Math 477 - Solutions to review problems - February 22, 2009

#1 In all the examples take the sample space $S = \{1, 2\}$.

(a) Sometimes: if $E = F = \{1, 2\}$ then $P(E \cup F) = P(E) = 1 < P(E) + P(F)$, while if $E = \{1\}, F = \{2\}$ then $P(E \cup F) = 1 = P(E) + P(F)$.

(b) Sometimes: the same examples as in (a) work.

(c) Never: $P(E \cup F) = P(E) + P(F) - P(EF) \leq P(E) + P(F)$.

(d) Sometimes: if $E = F = \{1, 2\}$ then $P(E \cap F) = P(E) = 1 > P(E)P(F)$, while if $E = \{1\}, F = \{2\}$ then $P(E \cap F) = 0 < P(E)P(F)$.

(e) Sometimes: if $E = S$, then $P(E \cap F) = P(F) = P(E)P(F)$ while if $E = F = \{1\}$ then $P(E \cap F) = P(E) = \frac{1}{2} > \frac{1}{4} = P(E)P(F)$.

(f) Sometimes: the same examples as in (e) work.

4 #2 For $k = 1$, the probability that all five women call one particular man is $(\frac{1}{3})^5$. Since there are 3 men, the probability that all five women call the same man is $(\frac{1}{3^4})$.

Identify the men by the numbers $1, 2, 3$ and let E_i denote the event that the i-th man is not called. Then the probability that at most 2 men are called is

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_1E_3) - P(E_2E_3).$$

Since$P(E_i) = (\frac{2}{3})^5$ and, for $i \neq j, P(E_iE_j) = (\frac{1}{3})^5$, we have that the probability that at most 2 men are called is

$$\frac{2^5}{3^4} - (\frac{1}{3})^4 = \frac{31}{81}$$

and so, using the result for $k = 1$, the probability that exactly 2 men are called is

$$\frac{31}{81} - \frac{1}{81} = \frac{30}{81}.$$

Finally, the probability that exactly 3 men are called is 1 minus the sum of the probability that exactly 1 man is called and the probability that exactly 2 men are called.

#3 $P(E) = \frac{1}{6}$, $P(F) = P(G) = \frac{1}{2}$, $P(EF) = \frac{1}{12}, P(EG) = \frac{1}{12}$, and $P(FG) = \frac{1}{4}$. Thus E and F are independent, E and G are independent, and F and G are independent. However $P(EFG) = \frac{1}{12} \neq P(E)P(F)P(G)$ and so E, F and G are not independent.

$4 Let R be the event that the selected ball is red and E_i be the event that it is drawn fro the i-th urn. We want $P(E_3|R)$. Now $P(R|E_i) = \frac{k}{5}$ and $P(RE_i) = P(R|E_i)P(E_i) = (\frac{i}{5})(\frac{1}{5}) = \frac{i}{25}$. Thus $P(R) = P(RE_1) + P(RE_2) + P(RE_3) + P(RE_4) + P(RE_5) = \frac{1+2+3+4+5}{25} = \frac{15}{25}$. Then

$$P(E_3|R) = \frac{P(RE_3)}{P(R)} = \frac{(\frac{3}{5})(\frac{1}{5})}{\frac{15}{25}} = \frac{\frac{3}{25}}{\frac{15}{25}} = \frac{1}{5}.$$

#5 This is really just a more general version of #2. Let E_i be the event that Fred never travels the i-th route. Then the probabiity that he takes each route at least once is

$$1 - P(E_1 \cup E_2 \cup E_3).$$

By inclusion-exclusion this is

$$1 - P(E_1) - P(E_2) - P(E_3) + P(E_1E_2) + P(E_1E_3) + P(E_2E_3).$$

Now $P(E_i) = (\frac{2}{3})^n$ and, for $i \neq j, P(E_iE_j) = (\frac{1}{3})^n$. Thus the required quantity is

$$1 - 3(\frac{2}{3})^n + 3(\frac{1}{3})^n.$$

#6 Let E_i be the event that the number i never appears. Then we want

$$P(E_1 \cup ... \cup E_6) = \sum_{1 \leq i \leq 6} P(E_i) - \sum_{1 \leq i_1 < i_2 \leq 6} P(E_{i_1}E_{i_2}) + \sum_{1 \leq i_1 < i_2 < i_3 \leq 6} P(E_{i_1}E_{i_2}E_{i_3}) -$$

$$\sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 6} P(E_{i_1}E_{i_2}E_{i_3}E_{i_4}) + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 < i_5 \leq 6} P(E_{i_1}E_{i_2}E_{i_3}E_{i_4}E_{i_5}) =$$

$$6(\frac{5}{6})^n - 15(\frac{4}{6})^n + 20(\frac{3}{6})^n - 15(\frac{2}{6})^n + 6(\frac{1}{6})^n.$$

#7 The definition of G is omitted so the problem cannot be completed.

#8 (a) Seat the first man and assign his seat the number 1. Number the remaining proceeding clockwise. Then no two men are in adjacent seats if and only if all the men are in odd numbered seats. Since the number of men still to be seated is $n-1$ the probability of this happening is

$$(\frac{n-1}{2n-1})(\frac{n-2}{2n-2})...\frac{1}{n+1}$$

and so the probability that at least one pair of men are in adjacent seats is

$$1 - \frac{n}{\binom{2n-1}{n}}.$$

(b) Seat the married woman. Then there are two seats adjacent to her and $2n-1$ vacant seats. Thus the probability that her husband is adjacent to her is $\frac{2}{2n-1}$. Similarly, there is one seat opposite her, so the probability that her husband is opposite to her is $\frac{1}{2n-1}$.

#9 Let E_i be the event that the had contains no cards of the i-th suit. Then the probability that a hand contains a card from every suit is

$$1 - P(E_1 \cup E_2 \cup E_3 \cup E_4) =$$

$$1 - \sum_{1 \le i \le 4} P(E_i) + \sum_{1 \le i_1 < i_2 \le 4} P(E_{i_1} E_{i_2}) - \sum_{1 \le i_1 < i_2 < i_3 \le 4} P(E_{i_1} E_{i_2} E_{i_3}) =$$

$$1 - 4\frac{\binom{39}{5}}{\binom{52}{5}} + 6\frac{\binom{26}{5}}{\binom{52}{5}} + 4\frac{\binom{13}{5}}{\binom{52}{5}}.$$

If at least two cards are hearts. Then we need to choose 3 cards from among the remaining 50 in such a way that each remaining suit occurs once. The probability that the first of these cards is not a heart is $\frac{39}{50}$. Then the probability the second of these card is from a new suit is $\frac{26}{49}$ and the probability that the 3-rd card is from the remaining suit is $\frac{13}{48}$. Thus the probability that all four suits occur in the hand is

$$(\frac{39}{50})(\frac{26}{49})(\frac{13}{48}).$$

#10 This will be added later.

#11 (a)

$$P(X = k) = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}};$$

and

$$P(Y = k) = \binom{5}{k}(\frac{4}{52})^k(\frac{48}{52})^{5-k}$$

#12 $P(X = 2) = \frac{1}{6}, P(X = 3) = \frac{1}{3}, P(X = 4) = \frac{1}{2}$. Thus $E[X] = 2\frac{1}{6} + 3\frac{1}{3} + 4\frac{1}{2} = \frac{10}{3}$. We have not discussed the cumulative distribution function. $E[X^2] = 2^2\frac{1}{6} + 3^2\frac{1}{3} + 4^2\frac{1}{2} = \frac{10}{3} = \frac{35}{3}$, and so

$$Var(X) = E[x^2] - E[X]^2 = \frac{5}{9}.$$