Solutions (and corrections) to Practice problems - Math 552

May 5, 2011

#1. Let R be a ring and, $A \in mod - R$, and $B \in R - mod$. Let A' be a submodule of A and B' be a submodule of B. Show that $(A/A') \otimes_R (B/B')$ is isomorphic to $(A \otimes_R B)/C$ where C is the subgroup of $A \otimes_R B$ generated by all $a' \otimes b$ and $a \otimes b'$ for $a \in A, b \in B, a' \in A', b' \in B'$.

Solution: Let $f : A \to A/A'$ and $g : B \to B'$ be the canonical surjections. Then $f \otimes g$ is a a surjective homomorphism from $A \otimes B$ to $(a/A') \otimes (B/B')$ Clearly C is in the kernel of $f \otimes g$, so $f \otimes g$ induces a surjective homomorphism

$$\Phi : ((A \otimes B)/C) \to A/A' \otimes B/B'.$$

Note that

$$\Phi(a \otimes b + C) = (a + A') \otimes (b + B').$$

We will now construct the inverse map. Let

$$\Psi' : (A/A') \times (B/B') \to (A \otimes B)/C$$

be defined by

$$\Psi'(a + A', b + B') = a \otimes b + C.$$

One checks that this is a well-defined balanced map. Therefore there is a homomorphism

$$\Psi : (A/A') \otimes (B/B') \to (A \otimes B)/C$$

satisfying

$$\Psi((a + A') \otimes (b + B')) = a \otimes b + C.$$

Then Φ and Ψ are clearly inverses.

#2. Let R be a ring and $M \in R - mod$ be both artinian and noetherian. Let $f \in End_R(M)$. Recall that f^∞M is defined to be $\cap_{n \geq 1} f^i(M)$ and $f^{-\infty}0$ is defined to be $\cup_{n \geq 1} ker(f^n)$. Prove that

$$M = f^\infty M \oplus f^{-\infty}0.$$

(This is Fitting's Lemma.)

Solution: This is in the text: BA-II page pp 113-114.

#3 State the definition of a projective resolution of an R-module M and show that any module has a projective resolution.

Solution: See BA-II, Definition 6.5 for the definition and pages 340,341 for the construction of a free (hence projective) resolution.

#4 Let $(C', d'), (C, d)$, and (C'', d'') be complexes. Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence (where the chain homomorphism from C' to C is denoted α and the chain homomorphism from C to C'' is denoted β). Suppose that there exist module homomorphisms $S_i : C'_i \to C_{i+1}$ for all $i \in \mathbf{Z}$ such that

$$\alpha_i = d_{i+1}s_i + s_{i-1}d'_i$$

for all i. Prove that if C'' is exact then C and C' are exact.

Solution: The condition on the α_i says that the chain map α is homotopic to the zero map. Thus in the long exact homology sequence we have

$$0 \to H_i(C) \to H_i(C'') \to H_{i-1}(C') \to 0$$

for all i. Then if C'' is exact, all $H_i(C'') = 0$ for all i and hence $H_i(C) = 0$ and $H_{i-1}(C') = 0$ for all i.

#5 Show that the ideal $(9, 3x + 3)$ has infinitely many primary decompositions.

Correction and Solution: The assertion that there are infinitely many is incorrect. Here is one: $(3) \cap (9, x + 1)$.

#6 If R is a commutative ring, $B \neq 0$ is an R-module, and P is maximal in the set of ideals

$$\{ann\ x\ | 0 \neq x \in B\}$$

then P is prime. (Recall that $ann\ x\ = \{r \in r \mid rx = 0\}$.)

Solution: Let $P = ann\ x$ be maximal in the set of annihilators. Suppose $u, v \in P, uv \in P, v \notin P$. Then $uvx = 0$ and $vx \neq 0$ Since $ann\ vx \supseteq ann\ x$ and $u \in ann\ vx$, the maximality of P implies $u \in ann\ x = P$, giving the result.

#7 Let R be noetherian and let S be a submonoid of the multiplicative monoid of R. Show that R_S is noetherian.

Solution: This is Theorem 7.12 of BA-II.

#8 Determine the Galois groups of $x^5 - 6x + 3$ and of $(x^3 - 2)(x^2 - 5)$ over the rational numbers.

Solution: Let $f(x) = x^5 - 6x + 3$. Then $f(x)$ is irreducible by Eistenstein's criterion. Also, $f'(x) = 0$ has only two real roots, so $f(x)$ has at most 3 real roots and since $f(-1) > 0$ and $f(1) < 0$ it has exactly 3 real roots.

Then, as we have seen in class, the Galois group is S_5. The Galois group of $(x^3-2)(x^2-5)$ os the direct product of S_3 and the cyclic group of order 2.

#9 Let $E \supseteq F$ be fields and $u, v \in E$. Suppose that v is algebraic over $F(u)$, and that v is transcendental over F. Show that u is algebraic over $F(v)$.

Correction: Note that inclusion between E and F in the originally posted version has been reversed.

Solution: Note that u must be transcendental over F (for, if not, v is algebraic over F). Let $f(x) = \sum_{i=0}^{n} a_i x_i \in F(u)[x]$ be a nonzero polynomial such that $f(v) = 0$. Then $a_i \in F(u)$ and so $a_i = b_i/c_i$ where $b_i, c_i \in F[u], c_i \neq 0$. Then we have

$$0 = (\prod_i c_i) f(v)$$

is a polynomial in u and v. We may view this as a polynomial $g(u)$ in u with coefficients from $F(v)$. Since v is transcendental over F, at least one of the coefficients of g is nonzero and so u is algebraic over $F(v)$.

#10 Let E, K, L, and F be fields with $E \supseteq K \supseteq F$, $E \supseteq L \supseteq F$ and $K \cap L = F$. Assume that E is generated by $K \cup L$. Suppose $[K : F] = n_1$, $[L : F] = n_2$, and that K and L are Galois extensions of F. What is $[E : F]$? Why?

Correction: Note that the inclusions in the originally posted version have all been reversed and that the hypothesis that L is Galois over F has been added.

Solution: Let H be the subgroup of $Gal(E/F)$ generated by $Gal(E/L)$ and $Gal(E/K)$. Then an element of E fixed by all elements of H must be in $Inv(Gal(E/L)) = L$ and in $Inv(Gal(E/K)) = K$ and hence must be in $K \cap L = F$. Thus $|H| = [E : F]$. But $Gal(E/L)$ and $Gal(E/K)$ are normal subgroups of $Gal(E/F)$ with trivial intersection (since an element of the intersection fixes all generators of E over F) and so $|H| = |Gal(E/L)||Gal(E/K)| = ([E : F]/[L : F])([E : F]/[K : F])$. Thus $[E : F] = [L : F][K : F]$.