Math 428 - Solutions torReview problems #1,3,5,6,7,9 for Exam #1 - October 9, 2009

#1 G is a connected graph of order n in which every trail is a path. What is the size of G? Why?

Solution: Any cycle in G is a trail which is not a path. Thus G must be acyclic. Hence G is a tree and so has size $n-1$.

#3 Show that if G is a disconnected graph containing exactly two odd vertices, then these odd vertices must be in the same component of G.

Solution: Suppose the two odd vertices of G are in different components. Let one of these components be A. Then A is a graph with exactly one vertex of odd degree and so the sum of the degrees of the vertices of A is odd. This contradists "The First Theorem of Graph Theory."

#5 Let G be a connected graph, T be a spanning tree in G and e an edge of G that is not in T. Show that $T+e$ contains a unique cycle.

Solution: Note that if H is any spanning graph of G, if H contains a cycle C, and if e is any edge of C, then $H-e$ is again a spanning subraph of G. For if $e=uv$ then C contains an edge different from e which is incident to u and also contains an edge dirrerent from e which is incident to v. Thus the set of vertices incident to an edge of $H-e$ is the same as the set of vertices incident to an edge of H, so $H-e$ is still a spanning subgraph.

Now if T is a spanning tree in G and e is not an edge of T, then $T+e$ cannot be a tree, for if $T+e$ is a tree, it must contain one more vertex than T, but T already contains all the vertices of G. Thus $T+e$ contains a cycle. We must show that it cannot contain more than one cycle. Suppose $T+e$ contains two different cycles, C_1 and C_2. Then there are edges e_1, e_2 such that e_1 is an edge of C_1 and e_2 is an edge of e_2. Then C_2 is a cycle in $T+e-e_1$ and hence$T+e-e_1-e_2$ is a spaning subgraph of G. But then $T+e-e_1-e_2$ contains a spanning tree, say J, and J is a spanning tree of G. But the size of J is at most the size of T minus 1.Thus the spanning trees T and J have different sizes, a contradiction.

#6 True or false: Any graph or order n with degree sequence $(2,2,...,2,1,1)$ is isomorphic to P_n. Explain.

Solution: This is false. One of the connected components must have degree sequence $2,...,2,1,1$ and all the others must have degree sequence $(2,2,...,2)$. Thus the graph can be $C_{n_1} \cup C_{n_2} \cup ... \cup C_{n_r} \cup P_{n_{r+1}}$ whenever $n_1 + ... + n_{r+1} = n$.

#7 True or false: Any 2-regular connected graph of order n is isomorphic to C_n. Explain.

Solution: True. Let G be such a graph and let $v_0, v_1, ..., v_k$ be a path of maximal length in G. Since $deg(v_k) = 2$ there must be a vertex w such that $w \neq v_{k-1}$ and $v_k w$ is an edge in G. Then if w is not one of $v_0, ..., v_{k-2}$ we can find a longer path: $v_0, v_1, ..., v_k, v_{k+1}$, contradicting our choice of the path. However, if $w = v_i$ for some $i, 1 \leq i \leq k-2$, then v_i has degree ≥ 3, contradicting our hypothesis. Thus $w = v_0$ and so $v_0, v_1, ..., v_k, v_0$ is a cycle. Since every vertex has degree 2, all the edges incident to any v_i are in this cycle. Since G is connected, this cycle is all of G.

#9 Which pairs of the following graphs are isomorphic? Give reasons.

Solution: The degree sequences of these graphs are: (a) $(4, 4, 4, 3, 3, 3, 3)$, (b) $(6, 3, 3, 3, 3, 3, 3)$, (c) $(4, 4, 4, 3, 3, 3, 3)$, (d) $(4, 4, 4, 3, 3, 3, 3)$. Thus the graph in (b) is not isomorphic to any of the others. In (c) two vertices are adjacent if and only if one has degree 3 and one has degree 4. Thus this graph is isomorphic to $K_{4,3}$, the graph in a. Finally, the graph in (d) contains a triangle. (For example, the top two vertices and the next vertex in the clockwise direction.) This is an odd cycle, so the graph in (d) is not bipartite and hence is not isomorphic to the graphs in (a) and (c).