Math 351

Review problems for Final Exam **December 12, 2007**

#1 Find $(78, 2340)$ and write it in the form $78a + 2340b$ where a and b are integers.

#2 Find $[12]^{-1}$ in $\mathbf{Z}_{25}$.

#3 Let R be a ring and A, B be ideals in R. Let $A + B$ denote $\{a + b | a \in A, b \in B\}$.

(a) Prove that $A + B$ is an ideal in R.

(b) Recall that if $n \in \mathbf{A}$, then (n) denotes $\{nk | k \in \mathbf{Z}\} = n\mathbf{Z}$. Prove that any ideal in $\mathbf{Z}$ is equal to (n) for some $n \in \mathbf{Z}, n \geq 0$.

(c) Let $m, n \in \mathbf{Z}, m, n > 0$. Prove that $(m) + (n) = ((m, n))$. (Recall that (m, n) denotes the greatest common divisor of m and n.)

#4 Find all the ideals in $\mathbf{Z}_{10} \times \mathbf{Z}$. Which of these are prime ideals? Which of these are maximal ideals?

#5 Find $[x^2 + x + 1]^{-1}$ in $\mathbf{Q}[x]/(x^3 + 2)$.

#6 Find $(x^3 + 2x^2 - x - 2, x^4 - 1)$ in $\mathbf{Q}[x]$ and exrpess it in the form $(x^3 + 2x^2 - x - 2)a + (x^4 - 1)b$ where $a, b \in \mathbf{Q}[x]$.

#7 (a) Let $R = \{A \in M_2(\mathbf{R}) | A \left|\begin{matrix} 1 \\ -1 \end{matrix}\right| = \left|\begin{matrix} 0 \\ 0 \end{matrix}\right|\}$. Show that A is a subring of $M_2(\mathbf{R})$ but that A is not an ideal.

(b) Let $S = \{B \in M_2(\mathbf{R}) | B \left|\begin{matrix} 1 \\ -1 \end{matrix}\right| \in \mathbf{R} \left|\begin{matrix} 1 \\ -1 \end{matrix}\right|\}$. Show that S is a subring of $M_2(\mathbf{R})$.

(c) Show that R is an ideal in S and that S/R is isomorphic to $\mathbf{R}$.

#8 Let F be a field and $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ be ideals of $F[x]$. Show that there is some k such that $I_k = I_{k+1} = ...$

#9 (a) Is $x^5 + 3x^4 + 6x^2 - 9x + 3$ irreducible over $\mathbf{Q}$? Why or why not?

(b) Is $x^5 + x^4 + 1$ irreducible over $\mathbf{Z}_2$? Why or why not?

#10 Let R be a ring and I be an ideal in R. Prove that every subring of R/I has the form J/I where J is a subring of R which contains I. Also show that J is an ideal in R if and only if J/I is an ideal in R/I.

#11 Let G be a group and N a normal subgroup of G. Prove that every subgroup of G/N has the form H/N where H is a subgroup of G which contains N. Also show that H is a normal subgroup of G if and only if H/N is a normal subgroup of G/N.

#12 Let G and H be groups, N be a normal subgroup of G, and f be a homomorphism from G to H.

(a) Let e_G be the identity element of G, e_H be the identity element of H, and let $g \in G$. Show that $f(e_G) = e_H$ and that $f(g^{-1}) = f(g)^{-1}$.

(b) Show that $ker(f)$ is a normal subgroup of G.

(c) Show that $f(G)$ is a subgroup of H.
(d) Give an example to show that $f(N)$ does not have to be a normal subgroup of H.
(e) Show that if f is surjective then $f(N)$ is a normal subgroup of $f(G)$.

#13 Write (137562)(234)(57) as a product of disjoint cycles.

#14 (a) Find $\sigma \in S_8$ such that $\sigma(87654321) = (12345678)$.
(b) Find $\tau \in S_8$ such that $\tau(87654321)\tau^{-1} = (12345678)$.

#15 Let $C(n)$ denote the cyclic group of order n.
(a) Show that $C(5) \times C(6)$ is isomorphic to $C(30)$.
(b) Show that $C(2) \times C(8)$ is not isomorphic to $C(8)$

#16 (a) Can S_{10} contain an element of order 14? Why or why not?
(b) Can S_{10} contain an element of order 16? Why or why not?

#17 Let G be a group, H be a subgroup of G, ad $a, b \in G$.
(a) Show that either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.
(b) Show that $|Ha| = |Hb|$.
(c) Suppose $|G|$ is finite. Prove that $|H|$ divides $|G|$.

#18 (a) Let $R = \mathbf{Z}[\sqrt{7}]$. Show that the quotient field of R is isomorphic to $\mathbf{Q}[\sqrt{7}]$.
(b) Prove that $\mathbf{Q}[\sqrt{7}]$ is a Euclidean domain with $\delta(a + b\sqrt{7}) = a^2 + 7b^2$.

#19 Let G be a group and H, K be subgroups of G. Assume $HK = KH$.
(a) Show that HK is a subgroup of G.
(b) Is H a normal subgroup of HK? (Think about subgroups of S_3.)
(c) Suppose $H \cap K = \{e\}$ and $hk = kh$ for all $h \in H, k \in K$. Prove that HK is isomorphic to $H \times K$.

#20 Let $K = \{f \in \mathbf{C}[x] | f(-2) = 0\}$ and $L = \{g \in \mathbf{C}[x] | g(-2) = g(5) = 0\}$.
(a) Show that K and L are ideals in $\mathbf{C}[x]$.
(b) What is the quotient $\mathbf{C}[x]/K$?
(c) What is the quotient $\mathbf{C}[x]/L$?