Math 477-02 - REVIEW PROBLEMS FOR FINAL EXAM - April 2009
REVIEW SESSION: WEDNESDAY, MAY 6, 4-7 PM in ARC-110 (BUSCH)

Calculators may not be used on the exam. You will be given a sheet containing a copy of table 5.1 of the the text and the following formulas:

Binomial: $P\{X = k\} = \binom{n}{k}p^k(1-p)^{n-k}, k = 0, 1, ..., n.$
$E[X] = np, Var(X) = np(1-p), M_X(t) = (pe^t + 1 - p)^n$

Geometric: $P\{X = k\} = p(1-p)^{k-1}, k = 1, 2, ...$
$E[x] = \frac{1}{p}, Var(X) = \frac{(1-p)}{p^2}, M_X(t) = \frac{pe^t}{1-(1-p)e^t}.$

Poisson: $P\{X = k\} = \frac{\lambda^k}{k!}, k = 0, 1, 2, ...$
$E[X] = \lambda, Var(X) = \lambda, M_X(t) = e^{\lambda(e^t-1)}.$

Uniform: $f_X(x) = \frac{1}{b-a}, a \le x \le b.$
$E[X] = \frac{a+b}{2}, Var(X) = \frac{(b-a)^2}{2}, M_X(t) = \frac{e^{bt}-e^{at}}{b-a}.$

Exponential: $f_X(x) = \lambda e^{-\lambda x}, x \ge 0,$
$E[X] = \frac{1}{\lambda}, Var(X) = \frac{1}{\lambda^2}, M_X(t) = \frac{\lambda}{\lambda-t}$

Normal: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$
$E[x] = \mu, Var(X) = \sigma^2, M_X(t) = e^{\mu t + \sigma^2 t^2/2}.$

Moment generating function formulas:
$M_X(t) = E[e^{tX}], M_{a+bX}(t) = e^{at}M_X(bt), E[X^n] = M_X^{(n)}(0).$

#1 A urn contains $r = r_1 + r_2$ packages of candy; of these, r_1 contain one piece and r_2 contain three pieces.

(a) If a package were to be drawn from the urn at random, what is the probability it would contain one piece of candy? Three pieces? If two packages were drawn (without repalcement), what are the probabilities of obtaining altogether two, four, or six pieces of candy?

(b) Suppose that n packages (where $n \le r$) are drawn from the urn, one at a time, without replacement. Let X_i be the number of pieces of candy in the ith package drawn. Find $E[X_i], Var(X_i)$, and $Cov(X_i, X_j)$ for $i \ne j$.

(c) Suppose that n packages (where $n \le r$) are drawn from the urn, one at a time, without replacement. Let X be the total number of pieces of candy obtained. Find $E[X]$

and $Var(X)$.

#2 Let X and Y be continuous random variables with joint density

$$f(x, y) = cx^2 ye^{-xy}, \quad if \quad 1 \le x \le 2 \quad and \quad y \ge 0,$$

$$f(x, y) = 0, \quad otherwise.$$

(a) If we know that X takes the value $3/2$, what is the (conditional) distribution of Y? (You can just write down the answer.)

(b) Find c.

(c) Find $E[X], E[Y]$, and $Cov(X, Y)$.

#3 (a) Alex and Bill play a game in which each, independently, chooses a real number between 0 and 2. Alex wins if his choice is at least twice Bill's. Suppose that both choose "at random", so that their choices are uniformly distributed on the interval $[0, 2]$. What is the probability that Alex wins?

(b) Suppose that each time the game is played Bill contributes $2.00 and Alex contributes $1.00 to a pot, with the winer of the game collecting the entire $3.00, and that they play the game 100 times in this fashion. Estimate the probability that Bill comes out at least $35.00 ahead. Explain how your approximation depends on the central limit theorem.

#4 Charlie and Don play a game using a tetrahedral die. This die has four sides; when it is rolled it rests on one side which we will refer to as the resulting side. All four sides are equally likely to occur as the resulting side. (Such "dice" actually exist and are used in various games including generalizations of "Dungeons and Dragons".) Three sides of the die are painted blue and one is painted red. Charlie rolls the die three times and then Don does also. The one whose rolls result in the most red sides (out of his three rolls) wins.

(a) What is the probability that the two players have a total of at least 5 resulting red sides?

(b) What is the probability that a tie occurs?

(c) What is the probability that Don has already won the game after his first roll?

#5 Three urns are numbered 1 through 3; urn k contains k balls numbered 1 through k. We select an urn at random, draw a ball from it, note the number of the ball, replace the ball, and then drqw again from the same urn. If it is known that the first ball drawn has number 1, find

(a) the probability mass function of the number of the selected urn;

(b) the expected value of the number of the second ball drawn.

#6 Let X and Y be independent continuous random variables with densities

$$f_X(x) = e^{-x}, \quad if \ \ x \geq 0,$$

$$f_X(x) = 0, \quad otherwise,$$

$$f_Y(y) = 3e^{-3y}, \quad if \ \ y \geq 0,$$

$$f_Y(y) = 0, \quad otherwise.$$

Find the density function $f_Z(z)$ for $Z = X + Y$.

#7 Let $X_1, ..., X_{25}$ be idependent random variables, each of which is uniformly distributed on the interval $[0, 2]$, and let $X = \sum_{i=1}^{25} X_i$.
  (a) Find the mean and variance of X.
  (b) use the central limit theorem to estimate the prbability that $|X - 24| \geq 3$.
  (c) Use Chebyshev's inequality to find a number a such that you are absolutely sure that $P\{|X - E[x]|/ge1\} \leq 0.1$.

#8 A single (ordinary) die is rolled. Denote the number shown by X. Then another (ordinary, cubical) die has X of its sides painted green and the rest painted yellow. Thus the probability that a green side shows when the second die is rolled is $X/6$. The second die is then rolled. Let $Y = 1$ if a green side shows, and $Y = 0$ otherwise.
  (a) Find the joint probability mass function, and the marginal probability mass functions of X and Y.
  (b) Find $E[X], E[Y], Var(X), Var(Y)$, and $Cov(X, Y)$.

#9 A modified form of poker is played with an ordinary deck of 52 playing cards, but the hands contain six cards. Calculate the probability that a randomly dealt hand contains:
  (i) four cards of the same rank and two from different ranks;
  (ii) three pairs, each of a different rank;
  (iii) two triples, each containing three cards of the same rank;
  (iv) only two ranks;
  (v) six cards in sequence;
  (vi) six cards, all of different ranks,
  (vii) a triple, a pair, and a single card, all of different ranks.

#10 (a) A hat contains n cards numbred 1 to n; one card is drawn at random and its number is denoted X. Calculate the moment generating function $M_X(t)$ of X.
  (b) If the experiment is repeated k times (with repalcement) and Y is the sum of the numbers of all cards drawn, find $M_Y(t)$.

#11 (a) State Markov's inequality as a theorem, defining all symbols used and including all necessary hypotheses.

(b) State Chebyshev's inequality as a theorem, defining all symbols used and including all necessary hypotheses.

#12 The Amalgamated Widgets Company has just declared bankruptcy. An honest investor would have had only a 20% chance of selling her stock in time to come out without loss, but there is a 70% chance that a dishonest investor would have had inside information and gotten out in time. If we know that Smith didn't lose any money, what is the probabilty that Smith is honesst.

#13 A large urn contains N balls of each of 20 different colors (that is, a total of $20N$ balls). 10 balls are selected at random; we let X be the total number of different colors obtained, and write $X = \sum_{i=1}^{20} X_i$ with X_i a Bernoulli random variable indicating whether or not the i-th color is obtained.

(a) Assume $N = 1$. Find $E[X]$ and $Var(X)$ by elementary reasoning.

(b) Let N be arbitrary and suppppose that the selection is without replacement. Find $E[x]$ and $Var(X)$ and show that your results are consistent with part (a). (Hint: Compute $E[X_i]$ and $E[X_iX_j]$.)

(c) Suppose that the selection is with replacement. Find $E[X]$ and $Var(X)$. Note that your answer should not depend on N.

(d) Show that the answers in part (c) are the $N \to \infty$ limit of your answers in part (b). Explain why this should be true.

#14 Each morning Ed makes a random choice of one of three routes to take to work. After n trips, (n ¿ 0), what is the probability that he has travele each route at least once?

#15 Five balls are randomly chosen, without replacement from an urn that contains 5 red, 6 white, and 7 blue balls. Find the probbility that at least one ball of each color is chosen.

#16 Player A flips a fair coin $n + 1$ times and player B flips a fair coin n times. Find the probability that A gets more heads than B.

#17 There are two factories that produce radios. Each radio produced at factory A is defective with probability .05, whereas each one produced at factory B is defective with probability .01. Suppose you purchase two radios that were produced at the same factory, which is equally likely to have been either factory A or factory B. If the first radio that you check is defective, what is the conditional probability that the other one is also defective?

#18 A urn contains n balls numbered $1, ..., n$. Suppose that a boy draws a ball from the urn, notes the number and returns the ball to the urn before drawing another ball. This

continues until the boy draws a ball he has previously drawn. Let X denote the number of draws. Find the probability mass function of X.