{2,6} = 5 \quad x_{3,7} = 4 \quad x_{4,9} = 3 \quad x_{5,8} = 1 \\ x_{6,8} = 5 \quad x_{7,10} = 4 \quad x_{8,10} = 6 \quad x_{9,10} = 3$$

is optimal.

#7 Convert the following linear programming problem to standard form. Then write the dual problem (using unrestricted variables and equalities where appropriate).

Minimize: $x_1 - 3x_2 + 4x_3$

Subject to

$$x_1 - 2x_2 + x_3 = 7$$

$$2x_1 + 5x_2 + 3x_3 \geq 5$$

$$x_1, x_2 \geq 0, x_3 \text{ unrestricted.}$$

Solution: In standard form (writing $x_3 = x_3^+ - x_3^-$) the problem is

Maximize: $-x_1 + 3x_2 - 4x_3^+ + 4x_3^-$

Subject to

$$x_1 - 2x_2 + x_3^+ - x_3^- \leq 7$$

$$\begin{aligned}
-x_1 + 2x_2 - x_3^+ + x_3^- &\leq -7 \\
-2x_1 - 5x_2 - 3x_3^+ + 3x_3^- &\leq -5 \\
x_1, x_2, x_3^+, x_3^- &\geq 0.
\end{aligned}$$

Then the dual problem is

$$\text{Minimize: } 7w_1 - 7w_2 - 5w_3$$

Subject to:

$$\begin{aligned}
w_1 - w_2 - 2w_3 &\geq -1 \\
-2w_1 + 2w_2 - 5w_3 &\geq 3 \\
w_1 - w_2 - 3w_3 &\geq -4 \\
-w_1 + w_2 + 3w_3 &\geq 4 \\
w_1, w_2, w_3 &\geq 0.
\end{aligned}$$

By setting $w_4 = w_1 - w_2$ and combining the last two constraints into a single inequality we may write the dual problem as

$$\text{Minimize: } 7w_4 - 5w_3$$

Subject to:

$$\begin{aligned}
w_4 - 2w_3 &\geq -1 \\
-2w_4 - 5w_3 &\geq 3 \\
w_4 - 3w_3 &= -4 \\
w_3 &\geq 0, w_4 \text{ unrestricted.}
\end{aligned}$$

#8 Consider the linear programming problem:

$$\text{Maximize: } 4x_1 + 3x_2 + 6x_3$$

Subject to:

$$\begin{aligned}
3x_1 - 4x_2 - 6x_3 &\leq 18 \\
-2x_1 - x_2 + 2x_3 &\leq 12 \\
x_1 + 3x_2 + 2x_3 &\leq 1 \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}$$

Use the revised simplex method to solve this problem, giving the values of B^{-1} and \mathbf{x}_B at each step and computing the $z_j - c_j$ from this information at each step.

Solution: After adding slack variables, the initial tableau is

$$\begin{bmatrix} 4 & 3 & 6 & 0 & 0 & 0 \\ 3 & -4 & -6 & 1 & 0 & 0 & 18 \\ -2 & -1 & 2 & 0 & 1 & 0 & 12 \\ 1 & 3 & 2 & 0 & 0 & 1 & 1 \\ -4 & -3 & -6 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Initially, } B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{x}_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ 1 \end{bmatrix}.$$

We pivot on the (3,2) position. Thus we perform row operations to change $\begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then we get the new values $B^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, $\mathbf{x}_B =$

$$\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 21 \\ 11 \\ \frac{1}{2} \end{bmatrix}, \mathbf{c}_B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}. \text{ Then } \mathbf{c}_B B^{-1} = [0 \ 0 \ 3], \text{ so } [z_1 \ \dots \ z_6] = [3 \ 9 \ 6 \ 0 \ 0 \ 3]. \text{ Thus } [z_1 - c_1 \ \dots \ z_6 - c_6] = [-1 \ 6 \ 0 \ 0 \ 0 \ 3].$$

This means that we must pivot on the first column. Thus we need to compute $\mathbf{t}_1 = B^{-1}A_1 = \begin{bmatrix} 6 \\ -3 \\ \frac{1}{2} \end{bmatrix}$. Then the θ -ratio for the first entry is $\frac{21}{6}$ and for the third entry is 1. Thus we pivot on the (3,1) position. This means

that we need to change $\begin{bmatrix} 6 \\ -3 \\ \frac{1}{2} \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\text{Then we get the new values } B^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{x}_B = \begin{bmatrix} x_4 \\ x_5 \\ x_1 \end{bmatrix} =$$

$$\begin{bmatrix} 15 \\ 14 \\ 1 \end{bmatrix}, \mathbf{c}_B = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \mathbf{c}_B^T B^{-1} = [0 \ 0 \ 4], \text{ so } [z_1 \ \dots \ z_6] = [4 \ 12 \ 8 \ 0 \ 0 \ 4].$$

Thus $[z_1 - c_1 \ \dots \ z_6 - c_6] = [0 \ 9 \ 2 \ 0 \ 0 \ 34]$. Since all the $z_j - c_j$ are ≥ 0 , the current solution is optimal.

#9 Solve problem #7 of Section 4.2 using

- (a) the cutting plane method;
- (b) the branch and bound method.

Solution: For either (a) or (b) we begin by solving the problem without the integrality constraints. After adding slack variables, the initial tableau is

$$\begin{bmatrix} 5 & 2 & 0 & 0 & \\ 12 & -7 & 1 & 0 & 84 \\ 6 & 10 & 0 & 1 & 69 \\ -5 & -2 & 0 & 0 & 0 \end{bmatrix}.$$

We first pivot on the (1,1) position to get

$$\begin{bmatrix} 5 & 2 & 0 & 0 & \\ 1 & \frac{-7}{12} & \frac{1}{12} & 0 & 7 \\ 0 & \frac{27}{2} & \frac{-1}{2} & 1 & 27 \\ 0 & \frac{-59}{12} & \frac{5}{12} & 0 & 35 \end{bmatrix}.$$

We then pivot on the (2,2) position to get

$$\begin{bmatrix} 5 & 2 & 0 & 0 & \\ 1 & 0 & \frac{10}{162} & \frac{7}{162} & \frac{49}{6} \\ 0 & 1 & \frac{-6}{162} & \frac{12}{162} & 2 \\ 0 & 0 & \frac{38}{162} & \frac{59}{162} & \frac{269}{6} \end{bmatrix}.$$

Thus the optimal solution without the integrality constraints is

$$x_1 = \frac{49}{6}, x_2 = 2, z = \frac{269}{6}.$$

- (a) We now impose the cutting plane constraint

$$\frac{10}{162}x_3 - \frac{7}{162}x_4 + u_1 = -\frac{1}{6}.$$

This gives the tableau

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & \\ 1 & 0 & \frac{10}{162} & \frac{7}{162} & 0 & \frac{49}{6} \\ 0 & 1 & \frac{-6}{162} & \frac{12}{162} & 0 & 2 \\ 0 & 0 & \frac{-10}{162} & \frac{-7}{162} & 1 & \frac{-1}{6} \\ 0 & 0 & \frac{38}{162} & \frac{59}{162} & 0 & \frac{269}{6} \end{bmatrix}.$$

Using the dual simplex method, we pivot on the (3, 3) position to get

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 8 \\ 0 & 1 & 0 & \frac{1}{10} & \frac{-6}{10} & \frac{21}{10} \\ 0 & 0 & 1 & \frac{1}{10} & \frac{-162}{10} & \frac{27}{10} \\ 0 & 0 & 0 & \frac{2}{10} & \frac{38}{10} & \frac{442}{10} \end{bmatrix}.$$

Next we impose the cutting plane constraint

$$-\frac{1}{10}x_4 - \frac{4}{10}u_1 + u_2 = 1\frac{1}{10}.$$

This gives the tableau

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 1 & 0 & \frac{1}{10} & \frac{-6}{10} & 0 & \frac{21}{10} \\ 0 & 0 & 1 & \frac{1}{10} & \frac{-162}{10} & 0 & \frac{27}{10} \\ 0 & 0 & 0 & \frac{-1}{10} & \frac{-4}{10} & 1 & \frac{-1}{10} \\ 0 & 0 & 0 & \frac{2}{10} & \frac{38}{10} & 0 & \frac{442}{10} \end{bmatrix}.$$

Using the dual simplex method, we pivot on the (4, 4) position to get

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & -19 & 7 & 2 \\ 0 & 0 & 0 & 1 & 4 & -10 & 1 \\ 0 & 0 & 0 & 0 & 3 & 2 & 44 \end{bmatrix}.$$

Hence the optimal solution to the integer programming problem is $x = 8, y = 2, z = 44$.

(b) To apply the brach and bound method, we start with the final tableau for the problem without the integrality constraints.

$$\begin{bmatrix} 5 & 2 & 0 & 0 & & \\ 1 & 0 & \frac{10}{162} & \frac{7}{162} & \frac{49}{6} & \\ 0 & 1 & \frac{-6}{162} & \frac{12}{162} & 2 & \\ 0 & 0 & \frac{38}{162} & \frac{59}{162} & \frac{269}{6} & \end{bmatrix}.$$

As noted before the optimal solution to this problem is

$$x_1 = \frac{49}{6}, x_2 = 2, z = \frac{269}{6}.$$

We will now add either the constraint $x_1 \geq 9$ or the constraint $x_1 \leq 8$. We will treat the case where we add the constraint $x_1 \geq 9$ first. We may write this as $x_1 - u_1 \geq 9$ or as $-x_1 + u_1 = -9$. This gives us the tableau

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & \\ 1 & 0 & \frac{10}{162} & \frac{7}{162} & 0 & \frac{49}{6} \\ 0 & 1 & \frac{-6}{162} & \frac{12}{162} & 0 & 2 \\ 0 & 0 & \frac{10}{162} & \frac{7}{162} & 1 & \frac{-5}{6} \\ 0 & 0 & \frac{38}{162} & \frac{59}{162} & 0 & \frac{269}{6} \end{bmatrix}.$$

Since the last entry in the third row is negative and all the other entries are ≥ 0 , this problem is infeasible (by the dual simplex method).

We now treat the case where we add the constraint $x_1 \leq 8$. We may write this as $x_1 + u_1 = 8$. This gives us the tableau

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & \\ 1 & 0 & \frac{10}{162} & \frac{7}{162} & 0 & \frac{49}{6} \\ 0 & 1 & \frac{-6}{162} & \frac{12}{162} & 0 & 2 \\ 0 & 0 & \frac{-10}{162} & \frac{-7}{162} & 1 & \frac{-1}{6} \\ 0 & 0 & \frac{38}{162} & \frac{59}{162} & 0 & \frac{269}{6} \end{bmatrix}.$$

Using the dual simplex method we pivot on the (3, 3) position and obtain

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 8 \\ 0 & 1 & 0 & \frac{1}{10} & \frac{-6}{10} & \frac{21}{10} \\ 0 & 0 & 1 & \frac{1}{10} & \frac{-162}{10} & \frac{27}{10} \\ 0 & 0 & 0 & \frac{2}{10} & \frac{38}{10} & \frac{442}{10} \end{bmatrix}.$$

This gives the (non-integral) solution $x = 8, y = \frac{21}{10}, z = \frac{442}{10}$. Therefore we will impose either the constraint $y \leq 2$ or the constraint $y \geq 3$.

We will treat the case where we add the constraint $y \leq 2$ first. We may write this as $y + u_2 = 2$. This gives the tableau

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 1 & 0 & \frac{1}{10} & \frac{-6}{10} & 0 & \frac{21}{10} \\ 0 & 0 & 1 & \frac{1}{10} & \frac{-162}{10} & 0 & \frac{27}{10} \\ 0 & 0 & 0 & \frac{-1}{10} & \frac{6}{10} & 1 & \frac{-1}{10} \\ 0 & 0 & 0 & \frac{2}{10} & \frac{38}{10} & 0 & \frac{442}{10} \end{bmatrix}.$$

Using the dual simplex method we pivot on the (4, 4) position and obtain

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & -19 & 7 & 2 \\ 0 & 0 & 0 & 1 & 4 & -10 & 1 \\ 0 & 0 & 0 & 0 & 3 & 2 & 44 \end{bmatrix}.$$

This gives the integral solution $x = 8, y = 2, z = 44$.

We still need to consider the case where we add the constraint $y \geq 3$. We may write this as $y - u_2 = 3$ or as $-y + u_2 = -9$. This gives the tableau

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 1 & 0 & \frac{1}{10} & \frac{-6}{10} & 0 & \frac{21}{10} \\ 0 & 0 & 1 & \frac{1}{10} & \frac{-162}{10} & 0 & \frac{27}{10} \\ 0 & 0 & 0 & \frac{1}{10} & \frac{-6}{10} & 1 & \frac{-9}{10} \\ 0 & 0 & 0 & \frac{2}{10} & \frac{38}{10} & 0 & \frac{442}{10} \end{bmatrix}.$$

Using the dual simplex method we pivot on the (4, 5) entry we obtain

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{6} & 0 & \frac{10}{6} & \frac{13}{2} \\ 0 & 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & -2 & 0 & -27 & 27 \\ 0 & 0 & 0 & \frac{-1}{6} & 1 & \frac{-10}{6} & \frac{3}{2} \\ 0 & 0 & 0 & \frac{5}{6} & 0 & \frac{38}{6} & \frac{77}{3} \end{bmatrix}.$$

This gives a (non-integral) solution $x = \frac{13}{2}, y = 3, z = \frac{77}{2}$. Since the value of the objective function is less than the value of the objective function for a previously obtained integral solution we do not need to consider this case further. Therefore the optimal solution to the integer programming problem is $x = 8, y = 2, z = 44$.

#10 Solve the assignment problem with matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 7 & 2 & 4 & 6 \\ 1 & 4 & 3 & 2 & 9 & 4 \\ 2 & 1 & 1 & 3 & 5 & 7 \\ 8 & 6 & 2 & 4 & 9 & 3 \\ 5 & 5 & 7 & 8 & 4 & 2 \end{bmatrix}.$$

Solution: We create at least one zero in every row and in every column by subtracting 1 from every entry in the first row, 2 from every entry in the second row, 1 from every entry in the thire row, 1 from every entry in the

fourth row, 2 from every entry in the fifth row, 2 from every entry in the sixth row and 2 from every entry in the fifth column. This gives the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 2 & 5 \\ 1 & 3 & 5 & 0 & 0 & 4 \\ 0 & 3 & 2 & 1 & 6 & 3 \\ 1 & 0 & 0 & 2 & 2 & 6 \\ 6 & 4 & 0 & 2 & 5 & 1 \\ 3 & 3 & 5 & 6 & 0 & 0 \end{bmatrix}.$$

Marking zeros in the usual fashion gives the matrix

$$\begin{bmatrix} 0* & 1 & 2 & 3 & 2 & 5 \\ 1 & 3 & 5 & 0* & 0 & 4 \\ 0 & 3 & 2 & 1 & 6 & 3 \\ 1 & 0* & 0 & 2 & 2 & 6 \\ 6 & 4 & 0* & 2 & 5 & 1 \\ 3 & 3 & 5 & 6 & 0* & 0 \end{bmatrix}.$$

Note that there is no marked zero in the third row. Then the path starting at the (3,1) position and going to the (1,1) position is maximal and ends at a marked zero. Therefore the first column and the second, fourth, fifth and sixth rows are necessary. Subtracting 1 from every entry that is in neither a necessary row nor a necessary column and adding 1 to every entry that is in both a necessary row and a necessary column gives the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0 & 0 & 4 \\ 0 & 2 & 1 & 0 & 5 & 2 \\ 2 & 0 & 0 & 2 & 2 & 6 \\ 7 & 4 & 0 & 2 & 5 & 1 \\ 4 & 3 & 5 & 6 & 0 & 0 \end{bmatrix}.$$

Marking zeros gives

$$\begin{bmatrix} 0* & 0 & 1 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0* & 0 & 4 \\ 0 & 2 & 1 & 0 & 5 & 2 \\ 2 & 0* & 0 & 2 & 2 & 6 \\ 7 & 4 & 0* & 2 & 5 & 1 \\ 4 & 3 & 5 & 6 & 0* & 0 \end{bmatrix}.$$

There is still no marked zero in the third row. Starting with the zero in the (3,1) position we construct the path going from (3,1) to (1,1) to (1,2), to (4,2) to (4,3) to (5,3). This ends at a marked zero. If the third column is deleted, the path cannot be extended past (4,2) and if the second column is deleted the path cannot be extended past (1,1). Thus the first three columns will be necessary. However, there is another zero in the third row (in the (3,4) position). Starting at this position we can construct a path from (3,4) to (2,4) to (2,5) to (6,5) to (6,6). This path ends at an unmarked zero, so remove the marks from all marked zeros on this path and mark all the unmarked zeros on the path. This gives the matrix

$$\begin{bmatrix} 0* & 0 & 1 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0 & 0* & 4 \\ 0 & 2 & 1 & 0* & 5 & 2 \\ 2 & 0* & 0 & 2 & 2 & 6 \\ 7 & 4 & 0* & 2 & 5 & 1 \\ 4 & 3 & 5 & 6 & 0 & 0* \end{bmatrix}.$$

There is now one mark in each row (and in each column). The corresponding optimal solution is $x_{1,1} = x_{2,5} = x_{3,4} = x_{4,2} = x_{5,3} = x_{6,6} = 1$, all other $x_{i,j} = 0$. The value of the objective function is 12.

#11 State and prove the Weak Duality Theorem.

Solution: The Weak Duality Theorem states that if \mathbf{x}_0 is a feasible solution of the primal problem

Maximize: $\mathbf{c}^T \mathbf{x}$

Subject to:

$$A\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

and \mathbf{w}_0 is a feasible solution of the dual problem

Minimize: $\mathbf{b}^T \mathbf{w}$

Subject to:

$$A^T \mathbf{w} \geq \mathbf{c}$$

$$\mathbf{w} \geq \mathbf{0}$$

then $\mathbf{c}^T \mathbf{x}_0 \leq \mathbf{b}^T \mathbf{w}_0$.

The proof consists of showing that

$$\mathbf{c}^T \mathbf{x}_0 \leq \mathbf{w}_0^T A \mathbf{x}_0 \leq \mathbf{b}^T \mathbf{w}_0.$$

The first inequality holds since $A^T \mathbf{w}_0 \geq \mathbf{c}$ and so, since $\mathbf{x}_0 \geq \mathbf{0}$, we have $\mathbf{x}_0^T A^T \mathbf{w}_0 \geq \mathbf{x}_0^T \mathbf{c}$. The second inequality holds since $A \mathbf{x}_0 \leq \mathbf{b}$ and so, since $\mathbf{w}_0 \geq \mathbf{0}$, we have $\mathbf{w}_0^T A \mathbf{x}_0 \leq \mathbf{w}_0^T \mathbf{b} = \mathbf{b}^T \mathbf{w}_0$.

#12 Consider the linear programming problem in standard form:

Maximize $\mathbf{c}^T \mathbf{x}$

subject to

$$A \mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}.$$

State and prove the principle of complementary slackness relating the optimal solutions of this problem and the dual problem.

Solution: Suppose that A is m by n and that m slack variables are added so that the first constraint becomes

$$[A \quad I] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}' \end{bmatrix} = \mathbf{b}.$$

Let $\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}'_0 \end{bmatrix}$ be an optimal solution to the primal problem, where $\mathbf{x}' =$

$\begin{bmatrix} x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ x_{n+m} \end{bmatrix}$. Let $\mathbf{w}_0 = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_m \end{bmatrix}$ be an optimal solution the dual problem.

The Principle of Complimentary Slackness states that $x_{n+i}w_i = 0$ for all $i, 1 \leq i \leq m$. (In words, if the i -th slack variable is non-zero then the i -th dual variable is equal to zero.) See page 179 of the text for a proof.

#13 Explain how the v_i and w_j used in the solution of the transportation problem are related to the dual problem and why they may be used to compute the $z_j - c_j$.

Solution: Let A be the (m by n) coefficient matrix for the transportation problem when it is written out as a linear programming problem in standard form. Convert this problem to canonical form by adding $m + n$ slack variables, and suppose that the tableau corresponding to some basic feasible solution \mathbf{x} is obtained from the original tableau by multiplying by a matrix B^{-1} . Then for each $i, 1 \leq i \leq m$ and $j, 1 \leq j \leq n$ there is a column (which I will denote by $A(i, j)$) with a 1 in the row corresponding to the i -th demand constraint, a 1 in the row corresponding to the j -th demand constraint and 0 in all other positions. The corresponding entry in the objective row (which I will denote by $z(i, j)$) will be $\mathbf{c}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_B^T B^{-1} A(i, j)$. Let the v_i be the dual variables corresponding to the supply constraints (that is, corresponding to the rows in the matrix defining the problem) and the w_j be the dual variables corresponding to the demand constraints (that is, corresponding to the columns in the matrix defining the problem). Then

$$[v_1 \quad . \quad . \quad . \quad v_m \quad w_1 \quad . \quad . \quad . \quad w_m] = \mathbf{c}_B^T B^{-1}$$

$$\begin{aligned} \text{and so } v_i + w_j &= [v_1 \quad . \quad . \quad . \quad v_m \quad w_1 \quad . \quad . \quad . \quad w_m] A(i, j) \\ &= \mathbf{c}_B^T B^{-1} A(i, j) = z(i, j). \end{aligned}$$

\$14 Explain what changes can be made in the matrix for an assignment problem without changing the optimal solution and why these changes can be made.

Solution: The matrix can be changed by adding some constant to every entry in a row (or to every entry in a column). Here is the argument that

shows this for rows: Let $A = [a_{i,j}]$ be the matrix of the problem. Suppose a new matrix $A' = [a'_{i,j}]$ is obtained from A by adding some constant a to every entry in the k -th row of A . If $\mathbf{x} = [x_{i,j}]$ is a feasible solution of the assignment problem, let $z_A(\mathbf{x})$ denote the value of the objective function for the problem with matrix A (which is $\sum_{i,j} a_{i,j}x_{i,j}$) and $z_{A'}(\mathbf{x})$ denote the value of the objective function for the problem with matrix A' (which is $\sum_{i,j} a'_{i,j}x_{i,j}$). Then, since $x_{k,j} = 1$ for exactly one value of j and is 0 for all other values of j , we see that $z_{A'}(\mathbf{x}) = z_A(\mathbf{x}) + a$. Thus \mathbf{x} is optimal for the problem with matrix A if and only if it is optimal for the problem with matrix A' .

Then for any feasible solution, the value of the objective function will

#15 Consider the linear programming problem

Maximize: $x_1 + x_2 + 2x_3 + x_4$

subject to

$$-11x_1 - 3x_3 + 2x_4 \leq 1$$

$$7x_1 + x_2 + 2x_3 - x_4 \leq 1$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

(a) Find the optimal solution.

(b) Find all values of Δc_3 such that the optimal solution \mathbf{x}_B remains unchanged if the coefficient of x_3 in the objective function is replaced by $2 + \Delta c_3$.

(c) Suppose the constant on the right hand side of the first constraint is replaced by -2 . Use your final tableau from part (a) and the dual simplex method to find the optimal solution to the new problem.

Solution: (a) After adding slack variables, the initial tableau is

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & \\ -11 & 0 & -3 & 2 & 1 & 0 & 1 \\ 7 & 1 & 2 & -1 & 0 & 1 & 1 \\ -1 & -1 & -2 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Applying the simplex method, we pivot on the (2, 3) position to get

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{1}{2} & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{bmatrix}.$$

Applying the simplex method again, we pivot on the $(1, 4)$ position to get

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 1 & 2 & 3 & 5 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 4 & 6 & 0 & 0 & 4 & 4 & 11 \end{bmatrix}.$$

Since all the entries in the objective row are positive, the solution $x_1 = x_2 = 0, x_3 = 3, x_4 = 5$ is optimal.

(b) We may modify the final tableau by changing the entry 2 in the top row (representing the coefficient of x_3 in the objective function by $2 + \Delta c_3$ and making the corresponding changes in the objective row. We get

$$\begin{bmatrix} 1 & 1 & 2 + \Delta c_3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 1 & 2 & 3 & 5 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 4 + 3\Delta c_3 & 6 + 2\Delta c_3 & 0 & 0 & 4 + \Delta c_3 & 4 + 2\Delta c_3 & 11 + \Delta c_3 \end{bmatrix}.$$

This represents an optimal solution if $4 + 3\Delta c_3 \geq 0, 6 + 2\Delta c_3 \geq 0, 4 + \Delta c_3 \geq 0$, and $4 + 2\Delta c_3 \geq 0$. This is equivalent to $\Delta c_3 \geq \frac{-4}{3}$.

(c) Note that the final tableau is B^{-1} times the initial tableau where $B^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Thus if when the change in the constraints is made, the last column of the final tableau will be replaced by

$$B^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Thus the tableau becomes

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & \\ -1 & 3 & 0 & 1 & 2 & 3 & -1 \\ 3 & 2 & 1 & 0 & 1 & 2 & 0 \\ 4 & 6 & 0 & 0 & 4 & 4 & -1 \end{bmatrix}.$$

Applying the dual simplex method, we pivot on the (1,1) position and get

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & \\ 1 & -3 & 0 & -1 & -2 & -3 & 1 \\ 0 & 11 & 1 & 3 & 7 & 11 & -3 \\ 0 & 18 & 0 & 4 & 12 & 16 & -5 \end{bmatrix}.$$

The dual simplex method now shows that the problem is infeasible.