Math 477-02 - Solutions to Review problems for Exam #2 - April 15, 2009
Review Session: Monday, April 13, 6:30-8:30 M in SEC-211 (BUSCH)

Calculators may not be used on the exam. You will be given a sheet containing a copy of table 5.1 of the the text and the following formulas:
Binomial: $P\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, ..., n. E[X] = np, Var(X) = np(1-p)$.
Geometric: $P\{X = k\} = p(1-p)^{k-1}, k = 1, 2, ... E[x] = \frac{1}{p}, Var(X) = \frac{(1-p)}{p^2}$.
Poisson: $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}, k = 0, 1, 2, ... E[X] = \lambda, Var(X) = \lambda$.
Exponential: $f_X(x) = \lambda e^{-\lambda x}, x \geq 0, E[X] = \frac{1}{\lambda}, Var(X) = \frac{1}{\lambda^2}$.
Normal: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} . E[x] = \mu, Var(X) = \sigma^2$.

#1 A continuous random variable X has density

$$f_X(x) = cx, \quad if \quad 0 \leq x \leq 1,$$

$$f_X(x) = 0, \quad if x < 0 \quad or \quad x > 1$$

for some constant c.
(a) Find c.
(b) Find $P\{X \geq \frac{1}{3}\}$ and $P\{X = \frac{1}{3}\}$.
(c) Find $E[X]$ and $Var(X)$.
Now assume that Y is a second continuous random variable which is uniforly distributed on the interval $[0, 1]$ an that X and Y are independent.
(d) Find the joint density function $f(x, y)$ being careful to specify where $f(x, y) = 0$ and giving its value where it is non-zero.
(e) Find $P\{X \geq Y\}$

Solution: (a) Since

$$1 = \int_{-\infty}^{\infty} f_X(x)dx = \int_0^1 cxdx =$$

$$[\frac{cx^2}{2}]_0^1 = \frac{c}{2}$$

we have $c = 2$.
(b) $P\{X \geq \frac{1}{3}\} = \int_{\frac{1}{3}}^1 2xdx = \frac{8}{9}$ and $P\{X = \frac{1}{3}\} = \int_{\frac{1}{3}}^{\frac{1}{3}} 2xdx = 0$.
(c) $E[X] = \int_0^1 x(2x)dx = \frac{2}{3}$, and $E[X^2] = \int_0^1 x^2(2x)dx = \frac{1}{2}$. Thus $Var(X) = E[x^2] - E[X]^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.
(d) $F_{X,Y}(x, y) = 2x$ if $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and is equal to 0 otherwise.
(e) $P\{X \geq Y\} = \int_0^1 \int_{y=0}^{y=x} 2xdydx = \int_0^1 [2xy]_{y=0}^{y=x} dx = \int_0^1 2x^2 dx = [\frac{2x^3}{3}]_0^1 = \frac{2}{3}$.

#2 Ellen plays a game in which her chance of winning is $\frac{1}{5}$. Using a normal approximation, estimate the probability of her winning exactly 25 times if she plays the game 100 times.

Solution: The exact solution comes from the binomial distribution with $n = 100, p = \frac{1}{5}$. Thus the mean is $np = 20$ and the variance is $np(1-p) = 16$. We may approximate this by a normal random variable Y with $\mu = 20, \sigma^2 = 16$ and approximate the required quantity by

$$P\{24.5 < Y < 25.5\} = P\{25.5 < Y\} - P(\{24.5 < Y)\}.$$

Now $Z = (Y - 20)/4$ is a standard normal random variable (mean 0, variance 1) and the quantity we want is

$$P\{1.375 < Z\} - P\{1.125 < Z\}.$$

Taking values from Table 5.1, we see that this isapproximately 0.0457.

#3 Alex, Bruce and Charlie are playing darts using the disk $x^2 + y^2 \leq 4$ as the target. They always hit the target, and the x and y components of their impact points have the following joint distributions ,denoted f_A, f_B, f_C respectively:

$$f_A(x, y) = c_A(4 - x^2 - y^2),$$

$$f_B(x, y) = c_B,$$

$$f_C(x, y) = c_C(x^2 + y^2).$$

(a) Find c_A, c_B, c_C. (You might want to use polar coordinates for Alex and Charlie.)

(b) The game is scored by giving 4 points for a hit inside the circle $x^2 + y^2 = 1$ and 1 point for a hit outside that circle. What is the expected value of the number of points scored by each player on a throw.

Solution: (a) Since

$$1 = \int\int_{x^2+y^2\leq 4} c_A(4 - x^2 - y^2)dxdy = c_A \int_{r=0}^{2}\int_{\theta=0}^{2\pi}(4 - r^2)rdrd\theta = 8\pi c_A$$

we have $c_A = \frac{1}{8\pi}$. Similarly,

$$1 = \int\int_{x^2+y^2\leq 4} c_B dxdy = c_B \int_{r=0}^{2}\int_{\theta=0}^{2\pi} rdrd\theta = 4\pi c_B$$

so $c_B = \frac{1}{4\pi}$ and

$$1 = \int\int_{x^2+y^2\leq 4} c_C(x^2+y^2)dxdy = c_B\int_{r=0}^{2}\int_{\theta=0}^{2\pi}(r^2)rdrd\theta = 8\pi c_C$$

so $c_C = \frac{1}{8\pi}$.

(b) Let D stand for either A, B or C. Then the expected value for the number of points scored by player D is

$$\int\int_{x^2+y^2\leq 1} 4f_D(x,y)dxdy + \int\int_{1\leq x^2+y^2\leq 4} f_D(x,y)dxdy.$$

Evaluating the integrals (again using polar coordinates) gives the expected values $\frac{37}{16}$ for Alex, $\frac{7}{4}$ for Bruce and $\frac{19}{16}$ for Charlie.

#4 A certain transistor has lifetime T, where T is a positive random variable, measured in days, with density$f(t) = Kte^{-2t}$.

(a) What is K?

(b) Suppose it is known that the component has lasted s days. What is the probability that it will last two more days?

Solution: (a) Using integration by parts we see that the

$$\int te^{-2t}dt = -\frac{(2t+1)e^{-2t}}{4} + C$$

(where C is an arbitrary constant). Then we have

$$1 = \int_0^\infty f(t)dt = K\int_0^\infty e^{-2t}dt = K[-\frac{(2t+1)e^{-2t}}{4}]_0^\infty = \frac{K}{4}.$$

Thus $K = 4$.

b) This is the conditional probability

$$P\{T\geq s+2|T\geq s\} = \frac{P\{T\geq s+2, T\geq s\}}{P\{T\geq s\}} =$$

$$\frac{P\{T\geq s+2\}}{P\{T\geq s\}} = \frac{\int_{t=s+2}^\infty 4te^{-2t}dt}{\int_{t=s}^\infty 4te^{-2t}dt} = \frac{2s+5}{2s+1}e^{-4}.$$

#5 Let X be a binomial random variable with $n = 3$ and $p = .5$ and let Y be a geometric random variable with parameter $p = .5$. Suppose that X and Y are independent.

(a) Give the values $p(x,y)$ of the joint probability mass function of X and Y for all x,y satisfying $0 \leq x \leq 3, 1 \leq y \leq 4$.
(b) Find $P\{X < Y\}$.
(c) Find the probability mass function for $X+Y$.

Solution: (a) $P\{X=i, Y=j\} = \binom{3}{i}(.5)^{k+3}$.
(b)

$$P\{X < Y\} = \sum_{j=1}^{\infty} P\{X=0, Y=j\} + \sum_{j=2}^{\infty} P\{X=1, Y=j\} +$$

$$\sum_{j=3}^{\infty} P\{X=2, Y=j\} + \sum_{j=4}^{\infty} P\{X=3, Y=j\} =$$

$$(.5)^3 + 3(.5)^4 + 3(.5)^5 + (.5)^6 = (8+12+6+1)/64 = 27/64.$$

(c)

$$P\{X+Y=1\} = P\{X=0, Y=1\} = (.5)^4 = 1/16,$$

$$P\{X+Y=2\} = P\{X=0, Y=2\} + P\{X=1, Y=1\} = (.5)^3 + 3(.5)^4 = 7/32,$$

$$P\{X+Y=3\} = P\{X=0, Y=3\} + P\{X=1, Y=2\} + P\{X=2, Y=0\} =$$

$$(.5)^6 + 3(.5)^5 + 3(.5)^4 = 19/64,$$

and, for $k \geq 4$,

$$P\{X=0, Y=k\} + P\{X=1, Y=k-1\} + P\{X=2, Y=k-2\} + P\{X=3, Y=k-3\} =$$

$$(.5)^{k+3} + 3(.5)^{k+2} + 3(.5)^{k+1} + (.5)^k = (.5)^{k+3}(1+6+12+8) = 27(.5)^{k+3}.$$

#6 Cars pass a certain point on a road according to a Poisson process, with an average rate of 3 cars per hour. Find the probability that an observer wil see exacty three cars pass in one hour of observation
(a) if three cars pass in the first half hour;
(b) if no cars pass in the first half hour.

Solution: The average number of cars passing in a half hour is $\frac{3}{2}$ and so, if X is the number of cars passing one half hour, then

$$PX = k = (\frac{3}{2})^k e^{-\frac{3}{2}}/k!.$$

Setting $k=0$ gives the answer $e^{-\frac{3}{2}}$ for (a), and setting $k=3$ gives the answer $(\frac{9}{16})e^{-\frac{3}{2}}$ for (b).

#7 By definition, a "hundred year flood" on a river is a flood which is so severe that it happens, on the average, once every hundred years. Find the probability that there will be exactly three hundred year floods on the Raritan River between 2010 and 2159 (inclusive), both exactly and by using a suitable Poisson approximation. Assume that at most one such flood can occur in any year and thet floods in different years are independent.

Solution: The average number of hundred year floods in a 150 year period is $\frac{3}{2}$. Thus, if X is the number of hundred year floods in a hundred year period, the Poisson approximation gives

$$P\{X=k\} = (\frac{3}{2})^k e^{-\frac{3}{2}}/k!.$$

The desired probability is then $P\{X=3\} = (\frac{9}{16})e^{-\frac{3}{2}}$. The exact value, using the binomial distribution is

$$\binom{150}{3}(.01)^3(.99)^{147}.$$

#8 Let X and Y be independent random variables, exponentially distributed with parameters λ and μ respectively.

(a) Find $P\{X > 2Y\}$.

(b) Find the probability density for the random variable $Z = X + Y$.

Solution: (a) $\mu/(\mu+2\lambda)$; (b) $\mu\lambda(e^{-\mu z} - e^{-\lambda z})/(\lambda-\mu)$.

#9 A total of n balls, numbered $1, 2, ..., n$ are put into n urns, also numbered $1, 2, ..., n$ in such a way that ball number i is equally likely to go into any one of the urns numbered $1, 2, ..., i$. Find:

(a) the expected number of urns that are empty;

(b) the probability that no urn is empty.

Solution: (a) Let $X_{i,j} = 1$ if the i-th ball is the first ball placed in the j-th urn and $X_{i,j} = 0$ otherwise. Then $\sum_{1\le j\le i\le n} X_{i,j}$ is the number of non-empty urns and so

$$\sum_{1\le j\le i\le n} E[X_{i,j}]$$

is the expected number of non-empty urns. Now $E[X_{1,1}] = 1$ while if $i > 1$ and $i \geq j \geq 1$ we have

$$E[X_{i,j}] = (\frac{j-1}{j})(\frac{j}{j+1})...(\frac{i-2}{i-1})(\frac{1}{i}) = \frac{j-1}{i(i-1)}.$$

Thus for $i > 1$ we have

$$\sum_{j=1}^{i} E[X_{i,j} = \frac{1}{i(i-1)} \sum_{j=1}^{i} j - 1 = \frac{1}{i(i-1)} \frac{i(i-1)}{2} = \frac{1}{2}.$$

Thus the expected number of non-empty urns is

$$\sum_{i=1}^{n} \sum_{j=1}^{i} E[X_{i,j}] = 1 + \sum_{i=2}^{n} \frac{1}{2} = 1 + \frac{(n-1)}{2} = \frac{n+1}{2}$$

and so the expected number of empty urns is $\frac{n-1}{2}$.

(b) If no urn is empty, then the i-th urn must contain the i-th ball. The probability of this is $\frac{1}{n!)}$.

#10 An entomologist is catching mosquitos in certain region which is inhabited by r distinct types of mosquitos. Each mosquito caught will, independently of the types of the previous catches, be of type i with probability P_i (where, of course, $\sum_{1 \leq i \leq r} P_i = 1$).

(a) Compute the mean number of mosquitos that are caught before the first type 1 catch.

(b) Compute the mean number of types of insects that are caught before the first type 1 catch.

Solution: (a) Let $X_i = 1$ if the i-th mosquito caught is first type 1 mosquito to be caught. Then the nuber of osquitos caught before the first type 1 mosquito is caught is

$$\sum_{i=1}^{\infty} (i-1) X_i$$

and so the expected number of mosquitos caught before the first type 1 mosquito is caught is

$$\sum_{i=1}^{\infty} (i-1) E[X_i].$$

Now $E[X_i] = (1-p_1)^{i-1} p_1$ and so (using $\sum_{i=0}^{\infty} i a^i = \frac{a}{(1-a)^2}$) we see that the expected number of mosquitos caught before the first type 1 mosquito is caught is

$$\frac{1-p_1}{p_1}.$$

(b) For $2 \le j \le r$, let $X_{i,j} = 1$ if the i-th mosquito is the first type j mosquito to be caughr and no type 1 mosquito has been caught yet. Then the number of types of mosquitos caught before the first type 1 mosquito is caught is $\sum_{j=2}^{r} \sum_{i=1}^{\infty} X_{i,j}$ and so the mean number of types caught before the first type 1 catch is

$$\sum_{j=2}^{r} \sum_{i=1}^{\infty} E[X_{i,j}].$$

Now $E[X_{i,j}] = (1 - p_1 - p_j)^{i-1} p_j$ and so

$$\sum_{i=1}^{\infty} E[X_{i,j}] = \frac{p_j}{p_1 + p_j}.$$

Thus the mean number of types of mosquitos caught before the first type 1 mosquito is caught is

$$\sum_{j=2}^{r} \frac{p_j}{p_1 + p_j}.$$