Math 428 - Solutions to review problems for exam #2.

#1 (a) u_2 is the only cut vertex

u_1u_2 is the only bridge.

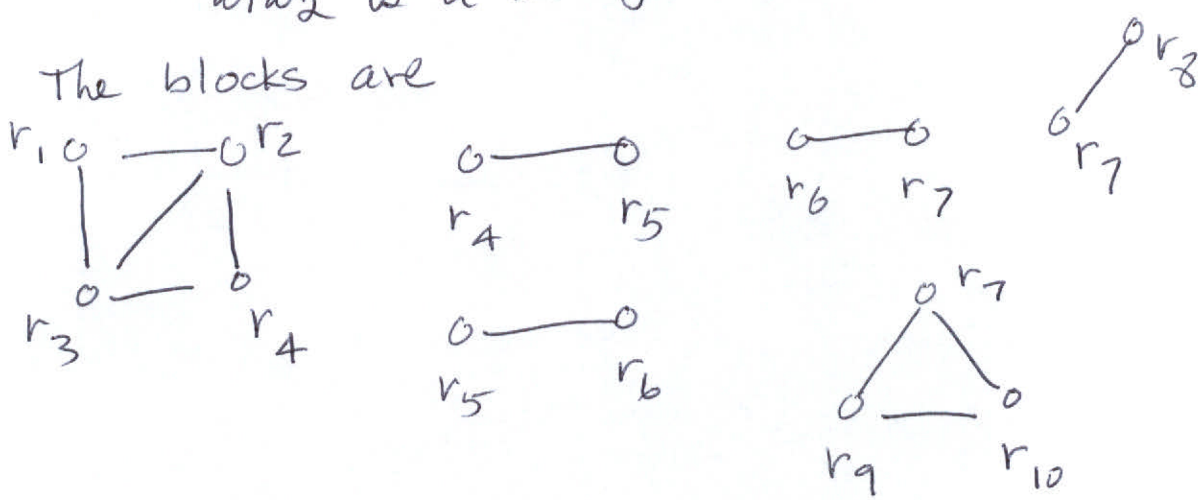
(b) v_3 and v_6 are the only cut vertices

every edge is a bridge.

(c) There are no cut vertices.

w_1w_2 is a bridge

#2 The blocks are



#3 (a) $\{u_5, u_6\}$ is a minimum vertex cut.

So are $\{u_5, u_8\}$ and $\{u_6, u_9\}$

$\{u_5u_7, u_7u_8\}$ is a minimum edge cut.

So is $\{u_6u_{10}, u_9u_{10}\}$,

#3 (b) $\{v_3, v_{10}\}$ is a minimum vertex cut

So is $\{v_4, v_9\}$.

$\{v_3v_4, v_3v_9, v_{10}v_4, v_{10}v_9\}$ is a minimum edge cut.

The intention was to have an edge from v_5 to v_8. This is why the note says the right-hand portion of the graph is a K_6. If this edge is present, the set given above is the only minimum edge cut.

For the graph as drawn

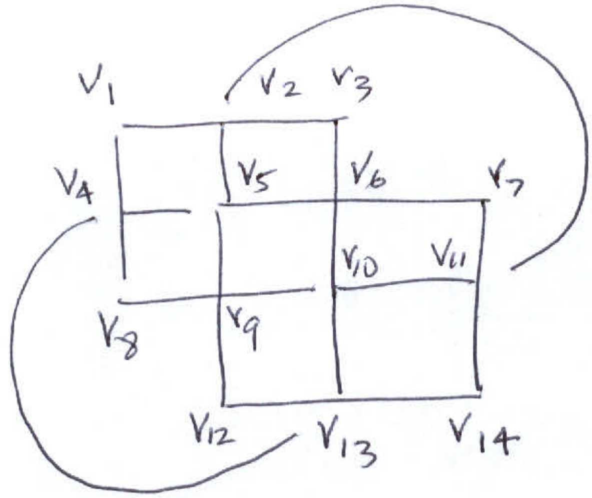
$\{v_4v_5, v_9v_5, v_7v_5, v_6v_5\}$ and

$\{v_4v_8, v_9v_8, v_7v_8, v_6v_8\}$ are also

minimum edge cuts.

#4

(a)

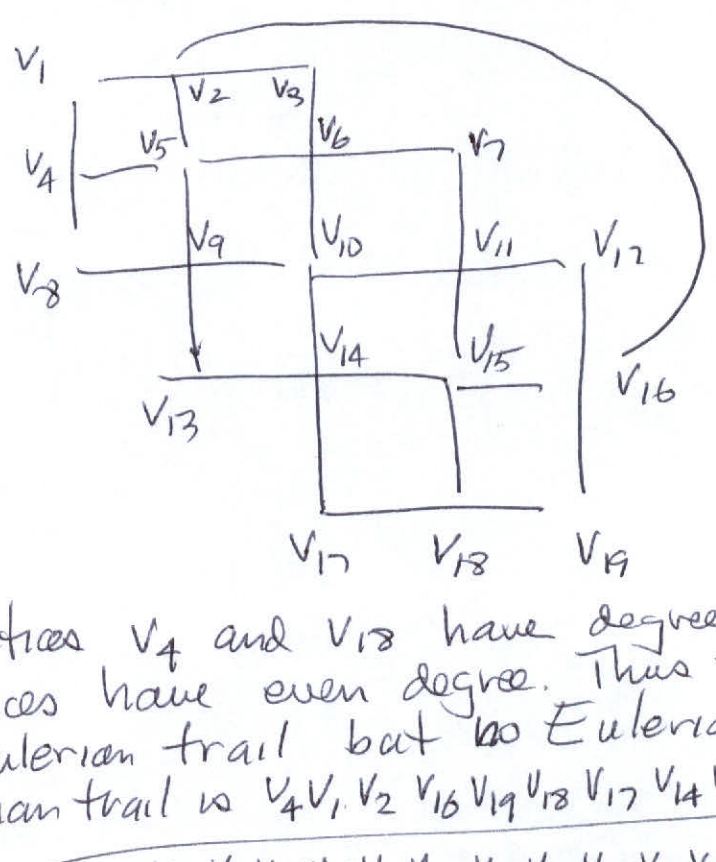


All vertices have even degree, so there is an Eulerian circuit but no Eulerian trail. One Eulerian circuit is

$V_1 V_2 V_3 V_6 V_7 V_{11} V_2 V_5 V_6 V_{10} V_{11} V_{14} V_{13} V_{10}$

$\to V_9 V_{12} V_{13} V_4 V_8 V_9 V_5 V_4 V_1$

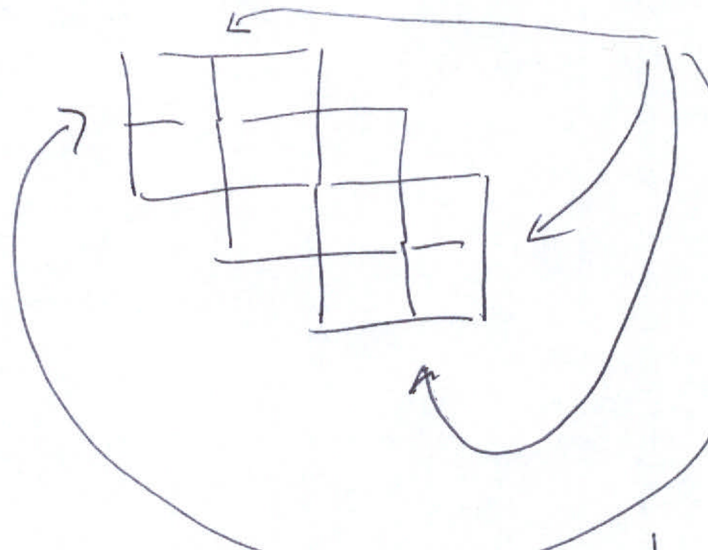
(b)



The vertices V_4 and V_{18} have degree 3. All other vertices have even degree. Thus there is an Eulerian trail but no Eulerian circuit. One Eulerian trail is $V_4 V_1 V_2 V_{16} V_{19} V_{18} V_{17} V_{14} V_{13} V_9 V_8 V_4$

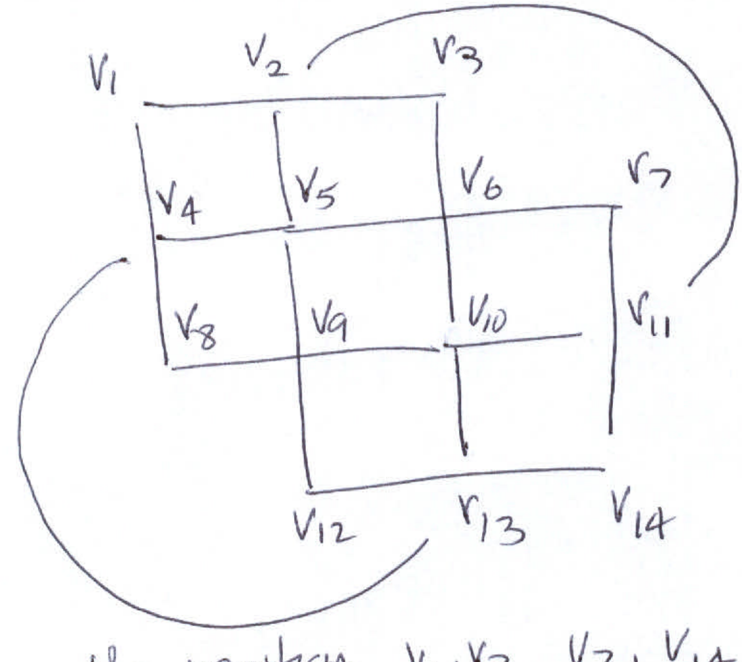
$\to V_5 V_2 V_3 V_6 V_5 V_9 V_{10} V_6 V_7 V_{11} V_{12} V_{16} V_{15} V_{14} V_{10} V_{11} V_{15} V_{18}$

#4 c) G_3 has 4 vertices of degree 3



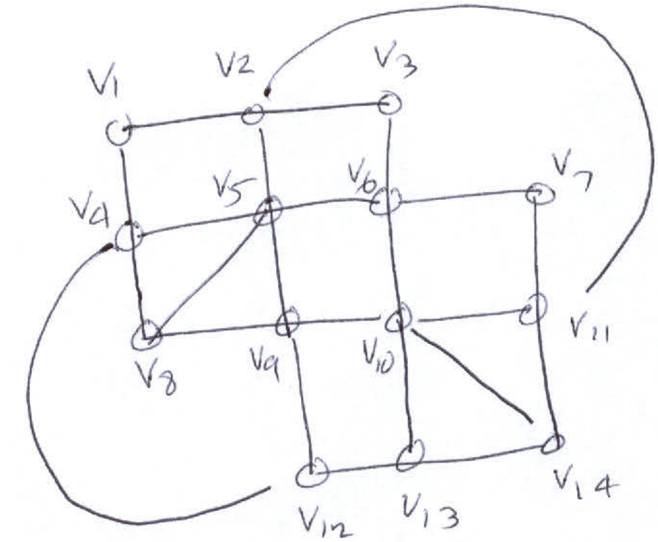
Hence it has neither an Eulerian circuit nor an Eulerian trail.

#5 a)



Since the vertices $v_1, v_3, v_7, v_{14}, v_{12}$ and v_8 all have degree 2 all the edges incident to them must be in any Hamiltonian cycle. Thus $v_1v_2, v_2v_3, v_3v_6, v_6v_7, v_7v_{11}, v_{11}v_{14}, v_{14}v_{13}, v_{13}v_{12}, v_{12}v_9, v_9v_8, v_8v_4, v_4v_1$ must all be in the cycle. But then no edge incident to v_5 can be in the cycle, so there is no Hamiltonian cycle.

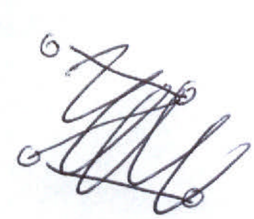
#5 (b)

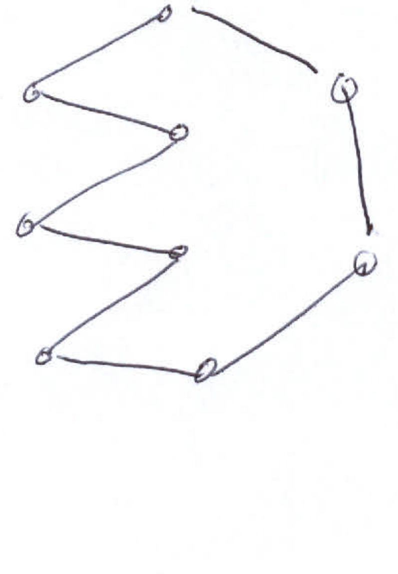


Since V_1, V_3, V_7 all have degree 2 any Hamiltonian cycle must contain $V_4 V_1 V_2 V_3 V_6 V_7 V_{11}$. One can continue this to get the Hamiltonian cycle

$$V_4 V_1 V_2 V_3 V_6 V_7 V_{11} V_{14} V_{10} V_{13} V_{12} V_9 V_5 V_8 V_4$$

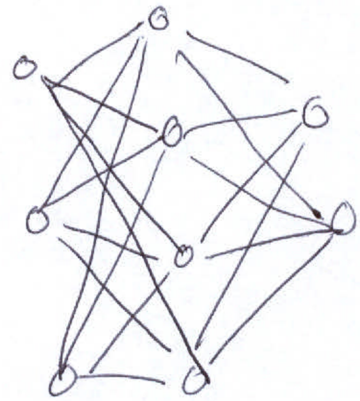
(c) $\delta(G_3) = 5$ and the order of G_3 is 9 so by Theorem 6.6 G_3 is Hamiltonian. One Hamiltonian cycle is





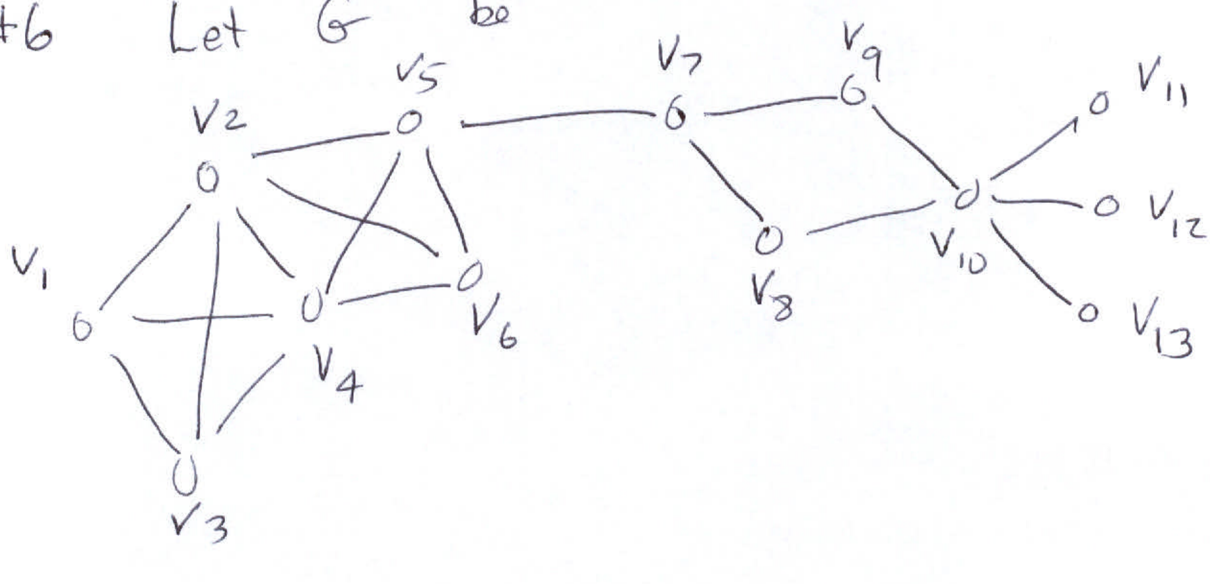
A variation on this problem is interesting. ~~Is~~ Let

G_4 is

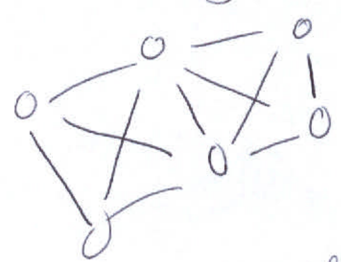


(This is G_3 with the two vertical edges on the left and the vertical edge on the right omitted. Then if S is the set of 4 vertices in the center $G-S$ has 5 components, so by (the contrapositive of) Theorem 6.5, G_4 is <u>not</u> Hamiltonian

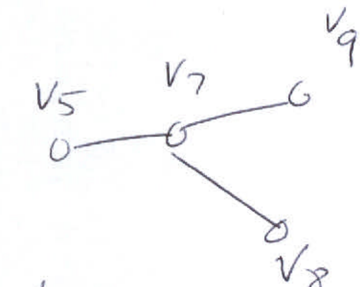
#6 Let G be



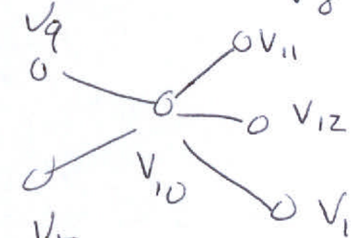
Then $\{v_1v_3, v_2v_4, v_5v_6, v_7v_9, v_{10}v_{11}\}$ is an independent set. It is a maximum independent set because no independent set can contain more than 3 edges from



or more than one edge from



or more than one edge from



This set is a maximum matching, so $\beta_1(G) = 5$. Also $\{v_1v_3, v_2v_4, v_5v_6, v_7v_9, v_7v_8, v_{10}v_{11}, v_{10}v_{12}, v_{10}v_{13}\}$ is an edge cover. This set contains 8 edges so (since $\alpha_1(G)+\beta_1(G) = 13$) it is a minimum edge cover.

The set $\{v_1, v_2, v_4, v_5, v_7, v_{10}\}$ is a vertex cover. It is minimum since any vertex cover must contain at least three of v_1, v_2, v_3, v_4; at least 3 of v_2, v_4, v_5, v_6, at least one of v_7, v_8, v_9 and at least one of $v_{10}, v_{11}, v_{12}, v_{13}$.

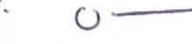
Thus $\alpha(G) = 6$.

The set $\{v_1, v_6, v_8, v_9, v_{11}, v_{12}, v_{13}\}$ is an independent set of vertices. Since $\beta(G) = 13 - \alpha(G) = 7$ this set is a maximum independent set of vertices.

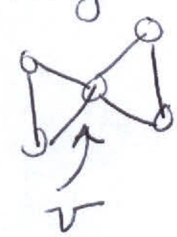
#7 $\{u_1, u_4\}$ is a minimal u-v separating set. $u = u_1 u_2 u_{12} u_{19} = v$ and $u = u_1 u_4 u_8 u_9 u_{13} u_{14} u_{17} u_{18} u_{19} = v$ are two internally disjoint u-v paths. By Menger's Theorem the set consisting of these two paths is maximum.

$\{u_2, u_{11}, u_{14}\}$ is a minimum u_{10}-u_{16} separating set.

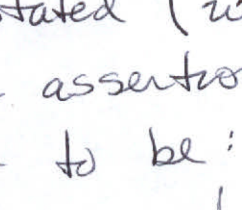
$\{u_{10} u_{11} u_{12} u_{16},\ u_{10} u_{14} u_{17} u_{18} u_{19} u_{16},\ u_{10} u_6 u_3 u_2 u_{16}\}$ is a set of three internally disjoint u_{10}-u_{16} paths. By Menger's Theorem this set is maximum

#8 (a) This is false. In P_2: 
the single edge is a bridge but neither vertex is a cut vertex.
Theorem 5.1 shows that in a connected graph a vertex incident to a bridge is a cut-vertex if and only if it has degree 1.

(b) As the problem is stated (with no hypothesis on v) the assertion is false.
I intended the statement to be:
"If v is a cut-vertex of G and e is incident to v then e is a bridge." This is still false, for example



(c) This is false. In

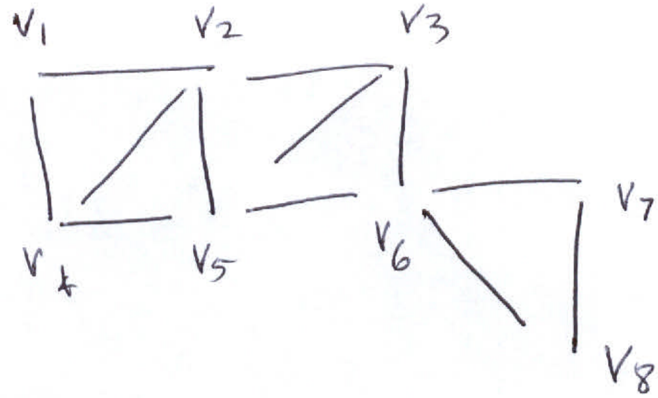


$\{v\}$ is a vertex cut and $G - \{v\}$ has 3 components.

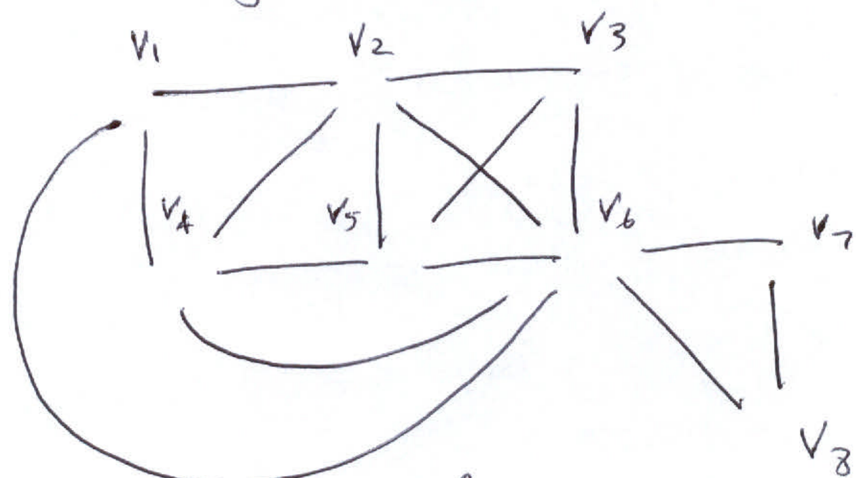
(d) This is false, since X is not required to be minimum. Thus if G is then $G - \{e,f\}$ has 3 components. If X is a minimum edge cut then this is true (and useful).

(e) Suppose two edges $e = vw$ and $f = vu$ are bridges incident to v. Then $G-e-f$ has 3 components. (To see this note that $G-e$ has two components, say G_1 and G_2 with $v \in V(G_1)$ and so $f \in E(G_1)$. Then f is a bridge in G_1 - for it is not contained in any cycle - so $G_1 - f$ has two components, say G_3 and G_4 with $v \in G_3$. Then G_2, G_3, G_4 are the components of $G-e-f$. We ~~m~~ have $v \in G_3$, $w \in G_2$, $u \in G_4$. Now there are only two vertices of odd degree in G, say v and x. The vertex x cannot be in both G_2 and G_4. Assume it is in G_4. Then in G_2 the vertex w has odd degree (for its degree in G is even, but the edge $e = vw$ has been removed from $G-e$). But all the other vertices in G_2 have the same degree as in G, hence have even degree. But this is impossible.

#9 G



G has order 8 and the vertices v_2 and v_6 have degree 4 so we can add an edge v_2v_6. This gives v_6 degree 5 so we can add an edge v_4v_6. But now v_6 has degree 6 so we can add an edge v_1v_6. The graph has now become



This is $C(G)$ since for any pair of nonadjacent vertices u, v we have $\deg u + \deg v < 8$.

#10 Choose u, v so that the distance from u to v is as large as possible. Let this distance be k and let

$u = u_0 u_1 \cdots u_k = v$ be a u-v path.

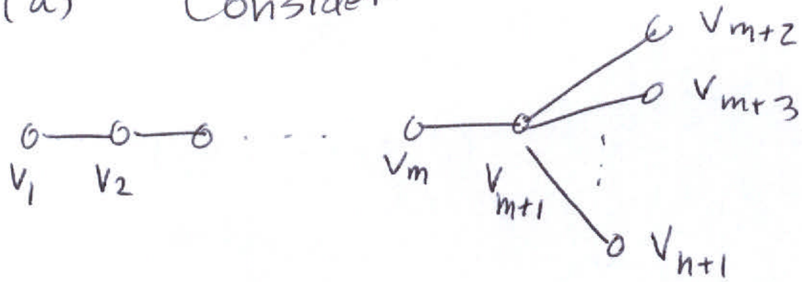
Since this path has maximal length v cannot be adjacent to any vertex except those in the path. But if v is adjacent to any vertex in the path except u_{k-1}, say to u_i then $u_i u_{i+1} \cdots u_k u_i$ is a cycle and so $v = u_k$ is not a cut vertex. Thus v can be adjacent only to u_{k-1}, Thus v can be adjacent only to u_{k-1}, i.e., $\deg v = 1$, so v is not a cut vertex. Similarly, u is not a cut vertex.

#11 ~~Assume~~ Let $u = u_0 u_1 \cdots u_k$ be a path of maximal length. Since $\deg u_k = 2$ it is adjacent to some vertex other than u_{k-1}. If this vertex, say w is not one of $u_0, u_1, \dots, u_{k-2}$ then $u_0 u_1 \cdots u_k w$ is a longer path, contradicting our choice of the path. If w is adjacent to u_i with $0 < i \leq k-2$ then $\deg u_i \geq 3$, contradicting our hypothesis. Hence w must be adjacent to u_0, so $u_0 u_1 \ldots u_k u_0$ is a cycle. Since every u_i is adjacent to two vertices in this cycle it is not adjacent to any other vertex. Thus this cycle is a component of the graph and, since the graph is connected, it is the entire graph.

#12 (a) Is Theorem 5.8

(b) If $e \neq f$ then they are contained in a common cycle C. But C is a nonseparable subgraph, so it is contained in a maximal nonseparable subgraph. But this is a block.

#13(a) Consider



This is a tree of order $n+1$, so has n edges and for any edge e the induced subgraph $\langle e \rangle$ is a block. Thus there are n blocks. The cut-vertices are $v_2, \dots, v_{m+1}$, so there are m cut-vertices

(b) If there are no cut vertices, there is one block so the result is true. We will proceed by induction on the order of G. Let v be a cut vertex and let $G_1, \dots, G_k$ be the components of $G - v$. ~~Then the induced subg.~~ Let H_i be the induced subgraph $\langle V(G_i) \cup \{v\} \rangle$. Then the cut vertices of $G - v$ are the cut vertices of the H_i and also v. The blocks of G are the blocks of the H_i

Let m_i be the number of cut vertices of H_i, n_i be the number of blocks of H_i. Then $m = m_1 + \dots + m_k + 1$, By induction this is $\leq (n_1 - 1) + \dots + (n_k - 1) + 1 = n_1 + \dots + n_k - k + 1 = n - k + 1$. Since $k \geq 2$, this is $\leq n - 1$.