Math 428 - Solutions to review problems for Final Exam - December 16, 2009
This is the complete set of review problems.

\# 1 Let G be the complete bipartite graph with partite sets $\{a, b, c, d\}$ and $\{e, f, g\}$.

i) Find a bf-path of maximal length.

Solution: Let $b = v_0, v_1, v_2, ..., v_k = f$ be a bf-path of length k. Since G is bipartite, v_i must belong to $\{a, b, c, d\}$ when i is even and must belong to $\{e, f, g\}$ when i is odd. Since the vertices in a path must be distinct, there can be at most 3 vertices v_i with i odd. Thus $k \le 5$. The path b, e, a, g, c, f has length 5 and so is of maximal length.

ii) Find a bc-path of maximal length.

Solution: Let $b = v_0, v_1, v_2, ..., v_k = c$ be a bc-path of length k. Since G is bipartite, v_i must belong to $\{a, b, c, d\}$ when i is even and must belong to $\{e, f, g\}$ when i is odd. Since the vertices in a path must be distinct, there can be at most 3 vertices v_i with i odd. Thus $k \le 6$. The path b, e, a, g, d, f, c has length 6 and so is of maximal length.

iii) Find a trail of maximal length

Solution: Note that the size of G is 12, so no trail can have more than 12 edges. Let P be any trail and S be the set of edges in P. Then P is an Eulerian trail or Eulerian circuit in the edge induced subgraph $< S >$. Hence this subgraph can contain at most two vertices of odd degree. Now in the graph G, the vertices a, b, c, d all have degree 3 and the vertices e, f, g all have degree 4. For there to be an Eulerian trail in $< S >$, S must be obtained by removing at least one edge incident each of two of the vertices of degree 3 in G. Thus, no trail in G can have more than 10 vertices. In fact, if $S = \{af, ag, bf, bg, ce, cf, cg, de, df, eg\}$ then $< S >$ has three vertices (a, b and e) of degree 2, two vertices (c and d) of degree 3 and two vertices (f and g) of degree 4 and so $< S >$ has an Eulerian trail. Using Fleury's algorithm we see that $c, g, d, f, b, g, a, f, c, e, d$ is one such trail and hence is a trail of maximal length in G.

\# 2 Let G and H be planar graphs.

a) Suppose $G + H$ is planar. Show that either G or H has order ≤ 2.

Solution: We will prove the contrapositive. Suppose that G and H both have order ≥ 3. Let a_1, a_2, a_3 be three vertices of G and b_1, b_2, b_3 be three vertices of H. Then $a_i b_j$ is an edge of $G + H$ for all $i, j, 1 \le i, j \le 3$. Let $S = \{a_i b_j | 1 \le i, j \le 3\}$. Then the edge induced subgraph $< S >$ of $G + H$ is isomorphic to $K_{3,3}$ and so, by Kuratowski's Theorem, $G + H$ cannot be planar.

b) Does the converse of (a) hold. That is, if G and H are planar and G has order ≤ 2 does $G + H$ have to be planar?

Solution: $G + H$ does not have to be planar. For example $K_1 + K_4$ is isomorphic to K_5 and so is not planar.

\# 3 For each of the following sequences, answer the following questions (giving reasons).

i) Is the sequence graphical?

ii) If the sequence is the degree sequence of a graph G, can G be connected? Does G have to be connected?

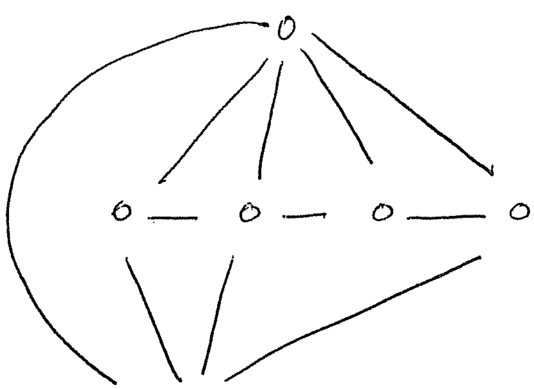
iii) If the sequence is the degree sequence of a graph G, can G be Eulerian? Does G have to be Eulerian?

iv) If the sequence is the degree sequence of a graph G, can G be Hamiltonian? Does G have to be Hamiltonian?

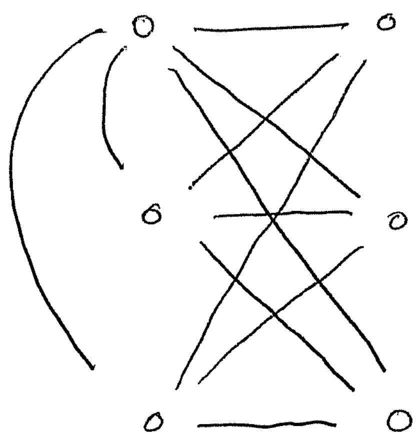
v) If the sequence is the degree sequence of a graph G, can G be planar? Can G be maximal planar? Does G have to be planar or maximal planar?

a) $(5, 4, 4, 3, 3, 3)$

Solution: This sequence is graphical if and only if $(3, 3, 2, 2, 2)$ is graphical and $(3, 3, 2, 2, 2)$ is graphical if and only if $(2, 2, 1, 1)$ is graphical. Since $(2, 2, 1, 1)$ is the degree sequence of P_4 we see that $(5, 4, 4, 3, 3, 3)$ is graphical. Suppose G is a graph with this degree sequence. Then since the vertex of degree 5 must be adjacent to 5 other vertices and since G has order 6 we see that the vertex of degree 5 is adjacent to every other vertex and so G must be connected. Since G has vertices of odd degree, it cannot be Eulerian. Since $\delta(G) = 3$ and so $2\delta(G)$ is $\geq$ the order of G, we see that G must be Hamiltonian. The plane graph G_1 shown below has the given degree sequence. However, the graph G_2 shown below contains a subgraph isomorphic to $K_{3,3}$ and so is not planar. No graph with this degree sequence can be maximal planar, since the order n is 6 and the size m is $(5+4+4+3+3+3)/2 = 11$ so $3n - 6 = 12 \neq m$.



G_1



G_2

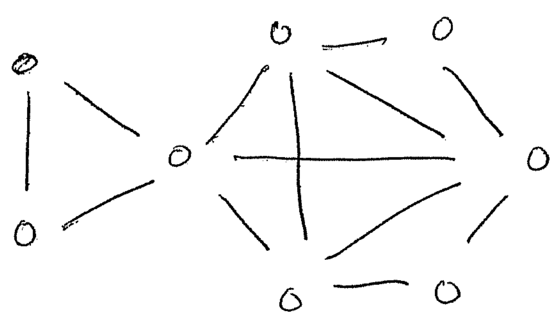
b) $(5, 4, 4, 4, 3, 3)$

Solution: This sequence is not graphical since it contains an odd number of odd entries.

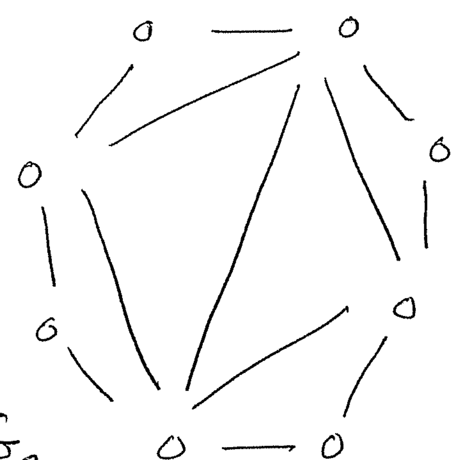
c) $(5, 4, 4, 1, 1, 1)$

Solution: This sequence is graphical if and only if $(3, 3, 0, 0, 0)$ is graphical. The sequence $(3, 3, 0, 0, 0)$ is not graphical since if it were there would be only two vertices of positive degree and hence no vertex could have degree greater than 1.

d) $(5, 5, 4, 4, 2, 2, 2, 2)$
Solution: This sequence is graphical if and only if $(4, 3, 3, 2, 2, 1, 1)$ is graphical and $(4, 3, 3, 2, 2, 1, 1)$ is graphical if and only if $(2, 2, 1, 1, 1, 1)$ is graphical. Since $P_4 \cup P_2$ has degree sequence $(2, 2, 1, 1, 1, 1)$ the given sequence is graphical. Let G be a graph with this degree sequence. Suppose G is not connected. Let G_1 be a component containing a vertex of degree 5 and G_2 be another component. Then, since G_1 contains a vertex of degree 5, its order is at least 6. Thus the order of G_2 is at most 2. However, any vertex of G_2 has degree ≥ 2, which is impossible. Thus G must be connected. Since G contains vertices of odd degree, it cannot be Eulerian. The graphs G_1 and G_2 below have the given degree sequence. Since G_1 has a cut-vertex, it is not Hamiltonian. The graph G_2 is Hamiltonian (since the perimeter of the octagon is a Hamilton cycle. Finally, if G were to contain a subdivision of K_5 as a subgraph, it would have to contain at least 5 vertices of degree ≥ 4. But G contains only 4 such vertices. Similarly, if G were to contain a subdivision of $K_{3,3}$ as a subgraph, it would have to contain at least 3 vertices of degree ≥ 3. But G contains only 4 such vertices. Thus, by Kuratowski's Theorem, G must be planar. Since G has order $n = 8$ and size $m = (5+5+4+4+2+2+2 = 2)/2 = 13$ we see that $3n - 6 = 18 \neq m$ and so G cannot be maximal planar.



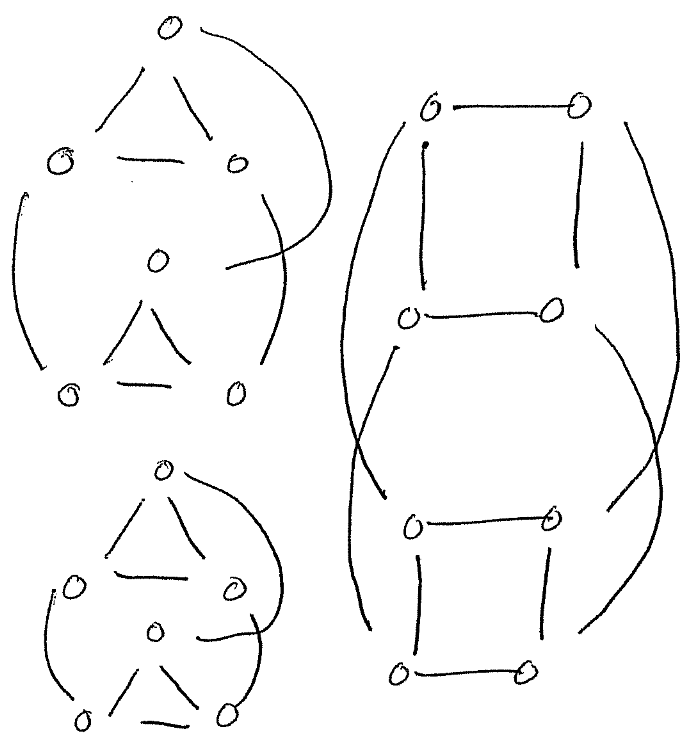
G_1



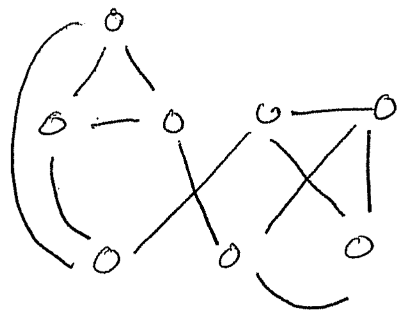
G_2

4 Let n be an integer, $n \geq 3$.
(a) How many isomorphism classes of $n-1$-regular graphs of order n are there? Why?
Solution: Such a graph must be K_n, so there is only one isomorphism class.
(b) How many isomorphism classes of $n-2$-regular graphs of order n are there? Why?
Solution: If n is odd, there are 0 isomorphism classes of such graphs, since such a graph would have an odd number of vertices of odd degree. If n is even, the complement would have degree sequence $(1, 1, ..., 1)$ and so would be isomorphic to $P_2 \cup ... \cup P_2$ where there are $n/2$ copies of P_2. Thus there is only one isomorphism class in this case.

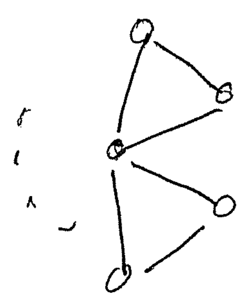
5 Give an embedding of $C_3 \cup P_2$ as an induced subgraph of a 3-regular graph.
Solution: Here are two such embeddings.



Or



6 Let v be a cut-vertex of a graph G. How many blocks of G can contain v?
Soluiton: Suppose v is contained in p blocks of G. Since every vertex of G is contained is some block we must have $p \geq 1$. Suppose v is contained in only one block, say B. Since v is a cut vertex there exist vertices u, w of G such that there is a $u - w$ path in G and that every $u - w$ path in G contains v. Let $u = u_0, u_1, \ldots, u_{k-1}, u_k = v = w_l, w_{l-1}, \ldots, w_1, w_0 = w$. be a $u - v$ path. Then every $u_{k-1} - w_{l-1}$ path in G must contain v. But, since B is the only block containing v, the edges $u_{k-1}v$ and vw_{l-1} must belong to B and so u_{k-1}, w_{l-1} are vertices of B. But then v is a cut-vertex of B, contradicting the fact that B is nonseparable. Thus $p \geq 2$. Considering the graph drawn below shows that p can be any integer ≥ 2.



p triangles

7 Prove that if B and C are blocks of a graph G (and $B \neq C$) then $|V(B) \cap V(C)| \leq 1$. **Solution:** Suppose $x, y \in V(B) \cap V(C)$ with $x \neq y$. Let D be the graph with $V(D) = V(B) \cup V(C), E(D) = E(B) \cup E(C)$. We will show that D is nonseparable. Since D properly contains B this contradicts the fact that B, being a block, is a maxial nonseparable subgraph. To see that D is nonseparable, let $v \in D$. Suppose $v \notin V(B)$. Then $v \in V(C)$. Now the block B is connected and $C - v$ is connected (since C is nonseparable. Since $V(B) \cap V(C - v) \neq \emptyset$ (since it contains x and y) we have that $D - v$ is connected. Similarly, if $v \notin C$ we see that $D - v$ is connected. Finally, suppose that $v \in V(B) \cap V(C)$. Then $B - v$ and $C - v$ are connected. Furthermore, $V(B - v) \cap V(C - v) \neq \emptyset$ (for it must contain at least one of the two vertices x and y. This in this case $D - v$ is connected. Hence $D - v$ is always connected an so v cannot be a cut vertex. The completes the proof.

8 Find all trees T such that $\overline{T}$ is planar. Explain.

Solution: If T has order ≤ 5 then $\overline{T}$ is planar (for any graph of order 4 is planar, the only nonplanar graph of order 5 is K_5, and $\overline{T}$ is not complete).

If T has order n then $\overline{T}$ has size $((n(n-1)/2) - (n-1)$. Thus if $\overline{T}$ is planar we have

$$3n - 6 \geq ((n(n-1)/2) - (n-1)$$

so

$$0 \geq n^2 - 9n + 14 = (n-2)(n-7).$$

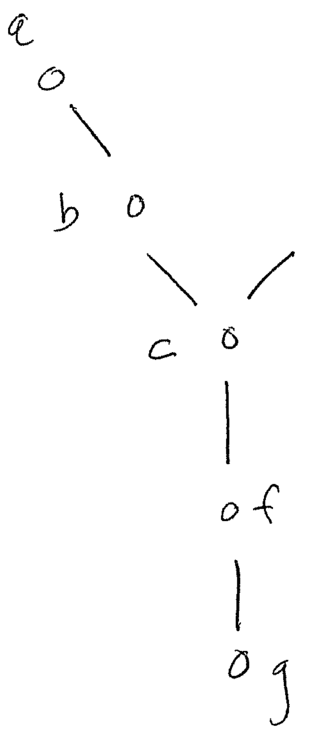
Thus if T has order ≥ 8, $\overline{T}$ is nonplanar.

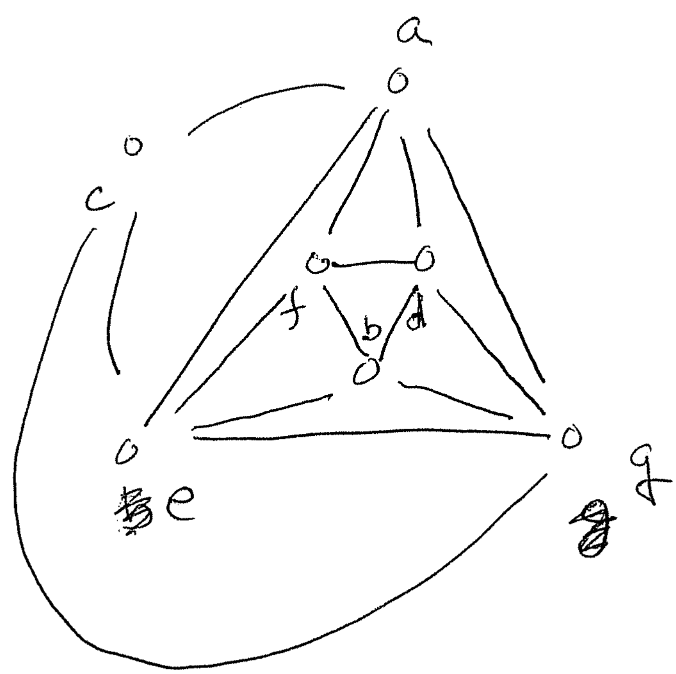
It remains to consider trees of order 6 and 7.

Suppose T is a tree of order 6 and that $\overline{T}$ contains a subgraph H isomorphic to $K_{3,3}$. Let A and B be the partite sets of H. Then every vertex in A is adjacent in $\overline{G}$ to every vertex in B. This means that in T no vertex in A is adjacent to any vetex in B. Since $A \cup B = V(T)$ this shows that T is not connected, a contradiction. Thus $\overline{T}$ cannot contain a subgraph isomorphic to $K_{3,3}$. Now suppose $\overline{T}$ contains a subgraph isomorphic to a subdivsion of K_5. Then $\overline{T}$ must contain at least 5 vertices of degree ≥ 4 and so T must contain at least 5 vertices of degree 1. This means that the degree sequence of T must be $(5, 1, 1, 1, 1, 1)$. Note that in this case $\overline{T}$ does contain a subgraph isomorphic to K_5 so $\overline{T}$ is nonplanar. In any other case (for T of order 6), Kuratowski's Theorem shows that $\overline{T}$ is planar.

Now suppose T is a tree of order 7. If T is P_7, let v be the middle vertex. Then $T - v$ has two components, say G_1 and G_2. Since, in T, no vertex of G_1 is adjacent to any vertex of G_2 we have that in $\overline{T}$ every vertex of G_1 is adjacent to every vertex of G_2. Thus $\overline{T}$ contains a subgraph isomorphic to $K_{3,3}$ and so is not planar. Thus if $\overline{T}$ is planar, T contains a vertex of degree ≥ 3. If T contains a vertex v of degree 5 or 6 then we may find 5 vertices v_1, v_2, v_3, v_4, v_5 adjacent to v in T. Then (as T is acyclic) no two of there vertices are adjacent in T and hence any two are adjacent in $\overline{T}$. Thus $\overline{T}$ contains a subgraph isomorphic to K_5 and so is nonplanar. Next suppose that T contains a vertex v of order 4. Then $G - v$ has 4 components. Say these components are G_1, G_2, G_3, G_4 and that they have orders a_1, a_2, a_3, a_4 where $a_1 \geq a_2 \geq a_3 \geq a_4$. Since $a_1 + a_2 + a_3 + a_4 = 6$, there are only two possibilities for the sequence (a_1, a_2, a_3, a_4). These are $(3, 1, 1, 1)$ and $(2, 2, 1, 1)$. In the first case, every vertex in G_1 is adjacent in $\overline{T}$ to every vertex of G_2, G_3 and G_4. In the second case, every vertex in $G_1 \cup G_3$ is adjacent in $\overline{T}$ to every vertex in $G_2 \cup G_4$. Thus in either ease there is a subgraph isomorphic to $K_{3,3}$ and so $\overline{T}$ is nonplanar. Finally, suppose that T contains a vertex v of degree 3. Then $G - v$ has 3 components. Say these components are G_1, G_2, G_3 and that they have orders a_1, a_2, a_3 where $a_1 \geq a_2 \geq a_3$. Since $a_1 + a_2 + a_3 = 6$, there are only three possibilities for the sequence (a_1, a_2, a_3). These are $(4, 1, 1), (3, 2, 1)$ and $(2, 2, 2)$. In the first case, v is adjacent to one vertex w in G_1 and no vertex in $G_1 - w$ is adjacent to any vertex in $\{v\} \cup V(G_2) \cup V(G_3)$. Thus $\overline{T}$ contains a subgraph isomorphic to $K_{3,3}$. In the second ease no vertex in G_1 is adjacent to any vertex in $G_2 \cup G_3$ and so again $\overline{T}$ contains a subgraph isomorphic to $K_{3,3}$ and so is nonplanar. In the last case, the tree T is represented by the left-hand diagram below and so $\overline{T}$ is represented by the right-hand diagram below. This shows that $\overline{T}$ is planar in this case.

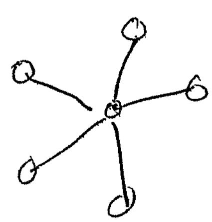
T



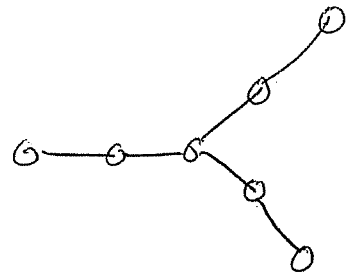


To summarize our results, if T is a tree of order n, then $\overline{T}$ is planar if and only if either:

(a) $n \leq 5$;

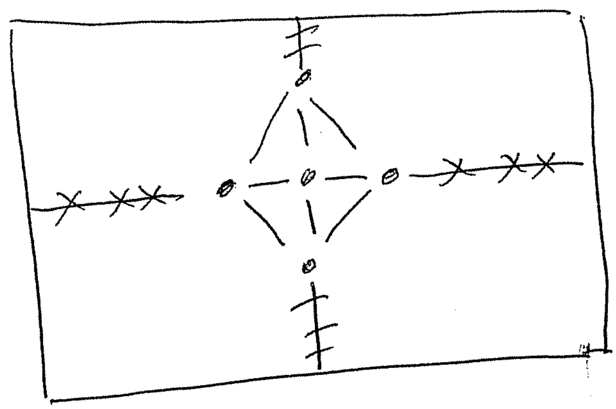
(b) $n = 6$ and T is not 

(c) $n = 7$ and T is



9 a) Show that $\gamma(K_5) = 1$.

Solution: We have seen in class that K_5 is nonplanar. (One way to see this is that K_5 has order $n = 5$ and size $m = 10$. Since $3n - 6 = 9 < m$ we know that K_5 is nonplanar.) Thus we only need to show that K_5 can be embbedded in S_1. Here is such an embedding.

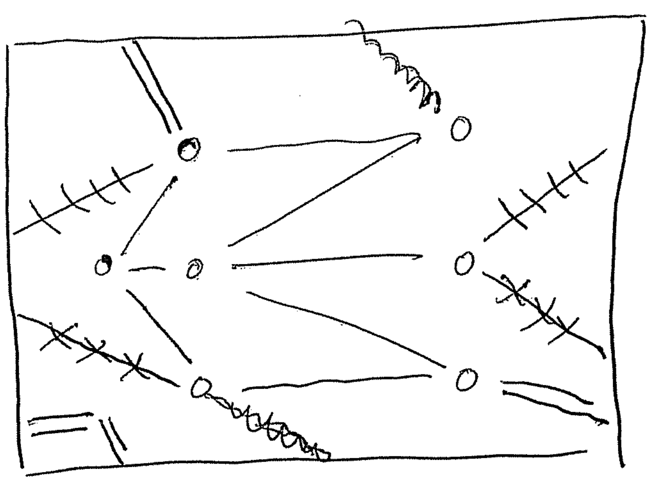


identify top and bottom

identify left and right edges

b) Show that $\gamma(K_{4,3}) = 1$.

Solution: Since $K_{4,3}$ contains a subgraph isomorphic to $K_{3,3}$ it is nonplanar by Kuratowski's Theorem. Thus we only need to show that $K_{4,3}$ can be embedded in S_+1. Here is such an embedding.



identify top and bottom

identify left and right edges

10 Suppose a graph G of order n has chromatic number n. Does G have to be complete? Why or why not?

Solution: G must be complete. We will establish this by proving the contrapositive. Suppose G is not complete. Then we may label the vertices of G as $v_1, v_2, ..., v_{n-1}, v_n$ with v_{n-1} not adjacent to v_n. Then we may color G with $n-1$ colors by giving vertex V_i color i for $i \leq n-1$ and giving vertex v_n color $n-1$. Thus $\chi(G) \leq n-1$.

11 Prove that a graph G of order ≥ 3 is connected if and only if G contains two distinct vertices u and v such that $G-u$ and $G-v$ are connected. (This is Theorem 1.10.)

Solution: See the text for this. The proof is split into two parts: Theorems 1.8 and 1.9.

12 Let G and$H$ be graphs with $V(G) \cap V(H) = \emptyset$. State the definitions of $G \cup H, G+H$, and $G \times H$.

Solution: See pages 23 and 24 of the text for these definitions.

13 Prove that two graphs G and H are isomorphic if and only if $\overline{G}$ and $\overline{H}$ are isomorphic.

Solution: See the text for this. It is Theorem 3.1.

14 Let F, G and H be graphs.

(a) Suppose that $F \cup H$ and $G \cup H$ are isomorphic. Do F and G have to be isomorphic. Explain your answer.

Solution: We know that if A and B are isomorphic graphs then A is connected if and only if B is connected. (This is Theorem 3.5(b).) Thus if ϕ is an isomorphism from a graph C to a graph D and C' is a component of C, its image $\phi(C')$ is a component of D. Now let G' be a component of G. Let a be the number of components of F that are isomorphic to G', b be the number of components of H that are isomorphic to G' and c be the number of components of G that are isomorphic to G'. Then $F \cup H$ has $a+b$ components isomorphic to G' and $G \cup H$ has $b+c$ components that are isomorphic to G'. Since $F \cup H$ and $G \cup H$ are isomorphic we see that $a+b = b+c$ so $a = c$. Thus we may find a one-to-one onto map of $V(F)$ onto $V(G)$ which restricts to an isomorphism on each component of F, so F and G are isomorphic.

(b) Suppose that $F+H$ and $G+H$ are isomorphic. Do F and G have to be isomorphic. Explain your answer. Hint: Consider using problem #13.

Solution: F and G do have to be isomorphic. To see this, suppose $F+H$ and $G+H$ are isomorphic. Then by Theorem 3.1 (which is problem #13), $\overline{F+H}$ and $\overline{G+H}$ are isomorphic. Now $\overline{F+H}$ is isomorphic to $\overline{F} \cup \overline{H}$. Thus we have $\overline{F} \cup \overline{H}$ is isomorphic to $\overline{G} \cup \overline{H}$. Then by part (a) we have that $\overline{F}$ is isomorphic to $\overline{G}$. Finally, using Theorem 3.1 again gives that F and G are isomorphic.

15(a) Suppose G has no cut vertices. How many bridges can G have? Explain.

Solution: Let G' be a component of G and suppose G' has a bridge. If G' has order ≥ 3

then G' has a cut vertex, so G has a cut vertex, contradicting our hypothesis. If G' has order 2 then G' is isomorphic to P_2 and so contains one bridge. If G' has order 1 then it has no bridges. Consequently, the number of bridges of G is equal to the number of components of G that are isomorphic to P_2. This number can be any nonnegative integer. Note that if the hypothesis that G is connected is added, then the number of bridges must be 0 or 1.

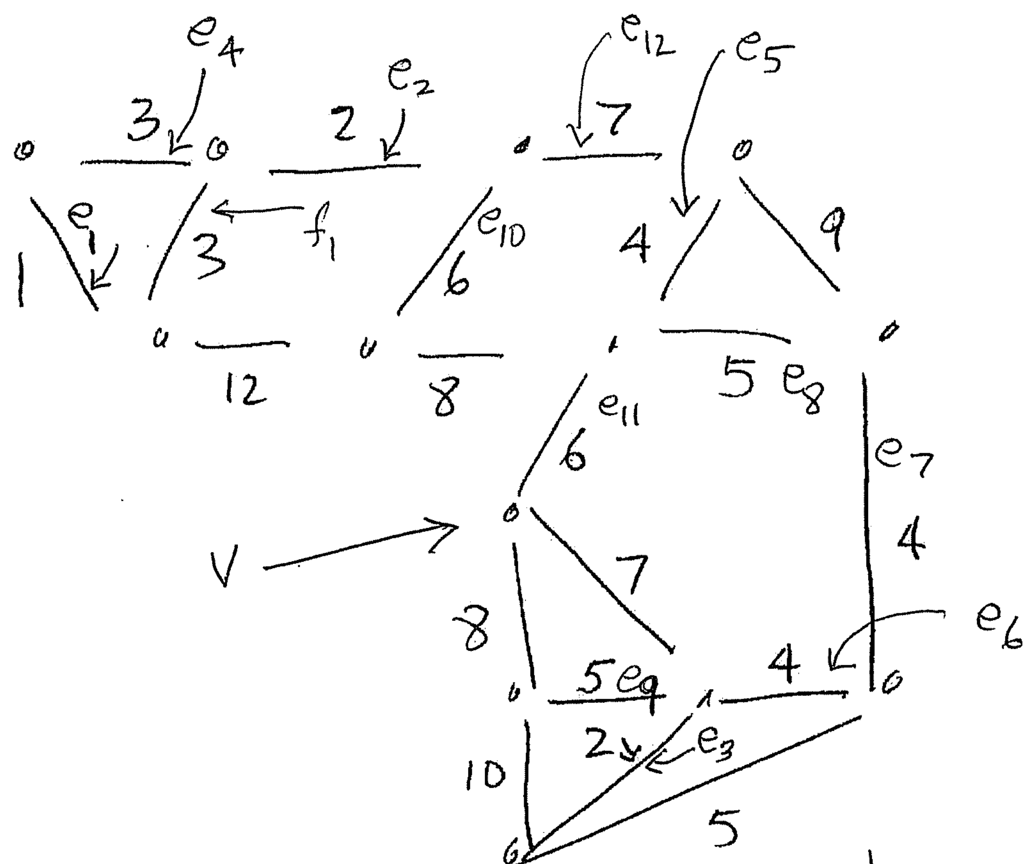
(b) Suppose G has exactly one cut vertex. How many bridges can G have? Explain.
Solution: The number of bridges can be any nonnegative integer, for the graph formed by joining two triangles at one vertex has one cut vertex and no bridges, the graph formed by joining a triangle and a P_2 at one vertex has one cut vertex and one bridge, and, for $n \geq 2$, the tree of order $n+1$ with degree sequence $(n, 1, 1, ..., 1)$ has one cut vertex and n bridges.

#16 Suppose that G is a graph of order n containing no isolated vertices. Prove that $\alpha(G) + \beta(G) = n$ and that $\alpha_1(G) + \beta_1(G) = n$.
Solution: The result for α_1 and β_1 is Theorem 8.7 in the text.

To prove the result for α and β note that a subset $S \subseteq V(G)$ is independent if and only if its complement $V(G) - S$ is a vertex cover. Thus the complement of an independent set of maximum cardinality is a vertex cover of minimum cardinality.

#17 Let G be a graph. Prove that $\gamma(G) \leq 1 + \Delta(G)$ and that $\gamma(G) \leq 1 + max\{\delta(G)\}$ where the maximum is taken over all induced subgraphs.
Soluiton These results are Theorems 10.7 and 10.9 in the text.

#18 Find a minimal spanning tree in the following weighted graph using Kruskal's algorithm and also using Prim's algorithm. Show your work

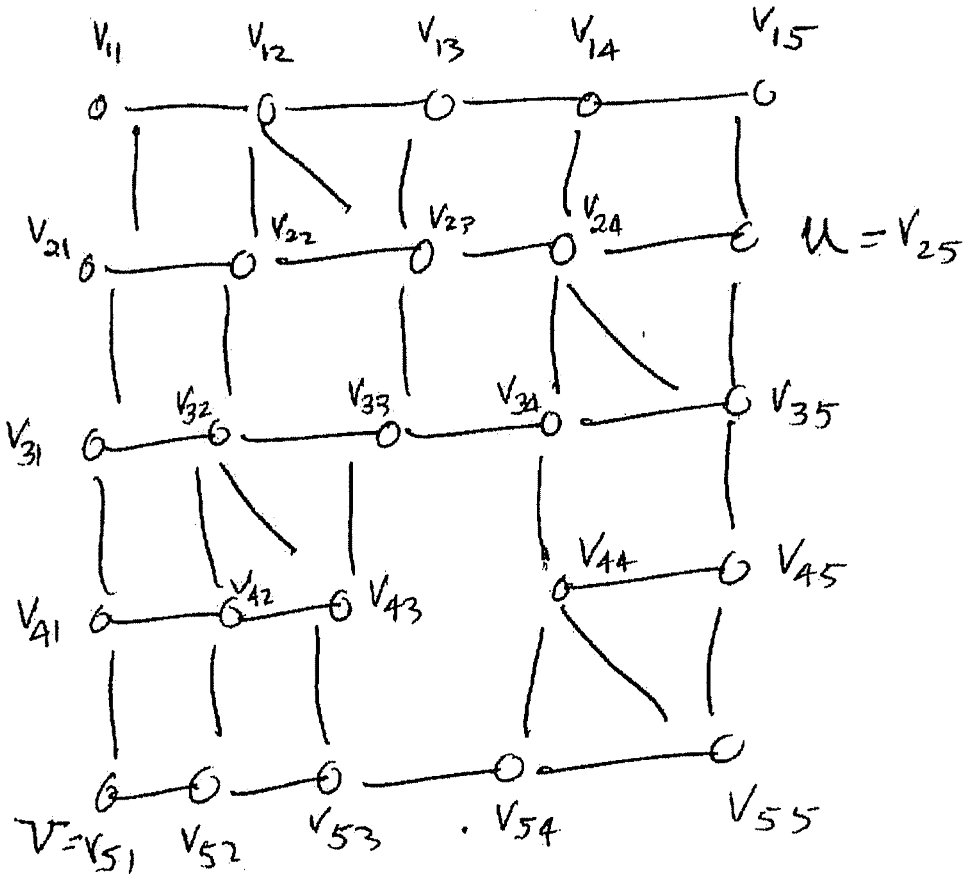


Solution: Note that there are 13 vertices, so any spanning tree will have 12 edges.

One application of Kruskal's algorithm selects the edges $e_1, e_2, \ldots, e_{12}$ in that order. (Other sequences are possible. due to different choices of edges with the same weight.

One application of Prim's algorithm (starting at v) selects the edges $e_{11}, e_5, e_8, e_7, e_6, e_3, e_9, e_{12}, e_2, f_1, e_1, e_{10}$ in that order.

#19 Find a minimal u-v separating set and a maximal set of internally disjoint u-v paths in the following graph. Explain how Menger's theorem can be used to justify your answer



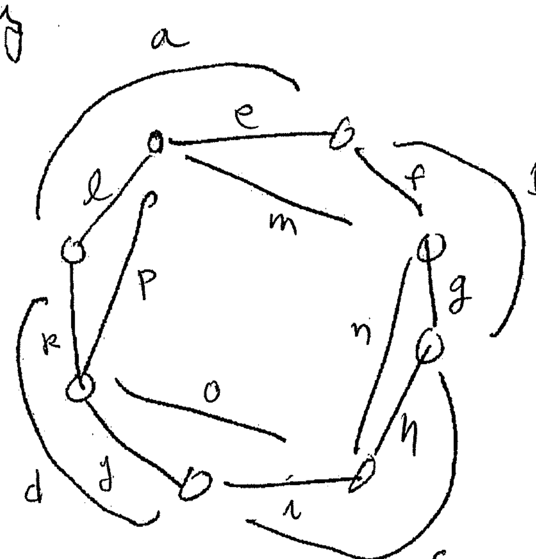
$\{v_{41}, v_{52}\}$ is a u-v separating set

$\{u\, v_{15} v_{14} v_{13} v_{12} v_{11} v_{21} v_{31} v_{41} v,\ u\, v_{35} v_{45} v_{55} v_{54} v_{53} v_{52} v\}$ is a set of 2 internally disjoint u-v paths.

By Menger's Theorem these are respectively minimum and maximum.

#20. In each of the following graphs find an Eulerian circuit if on exists, or an Eulerian trail if one exist. If none exists explain why

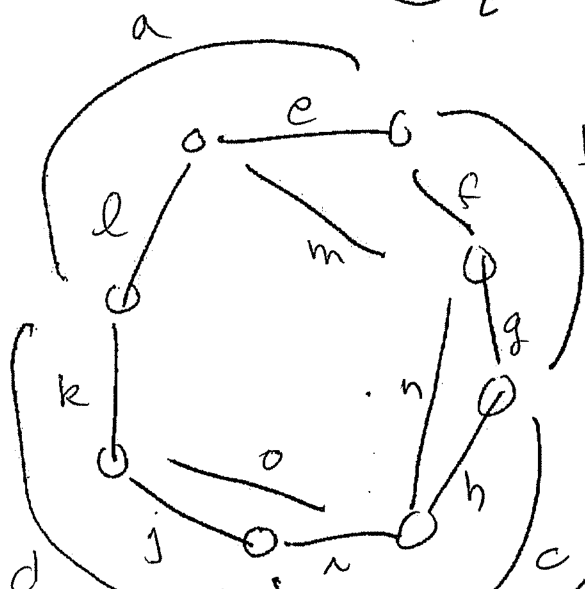
(a)



All vertices have even degree, so there is an Eulerian circuit.

abcdkjihgfemnop is such a circuit

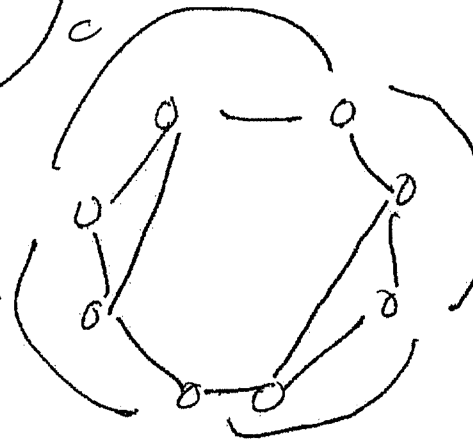
(b)



There are exactly two vertices of odd degree so there is an Eulerian trail

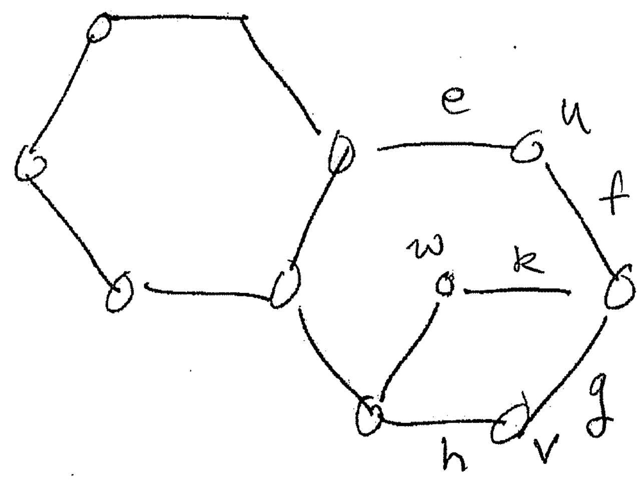
efghijonmlabcdk is such a trail

(c) There are 4 vertices of odd degree, so there is no Eulerian circuit or Eulerian trail.



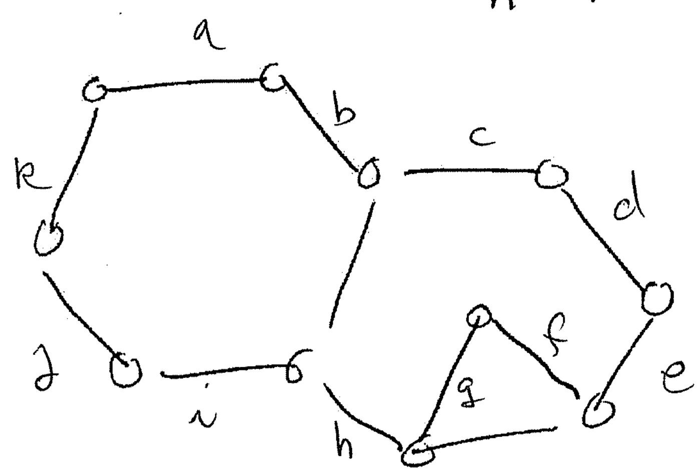
#21 Is each of the following graphs Hamiltonian. Why or why not

(a)



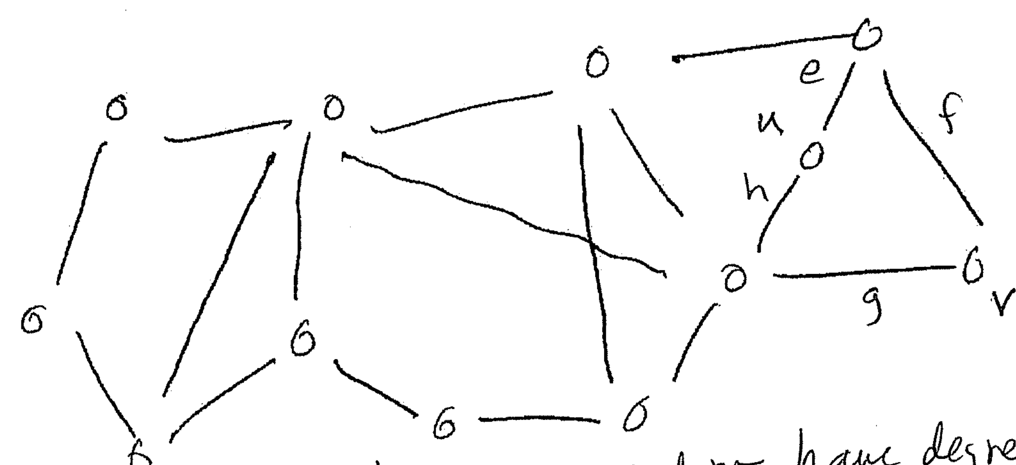
Since deg $u=2$, e and f must be in any Hamiltonian cycle. Similarly deg $v = 2$ so g and h must be in any Hamiltonian cycle, But then k cannot be in a Hamiltonian cycle, so k cannot be. Thus the graph is not Hamiltonian

(b)



This is Hamiltonian: abcdefghijk is a Hamiltonian cycle

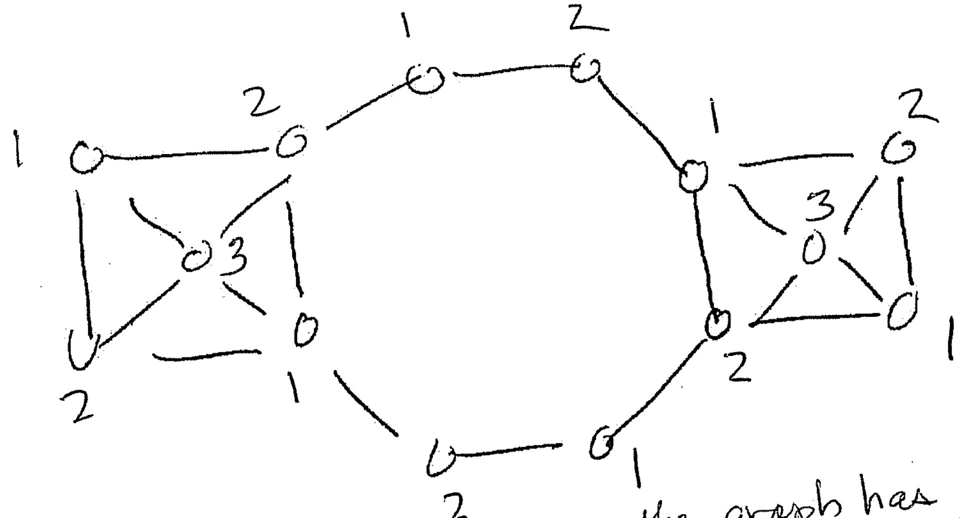
(c)



This is not Hamiltonian: u and v have degree 2 so e, f, g, h must be in any Hamiltonian cycle. But efgh is a cycle.

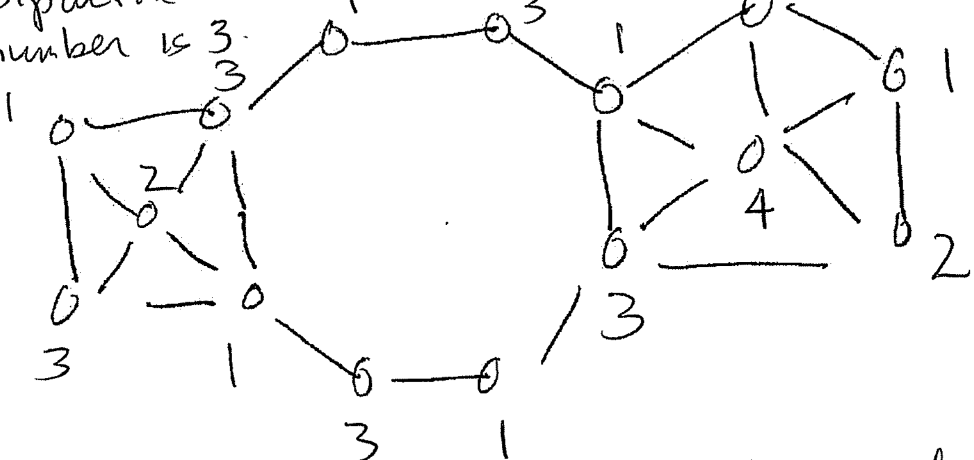
#22 Find the chromatic number of each of the following graphs

(a)



A 3-coloring is shown. Since the graph has 3-cycles it is not bipartite and so has no 2-coloring. Thus the chromatic number is 3.

(b)



A 4-coloring is shown. The (exterior of the) pentagon on the right requires at least 3 colors (as does any odd cycle). Then the vertex in the center must be a 4th color. Thus the chromatic number is 4.