Math 477-02 - REVIEW PROBLEMS FOR FINAL EXAM - April 2009
REVIEW SESSION: WEDNESDAY, MAY 6, 4-7 PM in ARC-110 (BUSCH)

Calculators may not be used on the exam. You will be given a sheet containing a copy of table 5.1 of the the text and the following formulas:

Binomial: $P\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, ..., n.$
$E[X] = np, Var(X) = np(1-p), M_X(t) = (pe^t + 1 - p)^n$

Geometric: $P\{X = k\} = p(1-p)^{k-1}, k = 1, 2, ...$
$E[x] = \frac{1}{p}, Var(X) = \frac{(1-p)}{p^2}, M_X(t) = \frac{pe^t}{1-(1-p)e^t}.$

Poisson: $P\{X = k\} = \frac{\lambda^k}{k!}, k = 0, 1, 2, ...$
$E[X] = \lambda, Var(X) = \lambda, M_X(t) = e^{\lambda(e^t-1)}.$

Uniform: $f_X(x) = \frac{1}{b-a}, a \leq x \leq b.$
$E[X] = \frac{a+b}{2}, Var(X) = \frac{(b-a)^2}{2}, M_X(t) = \frac{e^{bt}-e^{at}}{b-a}.$

Exponential: $f_X(x) = \lambda e^{-\lambda x}, x \geq 0,$
$E[X] = \frac{1}{\lambda}, Var(X) = \frac{1}{\lambda^2}, M_X(t) = \frac{\lambda}{\lambda-t}$

Normal: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$
$E[x] = \mu, Var(X) = \sigma^2, M_X(t) = e^{\mu t + \sigma^2 t^2/2}.$

Moment generating function formulas:
$M_X(t) = E[e^{tX}], M_{a+bX}(t) = e^{at} M_X(bt), E[X^n] = M_X^{(n)}(0).$

#1 A urn contains $r = r_1 + r_2$ packages of candy; of these, r_1 contain one piece and r_2 contain three pieces.

(a) If a package were to be drawn from the urn at random, what is the probability it would contain one piece of candy? Three pieces? If two packages were drawn (without repalcement), what are the probabilities of obtaining altogether two, four, or six pieces of candy?

Solution: $\frac{r_1}{r}; \frac{r_2}{r}; \frac{r_1(r_1-1)}{r(r-1)}; 2\frac{r_1 r_2}{r(r-1)}; \frac{r_2(r_2-1)}{r(r-1)}.$

(b) Suppose that n packages (where $n \leq r$) are drawn from the urn, one at a time, without replacement. Let X_i be the number of pieces of candy in the ith package drawn. Find $E[X_i], Var(X_i)$, and $Cov(X_i, X_j)$ for $i \neq j$.

Solution:

$$E[X_i] = (\frac{r_1}{r}) + 3(\frac{r_2}{r}) = \frac{r_1 + 3r_2}{r};$$

$$E[X_i^2] = (\frac{r_1}{r}) + 9(\frac{r_2}{r}) = \frac{r_1 + 9r_2}{r};$$

$$Var(X_i) = E[X_i^2] - E[X_i]^2 = \frac{r_1 + 9r_2}{r} - (\frac{r_1 + 3r_2}{r})^2 =$$

$$\frac{(r_1 + 9r_2)(r_1 + r_2)}{r^2} - (\frac{r_1 + 3r_2}{r})^2 =$$

$$\frac{4r_1 r_2}{r^2}.$$

Using the results from (a), we see that

$$E[X_i X_j] = \frac{r_1(r_1 - 1)}{r(r-1)} + 6\frac{r_1 r_2}{r(r-1)} + 9\frac{r_2(r_2 - 1)}{r(r-1)}.$$

Then

$$Cov(X_i X_j) = E[X_i X_j] - E[X_i]E[X_j] =$$

$$\frac{r_1(r_1 - 1) + 6r_1 r_2 + 9r_2(r_2 - 1)}{r(r-1)} - \frac{(r_1 + 3r_2)^2}{r^2} =$$

$$\frac{r(r_1^2 + 6r_1 r_2 + 9r_2^2) - r(r_1 + 9r_2) - r(r_1 + 3r_2)^2 + (r_1 + 3r_2)^2}{r^2(r-1)} =$$

$$\frac{-(r_1 + r_2)(r_1 + 9r_2) + (r_1 + 3r_2)^2}{r^2(r-1)} = \frac{-4r_1 r_2}{r^2(r-1)}.$$

(c) Suppose that n packages (where $n \le r$) are drawn from the urn, one at a time, without replacement. Let X be the total number of pieces of candy obtained. Find $E[X]$ and $Var(X)$.

Solution:

$$E[X] = \sum_{i=1}^{n} E[X_i] = \frac{n(r_1 + 3r_2)}{r};$$

$$Var(X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j) =$$

$$\frac{4nr_1 r_2}{r^2} - \frac{n(n-1)4r_1 r_2}{r^2(r-1)} = \frac{4nr_1 r_2(r - 1 - (n-1))}{r^2(r-1)} =$$

$$\frac{4nr_1r_2(r-n)}{r^2(r-1)}.$$

#2 Let X and Y be continuous random variables with joint density

$$f(x,y) = cx^2ye^{-xy}, \quad if \quad 1 \le x \le 2 \quad and \quad y \ge 0,$$

$$f(x,y) = 0, \quad otherwise.$$

(a) If we know that X takes the value 3/2, what is the (conditional) distribution of Y? (You can just write down the answer.)

Solution: We have $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$. Now $f_X(x) = \int_{y=0}^{\infty} cx^2ye^{-xy}dy$. Using integration by parts we see that this is $cx^2[-\frac{ye^{-xy}}{x} - \frac{e^{-xy}}{x^2}]_{y=0}^{\infty} = cx^2[\frac{1}{x^2}] = c$. Thus $f_{Y|X}(y|x) = x^2ye^{-xy}$ and so $f_{Y|X}(y|\frac{3}{2}) = (\frac{3}{2})^2ye^{-\frac{3y}{2}}$. Then the cumulative distribution is

$$\int_0^a (\frac{3}{2})^2ye^{-\frac{3y}{2}}dy.$$

(b) Find c.

Solution: Using integration by parts

$$\int_1^2\int_0^{\infty} cx^2ye^{-xy}dydx = c\int_1^2 x^2[-\frac{ye^{-xy}}{x} - \frac{e^{-xy}}{x^2}]_{y=0}^{\infty}dx =$$

$$c\int_1^2 dx = c.$$

Thus $c = 1$.

(c) Find $E[X], E[Y]$, and $Cov(X,Y)$.

Solution:

$$E[X] = \int_1^2\int_0^{\infty} x^3ye^{-xy}dydx = \int_1^2 x^3[-\frac{ye^{-xy}}{x} - \frac{e^{-xy}}{x^2}]_{y=0}^{\infty}dx =$$

$$\int_1^2 xdx = [\frac{x^2}{2}]_1^2 = \frac{3}{2}.$$

$$E[Y] = \int_1^2\int_0^{\infty} x^2y^2e^{-xy}dydx =$$

$$\int_1^2 x^2[-\frac{y^2e^{-xy}}{x} - \frac{2ye^{-xy}}{x^2} - \frac{2e^{-xy}}{x^3}]_{y=0}^{\infty}dx =$$

$$\int_1^2 x^2[\frac{2}{x^3}dx = \int_1^2 \frac{2}{x}dx = 2\ln(2).$$

$$E[XY] = \int_1^2 \int_0^{\infty} x^3y^2e^{-xy}dydx =$$

$$\int_1^2 x^3[-\frac{y^2e^{-xy}}{x} - \frac{2ye^{-xy}}{x^2} - \frac{2e^{-xy}}{x^3}]_{y=0}^{\infty}dx =$$

$$\int_1^2 x^3[\frac{2}{x^3}dx = \int_1^2 2dx = 2.$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 2 - 3\ln(2).$$

#3 (a) Alex and Bill play a game in which each, independently, chooses a real number between 0 and 2. Alex wins if his choice is at least twice Bill's. Suppose that both choose "at random", so that their choices are uniformly distributed on the interval $[0,2]$. What is the probability that Alex wins?

Solution: Let X denote Alex's number, Y denote Bill's. Then the probability that Alex wins is

$$P\{X \geq 2Y\} = \int_{x=0}^2 \int_{y=0}^{\frac{x}{2}} \frac{1}{4}dydx = \frac{1}{4}.$$

(b) Suppose that each time the game is played Bill contributes \$2.00 and Alex contributes \$1.00 to a pot, with the winer of the game collecting the entire \$3.00, and that they play the game 100 times in this fashion. Estimate the probability that Bill comes out at least \$35.00 ahead. Explain how your approximation depends on the central limit theorem.

Solution: If Bill wins a game, he wins 1. If he loses, loses 2. Let Z_i denote Bill's winnings in the i-th game. Then $E[Z_i] = (\frac{3}{4})(1) + (\frac{1}{4})(-2) = \frac{1}{4}$ and $E[Z_i^2] = (\frac{3}{4})(1) + (\frac{1}{4})(4) = \frac{7}{4}$. Thus $Var(Z_i) = \frac{7}{4} - \frac{1}{16} = \frac{27}{16}$.

Let $Z = \sum_{i=1}^{25} Z_i$. We want to estimate $P\{Z \geq 35\}$. This is the same as

$$P\{Z - 25 \geq 10\} = P\{\frac{Z-25}{10\sqrt{\frac{27}{16}}} \geq \frac{10}{10\sqrt{\frac{27}{16}}}\} =$$

$$P\{\frac{Z-25}{10\sqrt{\frac{27}{16}}} \geq \frac{4}{3\sqrt{3}}\}.$$

By the central limit theorem, this is approximately

$$1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{4}{3\sqrt{3}}} e^{-\frac{x^2}{2}}\, dx$$

and the numerical value can be found using the table on page 222.

#4 Charlie and Don play a game using a tetrahedral die. This die has four sides; when it is rolled it rests on one side which we will refer to as the resulting side. All four sides are equally likely to occur as the resulting side. (Such "dice" actually exist and are used in various games including generalizations of "Dungeons and Dragons".) Three sides of the die are painted blue and one is painted red. Charlie rolls the die three times and then Don does also. The one whose rolls result in the most red sides (out of his three rolls) wins.

(a) What is the probability that the two players have a total of at least 5 resulting red sides?

Solution: $\binom{6}{5}(\frac{1}{4})^5(\frac{3}{4}) + \binom{6}{6}(\frac{1}{4})$.

(b) What is the probability that a tie occurs? $\sum_{k=0}^{3}(\binom{3}{k}(\frac{1}{4})^k(\frac{3}{4})^{3-k})^2$.

Solution:

(c) What is the probability that Don has already won the game after his first roll?

Solution: $(\frac{3}{4})^3\frac{1}{4}$.

#5 Three urns are numbered 1 through 3; urn k contains k balls numbered 1 through k. We select an urn at random, draw a ball from it, note the number of the ball, replace the ball, and then drqw again from the same urn. If it is known that the first ball drawn has number 1, find

(a) the probability mass function of the number of the selected urn;

Solution: Let X be the number of the first ball drawn and U be the number of the urn it is drawn from. Then

$$P\{U = k | X = 1\} = \frac{P\{U = k, X = 1\}}{P\{X = 1\}} =$$

$$\frac{1}{3k}/(\frac{1}{3} + (\frac{1}{3})(\frac{1}{2}) + (\frac{1}{3})(\frac{1}{3})) = \frac{6}{11k}.$$

(b) the expected value of the number of the second ball drawn.

Solution: Let Y be the number of the second ball drawn. Then

$$E[Y|U=1]=1, E[Y|U=2]=3/2, E[Y|U=3]=2$$

and so

$$E[Y|X=1]=E[Y|U=1]P\{U=1|X=1\}+$$
$$E[Y|U=2]P\{U=2|X=1\}+E[Y|U=3]P\{U=3|X=1\}=$$
$$(1)(6/11)+(3/2)(3/11)+(2)(2/11)=29/22.$$

#6 Let X and Y be independent continuous random variables with densities

$$f_X(x)=e^{-x}, \quad if \quad x\geq 0,$$

$$f_X(x)=0, \quad otherwise,$$

$$f_Y(y)=3e^{-3y}, \quad if \quad y\geq 0,$$

$$f_Y(y)=0, \quad otherwise.$$

Find the density function $f_Z(z)$ for $Z=X+Y$.

Solution:

$$F_Z(a)=P\{X+Y\leq a\}=\int_{x=0}^{a}\int_{y=0}^{a-x}3e^{-x}e^{-3y}dydx=$$

$$\int_{x=0}^{a}e^{-x}[-e^{-3y}]_0^{a-x}dx=\int_{x=0}^{a}e^{-x}[1-e^{3x-3a}]dx=$$

$$[-e^{-x}-\frac{1}{2}e^{2x-3a}]_0^a=1-(\frac{3}{2})e^{-a}+(\frac{1}{2})e^{-3a}.$$

Then

$$f_Z(a)=\frac{d}{da}F_Z(a)=(\frac{3}{2})(e^{-a}-e^{-3a}).$$

#7 Let $X_1, ..., X_{25}$ be independent random variables, each of which is uniformly distributed on the interval $[0,2]$, and let $X=\sum_{i=1}^{25}X_i$.

(a) Find the mean and variance of X.

Solution: $E[X_i]=1$ and $Var(X_i)=\int_0^2(x-1)^3dx=\frac{2}{3}$. Thus $E[X]=25$ and $Var(X)=\frac{50}{3}$.

(b) Use the central limit theorem to estimate the probability that $|X-24|\geq 3$.

Solution:

$$P\{|X-24|\geq 3\} = P\{X>27\}+P\{X<21\} = P\{X-25>2\}+P\{X-25<-4\} =$$

$$P\{\frac{X-25}{5\sqrt{\frac{50}{3}}} > \frac{2}{5\sqrt{\frac{50}{3}}}\} + P\{\frac{X-25}{5\sqrt{\frac{50}{3}}} < -\frac{4}{5\sqrt{\frac{50}{3}}}\}.$$

By the central limit theorem this is approximately

$$(\frac{1}{\sqrt{2\pi}})(\int_a^\infty e^{-x^2/2}dx + \int_\infty^{-2a} e^{-x^2/2}dx)$$

where $a = \frac{2}{5\sqrt{\frac{50}{3}}}$. The numerical values may be found using the table on page 222.

(c) Use Chebyshev's inequality to find a number a such that you are absolutely sure that $P\{|X-E[x]|\geq a\}\leq 0.1$.

Solution: By Chebyshev's inequality, $P\{|X-E[x]|\geq a\}\leq \frac{\sigma^2}{a^2}$. Since $\sigma^2 = \frac{50}{3}$, it is sufficient to choose a so that $\frac{50}{3a^2}\leq 0.1$ or, equivalently, $\frac{500}{3}\leq a^2$. Thus $a=13$ works.

#8 A single (ordinary) die is rolled. Denote the number shown by X. Then another (ordinary, cubical) die has X of its sides painted green and the rest painted yellow. Thus the probability that a green side shows when the second die is rolled is $X/6$. The second die is then rolled. Let $Y=1$ if a green side shows, and $Y=0$ otherwise.

(a) Find the joint probability mass function, and the marginal probability mass functions of X and Y.

Solution: $P\{X=k, Y=1\} = \frac{k}{36}$ and $P\{X=k, Y=0\} = \frac{6-k}{36}$. Thus $p_Y(0) = 5/12, p_Y(1) = 7/12$, and $p_X(k) = 1/6$ for all k.

(b) Find $E[X], E[Y], Var(X), Var(Y)$, and $Cov(X,Y)$.

Solution:

$$E[X] = 7/2, E[Y] = P\{Y=1\} = 7/12,$$

$$E[X^2] = (1/6)(1^2+2^2+3^2+4^2+5^2+6^2) = 91/6, Var(X) = (91/6)-(7/2)^2 = 35/12,$$

$$E[Y^2] = 7/12, Var(Y) = (7/12)-(7/12)^2 = 35/144,$$

$$E[XY] = (1/36)1+(2/36)2+(3/36)3+(4/36)4+(5/36)5+(6/36)6 = 91/36,$$

$$Cov(X,Y) = E[XY]-E[X]E[Y] = (91/36)-(49/24) = 35/72.$$

#9 A modified form of poker is played with an ordinary deck of 52 playing cards, but the hands contain six cards. Calculate the probability that a randomly dealt hand contains:

(i) four cards of the same rank and two from different ranks;

Solution: In each part, the first factor represents the number of ways of choosing the ranks that occur, the second factor represents the number of ways of choosing the suits that appear at each rank and the denominator is the total number of 6 card hands.

$$[(13)\binom{12}{2}][\binom{4}{1}^2]/\binom{52}{6}.$$

(ii) three pairs, each of a different rank;

Solution:

$$[\binom{13}{3}][\binom{4}{2}^3]/\binom{52}{6}.$$

(iii) two triples, each containing three cards of the same rank;

Solution:

$$[\binom{13}{2}][\binom{4}{3}^2]/\binom{52}{6}.$$

(iv) only two ranks;

Solution:

$$\{[\binom{13}{2}][\binom{4}{3}^2] + [(13)(12)][\binom{4}{4}\binom{4}{2}]\}/\binom{52}{6}.$$

(v) six cards in sequence;

Solution:

$$[8][4^6]/\binom{52}{6}$$

(vi) six cards, all of different ranks,

Solution:

$$[\binom{13}{6}][4^6]/\binom{52}{6}$$

(vii) a triple, a pair, and a single card, all of different ranks.

$$[(13)(12)(11)][\binom{4}{3}\binom{4}{2}\binom{4}{1}]/\binom{52}{6}.$$

Solution:

#10 (a) A hat contains n cards numbred 1 to n; one card is drawn at random and its number is denoted X. Calculate the moment generating function $M_X(t)$ of X.

Solution:

$$M_X(t) = E[e^{tX}] = (\frac{1}{n})e^t + (\frac{1}{n})e^{2t} + ... + (\frac{1}{n})e^{nt} =$$

$$\frac{1}{n}e^t \frac{1 - e^{nt}}{1 - e^t}.$$

(b) If the experiment is repeated k times (with repalcement) and Y is the sum of the numbers of all cards drawn, find $M_Y(t)$.

Solution: $M_Y(t) = (M_X(t))^k$.

#11 (a) State Markov's inequality as a theorem, defining all symbols used and including all necessary hypotheses.

Solution: See page 430 of the text.

(b) State Chebyshev's inequality as a theorem, defining all symbols used and including all necessary hypotheses.

Solution: See page 431 of the text.

#12 Suppose 10% of the investors in the stock market are dishonest. The Amalgamated Widgets Company has just declared bankruptcy. An honest investor would have had only a 20% chance of selling her stock in time to come out without loss, but there is a 70% chance that a dishonest investor would have had inside information and gotten out in time. If we know that Smith didn't lose any money, what is the probabilty that Smith is honest.

Solution: Let S be the event that Smith is honest and W be the event that Smith sold her stock with out loss. Then

$$P\{W\} = (.9)(.2) + (.1)(.7) = .25$$

and

$$P\{S|W\} = P\{W|S\}\frac{P\{S\}}{P\{W\}} = (.2)\frac{.9}{.25} = .72.$$

Now

#13 A large urn contains N balls of each of 20 different colors (that is, a total of $20N$ balls). 10 balls are selected at random; we let X be the total number of different colors obtained, and write $X = \sum_{i=1}^{20} X_i$ with X_i a Bernoulli random variable indicating whether or not the i-th color is obtained.

(a) Assume $N = 1$. Find $E[X]$ and $Var(X)$ by elementary reasoning.

Solution: Since $N = 1$ the ten balls selected must have distinct numbers. Thus $E[X] = 10$ and $Var(X) = 0$.

(b) Let N be arbitrary and supppose that the selection is without replacement. Find $E[X]$ and $Var(X)$ and show that your results are consistent with part (a). (Hint: Compute $E[X_i]$ and $E[X_iX_j]$.)

Solution:

$$E[X_i] = 1 - \binom{19N}{10}/\binom{20N}{10}.$$

Note that if W is any random variable (such as X_i) that takes value 1 with probability p and value 0 with probability $1-p$, then $E[W] = p = E[W^2]$ and so $Var(W) = p^2 - p = p(1-p)$. Thus

$$Var(X_i) = [1 - \binom{19N}{10}/\binom{20N}{10}][\binom{19N}{10}/\binom{20N}{10}].$$

Also, using inclusion-exclusion,

$$E[X_iX_j] = 1 - 2\binom{19N}{10}/\binom{20N}{10} + \binom{18N}{10}/\binom{20N}{10}$$

and so

$$Cov(X_i, X_j) = 1 - 2\binom{19N}{10}/\binom{20N}{10} + \binom{18N}{10}/\binom{20N}{10} - [1 - \binom{19N}{10}/\binom{20N}{10}]^2.$$

Then

$$E[X] = 20E[X_i], Var(X) = 20Var(X_i) + (20)(19)Cov(X_i, X_j).$$

(c) Suppose that the selection is with replacement. Find $E[X]$ and $Var(X)$. Note that your answer should not depend on N.

Solution: Now

$$E[X_i] = 1 - (19/20)^{10}, Var(X_i) = (1 - (19/20)^{10})(19/20)^{10},$$

$$Cov(X_i, X_j) = 1 - 2(19/20)^{10} + (18/20)^{10} - (1 - (19/20)^{10})^2,$$

$$E[X] = 20E[X_i], Var(X) = 20Var(X_i) + (20)(19)Cov(X_i, X_j).$$

(d) Show that the answers in part (c) are the $N \to \infty$ limit of your answers in part (b). Explain why this should be true.

Solution: The quotient $\binom{19N}{10}/\binom{20N}{10}$ is

$$\frac{19N}{20N}\frac{19N-1}{20N-1}\cdots\frac{19N-9}{20N-9}$$

and as $N \to \infty$ this approaches $(\frac{19}{20})^{10}$, so the expressions in (b) converge to those in (c). This is to be expected, since for large N, the difference between choices with and without replacement is minor.

#14 Each morning Ed makes a random choice of one of three routes to take to work. After n trips, $(n > 0)$, what is the probability that he has travele each route at least once?

Solution: If $n < 3$ the probability is 0. Now assume $n \geq 3$. Let E_i be the event that Ed never takes the i-th route. Then we want $1 - P\{E_1 \cup E_2 \cup E_3\}$. By inclusion-exclusion this is

$$1 - P\{E_1\} - P\{E_2\} - P\{E_3\} + P\{E_1E_2\} + P\{E_1E_3\} + P\{E_2E_3\} = 1 - 3(2/3)^n + 3(1/3)^n.$$

#15 Five balls are randomly chosen, without replacement from an urn that contains 5 red, 6 white, and 7 blue balls. Find the probbility that at least one ball of each color is chosen.

Solution: Let E_R (respectively, E_W, E_B) be the event that no red (respectively, white, blue) ball is chosen. Then we want $1 - P\{E_R \cup E_W \cup E_B\}$. Use inclusion-exclusion to get that this is

$$1 - \binom{13}{5}/\binom{18}{5} - \binom{12}{5}/\binom{18}{5} - \binom{11}{5}/\binom{18}{5} +$$
$$\binom{5}{5}/\binom{18}{5} + \binom{6}{5}/\binom{18}{5} + \binom{7}{5}/\binom{18}{5}.$$

#16 Player A flips a fair coin $n+1$ times and player B flips a fair coin n times. Find the probability that A gets more heads than B.

Solution: For $1 \leq i \leq n+1$ let $X_i = 1$ if player A's i-th flip is a head and let X_i be 0 if player A's i-th flip is a tail. For $n+2 \leq i \leq 2n+1$ let $X_i = 1$ if player B's i-th flip is a tail and $X_i = 0$ if player B-s i-th flip is a head. Thus the number of heads that player A gets is $\sum_{i=1}^{n+1} X_i$ and the number of heads player B gets if $\sum_{i=n+2}^{2n+1}(1 - X_i) = n - \sum_{i=n+2}^{2n+1} X_i$. Thus we want to find

$$P\{\sum_{i=1}^{n+1} X_i > n - \sum_{i=n+2}^{2n+1} X_i\} = P\{\sum_{i=1}^{2n+1} X_i > n\}.$$

Thus, writing $X = \sum_{i=1}^{2n+1} X_i$, we want $P\{X > n\}$.

Now (using the binomial distibution) we have $P\{X = k\} = \binom{2n+1}{k}(\frac{1}{2})^{2n+1}$ and so, since

$$\binom{2n+1}{k} = \binom{2n+1}{2n+1-k}$$

we have

$$P\{X = k\} = P\{X = 2n+1-k\}.$$

Then

$$P\{X > n\} = \sum_{i=n+1}^{2n+1} P\{X = i\} = \sum_{i=n+1}^{2n+1} P\{X = 2n+1-i\}.$$

Then writing $j = 2n+1-i$ we see that

$$P\{X > n\} = \sum_{j=0}^{n} P\{X = j\}$$

and so

$$2P\{X > n\} = \sum_{j=0}^{n} P\{X = j\} + \sum_{j=n+1}^{2n+1} P\{X = j\} = \sum_{j=0}^{2n+1} P\{X = j\} = 1.$$

Thus $P\{X > n\} = \frac{1}{2}$.

#17 There are two factories that produce radios. Each radio produced at factory A is defective with probability .05, whereas each one produced at factory B is defective with probability .01. Suppose you purchase two radios that were produced at the same factory, which is equally likely to have been either factory A or factory B. If the first radio that you check is defective, what is the conditional probability that the other one is also defective?

Solution: Let A be the event that the radios are from factory A and B be the event that the radios are from factory B. Let X be the even that the first radio is defective and Y be the event that the second radio is defective. Then we want $P\{Y|X\}$.

We know that $P\{Y|A\} = .05$ and $P\{Y|B\} = .01$. Furthermore

$$P\{X\} = P\{X|A\}P\{A\} + P\{X|B\}P\{B\} = (.05)(.5) + (.01)(.5) = .03.$$

Then

$$P\{A|X\} = P\{X|A\}\frac{P\{A\}}{P\{X\}} = (.05)\frac{.5}{.03}$$

and

$$P\{B|X\} = P\{X|B\}\frac{P\{B\}}{P\{X\}} = (.01)\frac{.5}{.03}$$

Now

$$P\{Y|X\} = P\{Y|A\}P\{A|X\} + P\{Y|B\}P\{B|X\} = (.05)(.05)\frac{.5}{.03} + (.01)(.01)\frac{.5}{.03} =$$

$$\frac{1}{3}(.125 + .005) = \frac{1}{3}(.130) = .0433...$$

#18 A urn contains n balls numbered $1, ..., n$. Suppose that a boy draws a ball from the urn, notes the number and returns the ball to the urn before drawing another ball. This continues until the boy draws a ball he has previously drawn. Let X denote the number of draws. Find the probability mass function of X.

Solution: $P\{X = 1\} = 0$. Now let $k > 1$. Then $P\{X = k\}$ is the probability that the first $k - 1$ draws result in distinct values (an event with probability $(\frac{n}{n})(\frac{n-1}{n})...(\frac{n-k+2}{n})$ and the k draw results in a previously obtained value (an event with probability $\frac{k-1}{n}$). Thus,

$$P\{X = k\} = \frac{(k-1)n!}{n^k(n-k+1)!}.$$