MATH 354-03 **April 4, 2005**

Solutions to practice questions for exam #2

#1 Consider the linear programming problem

Maximize: $\mathbf{c}^T\mathbf{x}$
Subject to:
$A\mathbf{x} \leq \mathbf{b}$
$\mathbf{x} \geq \mathbf{0}$.

This problem is in standard form.

(a) State the dual problem.

(b) Show that if $\mathbf{x}$ is any feasible solution to the primal problem and $\mathbf{w}$ is any feasible solution to the dual problem then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{w}$.

(c) Show that if the primal problem is unbounded, then the dual problem is infeasible.

(d) Show that the dual of the dual of the given problem is the primal problem.

Solution: (a) The dual problem is:

Minimize: $\mathbf{b}^T\mathbf{w}$
Subject to:
$A^T\mathbf{w} \geq \mathbf{c}$
$\mathbf{w} \geq \mathbf{0}$.

(b) Since $A\mathbf{x} \leq \mathbf{b}$ and since $\mathbf{w} \geq \mathbf{0}$ we have

$$\mathbf{w}^T A\mathbf{x} \leq \mathbf{w}^T\mathbf{b}.$$

Also, $\mathbf{w}^T A\mathbf{x}$ is a 1 by 1 matrix and so is equal to its transpose $\mathbf{x}^T A^T\mathbf{w}$. Since $A^T\mathbf{w} \geq \mathbf{c}$ and $\mathbf{x} \geq \mathbf{0}$, this gives

$$\mathbf{w}^T A\mathbf{x} \geq \mathbf{x}^T\mathbf{c} = \mathbf{c}^T\mathbf{x}.$$

Combining these inequalities gives the result.

(c) If the dual problem is feasible, then (by (b)) for any feasible solution $\mathbf{w}$ to the dual problem, $\mathbf{b}^T\mathbf{w}$ is an upper bound for the values of the objective function for the primal problem. Thus if the dual problem is feasible, the primal problem must be bounded.

(d) When the dual problem is written in standard form it becomes:

Maximize: $-\mathbf{b}^T\mathbf{w}$
Subject to:
$-A^T\mathbf{w} \leq -\mathbf{c}$
$\mathbf{w} \geq \mathbf{0}$.

The dual of this (as in part (a)) is:

Minimize: $-\mathbf{c}^T\mathbf{u}$
Subject to:
$(-A^T)^T\mathbf{u} \geq -\mathbf{c}$
$\mathbf{u} \geq \mathbf{0}$.

Since $(A^T)^T = A$, when this is written in standard form it becomes:

Maximize: $\mathbf{c}^T\mathbf{u}$
Subject to:
$A\mathbf{u} \leq \mathbf{c}$
$\mathbf{u} \geq \mathbf{0}$,
which is the primal problem

#2 Consider the linear programming problem

Maximize: $\mathbf{r}^T\mathbf{x}$
Subject to:
$A\mathbf{x} = \mathbf{s}$
$\mathbf{x} \geq \mathbf{0}$

This problem is in canonical form. Find the dual of this problem, by writing the primal problem in standard form and using your answer to #1. Explain why the dual involves unrestricted variables.

Solution: We may write the problem in standard form as

Maximize: $\mathbf{r}^T\mathbf{x}$
Subject to:
$A\mathbf{x} \le \mathbf{s}$
$-A\mathbf{x} \le -\mathbf{s}$
$\mathbf{x} \ge \mathbf{0}$

We may rewrite this problem as

Maximize: $\mathbf{r}^T\mathbf{x}$
Subject to:
$$\begin{bmatrix} A \\ -A \end{bmatrix} \mathbf{x} \le \begin{bmatrix} \mathbf{s} \\ -\mathbf{s} \end{bmatrix}.$$
$\mathbf{x} \ge \mathbf{0}$.

Then (as in #1(a)), the dual is
$$\text{Minimize: } \begin{bmatrix} \mathbf{s} \\ -\mathbf{s} \end{bmatrix}^T \mathbf{w}$$
Subject to:
$$\begin{bmatrix} A \\ -A \end{bmatrix}^T \mathbf{w} \ge \mathbf{r}$$
$\mathbf{w} \ge \mathbf{0}$.

Here (if A is m by n), $\mathbf{w} = \begin{bmatrix} w_1 \\ . \\ . \\ . \\ w_{2m} \end{bmatrix}$ is a column vector in $\mathbf{R}^{2m}$. Then setting $u_i = w_i - w_{i+m}$ for $1 \le i \le m$ we have that $\mathbf{u}$ is unrestricted and

$$\begin{bmatrix} \mathbf{s} \\ -\mathbf{s} \end{bmatrix}^T \mathbf{w} = \mathbf{s}^T\mathbf{u}$$

and

$$\begin{bmatrix} A \\ -A \end{bmatrix}^T \mathbf{w} = A^T \mathbf{u}.$$

Thus the dual problem may be written:

Minimize: $\mathbf{s}^T \mathbf{u}$

Subject to:
$A^T \mathbf{u} \geq \mathbf{r}$
$\mathbf{u}$ unrestricted.

#3 Find the dual of the linear programming problem:

Minimize: $-3x_1 + 2x_2 + x_4$
Subject to:
$2x_1 + x_2 + x_3 + 2x_4 \geq 7$
$x_2 + 3x_4 = 5$
$x_1, x_2 \geq 0, x_3 \leq 0, x_4$ unrestricted.

Solution: We first write the problem in terms of positive variables by setting $x_3' = -x_3$ and $x_4 = x_4^+ - x_4^-$. This gives:
Minimize: $-3x_1 + 2x_2 + x_4^+ - x_4^-$
Subject to:
$2x_1 + x_2 - x_3' + 2x_4^+ - 2x_4^- \geq 7$
$x_2 + 3x_4^+ - 3x_4^- = 5$
$x_1, x_2, x_3', x_4^+, x_4^- \geq 0.$
Now we write this problem in standard form:
Maximize: $3x_1 - 2x_2 - x_4^+ + x_4^-$
Subject to:
$-2x_1 - x_2 + x_3' - 2x_4^+ + 2x_4^- \leq -7$
$x_2 + 3x_4^+ - 3x_4^- \leq 5$
$-x_2 - 3x_4^+ + 3x_4^- \leq -5$

$x_1, x_2, x_3', x_4^+, x_4^- \geq 0.$
The dual is therefore
Minimize: $-7w_1 + 5w_2 - 5w_3$
Subject to:
$-2w_1 \geq 3$
$-w_1 + w_2 - w_3 \geq -2$
$w_1 \geq 0$
$-2w_1 + 3w_2 - 3w_3 \geq -1$
$2w_1 - 3w_2 + 3w_3 \geq 1$
$w_1, w_2, w_3 \geq 0.$
We may combine the last two inequalities into a single equality and write $w_4 = w_2 - w_3$ to get
Minimize: $-7w_1 + 5w_4$
Subject to:
$-2w_1 \geq 3$
$w_1 + w_4 \geq -2$
$2w_1 - 3w_4 = 1$
$w_1 \geq 0, w_4$ unrestricted.

#4 Use the revised simplex method to solve the linear programming problem

Maximize: $2x_1 + x_2 + 3x_3 + x_6 + 2x_7 + 3x_8$
Subject to:
$2x_1 + x_2 + x_4 + 3x_5 + x_7 \leq 24$
$x_1 + 3x_3 + x_4 + x_5 + 2x_6 + 3x_8 \leq 30$
$5x_1 + 3x_2 + 3x_4 + 2x_5 + x_7 + 5x_8 \leq 18$
$3x_1 + 2x_2 + x_3 + x_6 + 3x_8 \leq 20$
$x_1, ..., x_8 \geq 0.$

Give the current B^{-1} and the current list of basic variables at each step.

Solution: The initial tableau is:

		2	1	3	0	0	1	2	3	0	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	$\mathbf{x}_B$
0	x_9	2	1	0	1	3	0	1	0	1	0	0	0	24
0	x_{10}	1	0	3	1	1	2	0	3	0	1	0	0	30
0	x_{11}	5	3	0	3	2	0	1	5	0	0	1	0	18
0	x_{12}	3	2	1	0	0	1	0	3	0	0	0	1	20
		-2	-1	-3	0	0	-1	-2	-3	0	0	0	0	0

Initially $B^{-1} = I_4$, the 4 by4 identity matrix, and $\mathbf{x}_B = \mathbf{b} = \begin{bmatrix} x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{bmatrix}$.

From the initial tableau we see that the first pivot should be on the $(2,3)$ position. Then the new B^{-1} is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

and the new $x_B = \begin{bmatrix} x_9 \\ x_3 \\ x_{11} \\ x_{12} \end{bmatrix}$. This is equal to the new B^{-1} times $\mathbf{b}$ which is $\begin{bmatrix} 24 \\ 10 \\ 18 \\ 10 \end{bmatrix}$. The new $\mathbf{c}_B = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$.

Now $\mathbf{c}_B^T B^{-1} = [0 \quad 1 \quad 0 \quad 0]$ and so

$$z_j = \mathbf{c}_B^T \mathbf{t}_j = \mathbf{c}_B^T B^{-1} A_j = [0 \quad 1 \quad 0 \quad 0]\, A_j.$$

Using this we compute the new values of $z_j - c_j$, getting the new objective row:

$$\begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We now pivot on the 7th column. To choose the pivot row we compute $\mathbf{t}_7 = B^{-1}A_7 = B^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. The θ-ratios are 24 for the first row and 18 for the third row. Thus we pivot on the $(3,7)$ position.

Then the new B^{-1} is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

and the new $\mathbf{x}_B = \begin{bmatrix} x_9 \\ x_3 \\ x_7 \\ x_{12} \end{bmatrix}$. This is equal to the new B^{-1} times $\mathbf{b}$ which is $\begin{bmatrix} 6 \\ 10 \\ 18 \\ 10 \end{bmatrix}$. The new $\mathbf{c}_B = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$.

Now $\mathbf{c}_B^T B^{-1} = \begin{bmatrix} 0 & 1 & 2 & 0 \end{bmatrix}$ and so

$$z_j = \mathbf{c}_B^T \mathbf{t}_j = \mathbf{c}_B^T B^{-1} A_j = \begin{bmatrix} 0 & 1 & 2 & 0 \end{bmatrix} A_j.$$

Using this we compute the new values of $z_j - c_j$, that is the new objective row:

$$\begin{bmatrix} 9 & 5 & 0 & 7 & 5 & 1 & 0 & 10 & 0 & 1 & 2 & 0 \end{bmatrix}.$$

Since all the entries in the new objective row are ≥ 0, the current solution is optimal. This solution is

$$x_1 = x_2 = 0, x_3 = 10, x_4 = x_5 = x_6 = 0, x_7 = 18, x_8 = 0.$$

#5 Consider the linear programming problem:

Maximize: $4x_1 + 3x_2 + 6x_3$
Subject to:
$3x_1 - 4x_2 - 6x_3 \leq 18$
$-2x_1 - x_2 + 2x_3 \leq 12$
$x_1 + 3x_2 + 2x_3 \leq 1$
$x_1, x_2, x_3 \geq 0.$

The optimal solution to this problem is $z = 4$ at the point $x_1 = 1, x_2 = 0, x_3 = 0$ and the final tableau for the simplex method is:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	−13	−12	1	0	−3	15
x_5	0	5	6	0	1	2	14
x_1	1	3	2	0	0	1	1
	0	9	2	0	0	4	4

(a) State the dual problem and find its optimal solution
(b) Find all values of Δc_2 such that the solution above remains optimal.
(c) Find all values of Δc_5 such that the solution above remains optimal.
(d) Find the optimal solution of the problem obtained by changing c_6 to 3.
(e) Suppose the final tableau is obtained from the initial tableau by multiplying by B^{-1}. Find B^{-1}.
(f) Find the optimal solution to the problem obtained by changing the constant term in the third constraint ($\mathbf{b}_3$) from 1 to 5.
(g) Find the optimal solution to the problem obtained by changing the constant term in the third constraint ($\mathbf{b}_3$) from 1 to 7.
(h) A further constraint $x_2 + x_3 \geq 1$ is added to the original problem. Use the dual simplex method to find an optimal solution to this new problem (if one exists).

(i) A different further constraint $2x_1 + x_2 \leq 1$ is added to the original problem (not to the modified problem in (h)). Use the dual simplex method to find an optimal solution to this new problem (if one exists).

Solution: (a) The dual problem is

Minimize: $18w_1 + 12w_2 + w_3$
Subject to:
$3w_1 - 2w_2 + w_3 \geq 4$
$-4w_1 - w_2 + 3w_3 \geq 3$
$6w_1 + 2w_2 + 2w_3 \geq 6$
$w_1, w_2, w_3 \geq 0.$

Since the initial tableau contains an identity matrix corresponding to the slack variables x_4, x_5, x_6, the optimal solution or the dual problem is given by the entries in the objective row of the final tableau that occur in the columns corresponding to slack variables. Thus the optimal solution of the dual problem is

$$\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}.$$

(b) Since x_2 is not basic, the current solution remains optimal if $z_2 - c_2 \geq \Delta c_2$. Taking the value of $z_2 - c_2$ from the final tableau, we see that the condition is that $\Delta c_2 \leq 9$.

(c) We copy the final tableau for the problem and add the (top) row recording the (original) values of the c_i and the (leftmost) column recording the $\mathbf{c}_B$:

		4	3	6	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	-13	-12	1	0	-3	15
0	x_5	0	5	6	0	1	2	14
4	x_1	1	3	2	0	0	1	1
		0	9	2	0	0	4	4

We now change the value of c_5 in this tableau and recompute the entries in the objective row to obtain:

		4	3	6	0	Δc_5	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	-13	-12	1	0	-3	15
Δc_5	x_5	0	5	6	0	1	2	14
4	x_1	1	3	2	0	0	1	1
		0	$9+5\Delta c_5$	$2+6\Delta c_5$	0	0	$4+2\Delta c_5$	$4+14\Delta c_5$

.

Now the condition that the solution is optimal is that every $z_j - c_j \geq 0$, that is,

$9+5\Delta c_5 \geq 0, 2+6\Delta c_5 \geq 0$, and $4+2\Delta c_5 \geq 0$. This is equivalent to $\Delta c_5 \geq -\frac{1}{3}$.

(d) The current solution remains optimal since $4 = z_j - c_j \geq \Delta c_6 = 3$.

(e) Since the 4-th through 6-th columns of the initial tableau form an identity matrix, the B^{-1} appears in the 4-th through 6-th columns of the final tableau. Thus $B^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(f) The right-hand column of the final tableau will be replaced by $B^{-1}\begin{bmatrix} 18 \\ 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 22 \\ 5 \end{bmatrix}$. Since the entries are all positive, this is a feasible solution and so is an optimal solution. Thus $x_1 = 5, x_2 = x_3 = 0, x_4 = 3, x_5 = 22, x_6 = 0$ is the optimal solution. The corresponding value of the objective function is $z = 20$.

(g) The right-hand column of the final tableau will be replaced by $B^{-1}\begin{bmatrix} 18 \\ 12 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 26 \\ 7 \end{bmatrix}$. Using this vector to replace the right-hand column of the final tableau we obtain:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	−13	−12	1	0	−3	−3
x_5	0	5	6	0	1	2	26 .
x_1	1	3	2	0	0	1	7
	0	9	2	0	0	4	28

One iteration of the dual simplex procedure (pivoting on the $(1,3)$ position) gives

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	0	$\frac{13}{12}$	1	$-\frac{1}{12}$	0	$\frac{1}{4}$	$\frac{1}{4}$
x_5	0	$-\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{49}{2}$
x_1	1	$\frac{5}{6}$	0	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{13}{2}$
	0	$\frac{59}{6}$	0	$\frac{1}{6}$	0	$\frac{5}{2}$	$\frac{55}{2}$

Thus $x_1 = \frac{13}{2}, x_2 = 0, x_3 = \frac{1}{4}$ is the optimal solution.

(h) The new constraint may be written $x_2 + x_3 - u_1 = 1$ or as $-x_2 - x_3 + u_1 = -1$. Adding this constraint to the final tableau gives

	x_1	x_2	x_3	x_4	x_5	x_6	u_1	
x_4	0	−13	−12	1	0	−3	0	15
x_5	0	5	6	0	1	2	0	14
x_1	1	3	2	0	0	1	0	1 .
u_1	0	−1	−1	0	0	0	1	−1
	0	9	2	0	0	4	4	

Now apply the dual simplex method, pivoting on the $(4,3)$ position to obtain a tableau whose 3-rd row is

$$[1 \quad 2 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad -1].$$

Since the last entry in this row is negative and all other entries are positive, there is no feasible solution.

(i) The new constraint may be written $2x_1 + x_2 + u_1 = 1$. Since each basic variable should appear in only one constraint and since x_1 is basic, we will subtract 2 times the 3-rd constraint from this constraint to obtain $-5x_2 - 4x_3 - 2x_6 + u_1 = -1$. Adding this constraint to the final tableau gives

$$\begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & u_1 & \\ x_4 & 0 & -13 & -12 & 1 & 0 & -3 & 0 & 15 \\ x_5 & 0 & 5 & 6 & 0 & 1 & 2 & 0 & 14 \\ x_1 & 1 & 3 & 2 & 0 & 0 & 1 & 0 & 1 \\ u_1 & 0 & -5 & -4 & 0 & 0 & -2 & 1 & -1 \\ & 0 & 9 & 2 & 0 & 0 & 4 & 0 & 4 \end{array}.$$

We now apply the dual simplex method and pivot on the $(4,3)$ position. This gives the tableau

$$\begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & u_1 & \\ x_4 & 0 & 2 & 0 & 1 & 0 & 3 & -3 & 18 \\ x_5 & 0 & -\frac{5}{2} & 0 & 0 & 1 & -1 & \frac{3}{2} & \frac{25}{2} \\ x_1 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ x_3 & 0 & \frac{5}{4} & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \\ & 0 & \frac{13}{2} & 0 & 0 & 0 & 3 & \frac{1}{2} & \frac{7}{2} \end{array}.$$

#6 Find an optimal solution to the following pure integer programming problem.

Maximize: $x_1 + 3x_2$
Subject to:
$x_1 - 2x_2 \geq 0$
$x_1 + 2x_2 \leq 42$
$x_1, x_2 \geq 0,\ x_1, x_2$ integers.

Solution: First we solve the problem without the integrality restriction. The initial tableau is

$$\begin{array}{cccccc} & 1 & 3 & 0 & 0 & \\ x_3 & -1 & 2 & 1 & 0 & 0 \\ x_4 & 1 & 2 & 0 & 1 & 42 \\ & -1 & -3 & 0 & 0 & 0 \end{array}.$$

We pivot on the $(1,2)$ position and obtain

$$\begin{array}{cccccc} & 1 & 3 & 0 & 0 & \\ x_2 & -\frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ x_4 & 2 & 0 & -1 & 1 & 42 \\ & -\frac{5}{2} & 0 & \frac{3}{2} & 0 & 0 \end{array}.$$

Now we pivot on the $(2,1)$ position and obtain

$$\begin{array}{cccccc} & 1 & 3 & 0 & 0 & \\ x_2 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{21}{2} \\ x_1 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 21 \\ & 0 & 0 & \frac{1}{4} & \frac{5}{4} & \frac{105}{2} \end{array}.$$

Thus $x_1 = 21, x_2 = \frac{21}{2}$ is a solution to the problem without the integrality condition.

Now we add the cutting plane constraint

$$-\frac{1}{4}x_3 - \frac{1}{4}x_4 + u_1 = -\frac{1}{2}.$$

Then the tableau becomes

$$\begin{array}{ccccccc} & 1 & 3 & 0 & 0 & 0 & \\ x_2 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{21}{2} \\ x_1 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 21 \\ u_1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{2} \\ & 0 & 0 & \frac{1}{4} & \frac{5}{4} & 0 & \frac{105}{2} \end{array}.$$

Using the dual simplex method we see that we must pivot on the $(3,3)$ position. Thus we obtain the tableau

	1	3	0	0	0	
x_2	0	1	0	0	1	10
x_1	1	0	0	1	-2	22 .
x_3	0	0	1	1	-4	2
	0	0	0	1	1	52

Thus the optimal integral solution is $x_1 = 22, x_2 = 10$.

#7 Find an optimal solution to the following pure integer programming problem.

Maximize: $x_1 + 2x_2 + x_3 + x_4$
Subject to:
$2x_1 + x_2 + 3x_3 + x_4 \leq 8$
$2x_1 + 3x_2 + 4x_4 \leq 12$
$3x_1 + x_2 + 2x_3 \leq 18$
$x_1, x_2, x_3, x_4 \geq 0,\ x_1, x_2, x_3, x_4$ integers

Solution:

First we solve the problem without the integrality restriction. The initial tableau is

	1	2	1	1	0	0	0	
x_5	2	1	3	1	1	0	0	8
x_6	2	3	0	4	0	1	0	12 .
x_7	3	1	2	0	0	0	1	18
	-1	-2	-1	-1	0	0	0	0

We pivot on the $(2, 2)$ position to obtain

	1	2	1	1	0	0	0	
x_5	$\frac{4}{3}$	0	3	$-\frac{1}{3}$	1	$-\frac{1}{3}$	0	4
x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	4 .
x_7	$\frac{7}{3}$	0	2	$-\frac{4}{3}$	0	$-\frac{1}{3}$	1	14
	$\frac{1}{3}$	0	-1	$\frac{5}{3}$	0	$\frac{2}{3}$	0	8

Now we pivot on the $(1,3)$ position to obtain

$$
\begin{array}{cccccccccc}
 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & \\
x_3 & \frac{4}{9} & 0 & 1 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & 0 & \frac{4}{3} \\
x_2 & \frac{2}{3} & 1 & 0 & \frac{4}{3} & 0 & \frac{1}{3} & 0 & 4 \\
x_7 & \frac{13}{9} & 0 & 0 & -\frac{10}{9} & -\frac{2}{3} & -\frac{1}{9} & 1 & \frac{34}{3} \\
 & \frac{7}{9} & 0 & 0 & \frac{14}{9} & \frac{1}{3} & \frac{5}{9} & 0 & \frac{28}{3}
\end{array}.
$$

Since the entries in the objective row are all positive, this gives an optimal solution to the problem without the integrality restriction.

Now we use the first row to impose a cutting plane constraint:

$$-\frac{4}{9}x_1 - \frac{8}{9}x_4 - \frac{1}{3}x_5 - \frac{8}{9}x_6 + u_1 = -\frac{1}{3}.$$

Adding this to our previous tableau gives

$$
\begin{array}{cccccccccc}
 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \\
x_3 & \frac{4}{9} & 0 & 1 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & 0 & 0 & \frac{4}{3} \\
x_2 & \frac{2}{3} & 1 & 0 & \frac{4}{3} & 0 & \frac{1}{3} & 0 & 0 & 4 \\
x_7 & \frac{13}{9} & 0 & 0 & -\frac{10}{9} & -\frac{2}{3} & -\frac{1}{9} & 1 & 0 & \frac{34}{3} \\
u_1 & -\frac{4}{9} & 0 & 0 & -\frac{8}{9} & -\frac{1}{3} & -\frac{8}{9} & 0 & 1 & -\frac{1}{3} \\
 & \frac{7}{9} & 0 & 0 & \frac{14}{9} & \frac{1}{3} & \frac{5}{9} & 0 & 0 & \frac{28}{3}
\end{array}.
$$

Using the dual simplex method, we pivot on the $(4,6)$ position to obtain

$$
\begin{array}{cccccccccc}
 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
x_3 & \frac{1}{2} & 0 & 1 & 0 & \frac{3}{8} & 0 & 0 & -\frac{1}{8} & \frac{11}{8} \\
x_2 & \frac{1}{2} & 1 & 0 & 1 & -\frac{1}{8} & 0 & 0 & \frac{3}{8} & \frac{31}{8} \\
x_7 & \frac{3}{2} & 0 & 0 & -1 & -\frac{5}{8} & 0 & 1 & -\frac{1}{8} & \frac{91}{8} \\
x_6 & \frac{1}{2} & 0 & 0 & 1 & \frac{3}{8} & 1 & 0 & -\frac{9}{8} & \frac{3}{8} \\
 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{8} & 0 & 0 & \frac{73}{8} &
\end{array}.
$$

Now we impose the cutting plane constraint (coming from the first row)

$$-\frac{1}{2}x_1 - \frac{3}{8}x_5 - \frac{7}{8}u_1 + u_2 = -\frac{3}{8}.$$

Adding this to our previous tableau gives

$$\begin{array}{cccccccccccc}
 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\
x_3 & \frac{1}{2} & 0 & 1 & 0 & \frac{3}{8} & 0 & 0 & -\frac{1}{8} & 0 & \frac{11}{8} \\
x_2 & \frac{1}{2} & 1 & 0 & 1 & -\frac{1}{8} & 0 & 0 & \frac{3}{8} & 0 & \frac{31}{8} \\
x_7 & \frac{11}{9} & 0 & 0 & -1 & -\frac{5}{8} & 0 & 1 & -\frac{1}{8} & 0 & \frac{91}{8} \\
x_6 & \frac{1}{2} & 0 & 0 & 1 & \frac{3}{8} & 1 & 0 & -\frac{9}{8} & 0 & \frac{3}{8} \\
u_2 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{3}{8} & 0 & 0 & -\frac{7}{8} & 1 & -\frac{3}{8} \\
 & 1 & 0 & 0 & 1 & \frac{1}{8} & 0 & 0 & \frac{5}{8} & 0 &
\end{array}.$$

Using the dual simplex method, we pivot on the $(5,5)$ position to obtain

$$\begin{array}{cccccccccccc}
 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\
x_3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\
x_2 & \frac{2}{3} & 1 & 0 & 1 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 4 \\
x_7 & \frac{2}{3} & 0 & 0 & -1 & 0 & 0 & 1 & \frac{4}{3} & -\frac{5}{3} & 12 \\
x_6 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 1 & 0 \\
x_5 & \frac{4}{3} & 0 & 0 & 0 & 1 & 0 & 0 & \frac{7}{3} & -\frac{8}{3} & 1 \\
 & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 9
\end{array}.$$