Math 135, Section C7 - Solutions to Review Problems for Exam #1
June 18, 2010

#1 Find all x such that $|x-3| < \frac{7}{2}$ and express your answer in interval notation.

Solution: Recall that $|x-3|$ is the distance from x to 3. Thus the given inequality says that the distance from x to 3 is less than $\frac{7}{2}$. The two points at distance $\frac{7}{2}$ from 3 are $3+\frac{7}{2}=\frac{13}{2}$ and $3-\frac{7}{2}=\frac{-1}{2}$. Thus if x satisfies the given inequality we must have

$$\frac{-1}{2} < x < \frac{13}{2}.$$

This says that x is in the interval $(\frac{-1}{2}, \frac{13}{2})$.

#2 Write an equation for a staight line:

(a) which passes throught the point $(1,-2)$ and has slope 3;

Solution: The "point-slope" form of the equation for the straight line of slope m passing through the point (a,b) is $y-b=m(x-a)$. Using this expression gives $y+2=3(x-1)$ for an equation for the given line.

(b) which passes through the points $(3,5)$ and $(5,-8)$

Solution: The straight line through the points $(3,5)$ and $(5,-8)$ has slope $\frac{(-8-5)}{(5-3)}=\frac{-13}{2}$. Then, using the point slope form for the equation (as in part (a)) we see that $y-5=(\frac{-13}{2})(x-3)$ is an equation of the line.

(c) which passes through the point $(-4,1)$ and is parallel to the the staight line with equation y = -2x + 7;

Solution: The line with equation $y=-2x+7$ has slope -2 and so a line parallel to this line also has slope -2. Thus, using the point-slope form for the equation (as in part (a)), we see that the desired line has equation $y-1=-2(x+4)$.

(d) which passes through the points $(-1,1)$ and is perpendicular to the line through $(7,4)$ and $(2,2)$.

Solution: The line segment connecting the points $(7,4)$ and $(2,2)$ has slope $\frac{4-2}{7-2}=\frac{2}{5}$. Now a line L_1 of slope m_1 is perpendicular to a line L_2 of slope m_2 if and only if $m_1=\frac{-1}{m_2}$ (where we assume $m_1 \neq 0, m_2 \neq 0$). Thus the desired line has slope $\frac{-5}{2}$. Thus, using the point slope form for the equation (as in part (a)), we see that the desired line has equation $y-1=(\frac{-5}{2})(x+1)$.

#3 Write an equation of the circle with center $(3,2)$ and radius 5.

Solution: The distance from the point (x, y) to the point (a, b) is $\sqrt{(x-a)^2+(y-b)^2}$. Thus the point (x, y) is on the circle with center (a, b) and radius r if and only if

$$\sqrt{(x-a)^2+(y-b)^2} = r.$$

Squaring both sides puts the equation in the simpler form $(x-a)^2+(y-b)^2 = r^2$. Thus the circle described in the problem has equation $(x-3)^2+(y-2)^2 = 25$.

#4 (a) The graph of the equation $x^2+y^2-2x+4y-4 = 0$ is a circle. What are the center and raduis of this circle?

Solution: If k is any real number, we may write out $(x+k)^2$ and see that $(x+k)^2-k^2 = x^2+2kx$. This is known as the formula for "completing the square". Now the given equation may be rewritten as $(x^2-2x)+(y^2+4y)-4 = 0$. Using the completing the square formula, we see that we may rewrite x^2-2x as $(x-1)^2-1^2 = (x-1)^2-1$ and we may rewrite y^2+4y as $(y+2)^2-2^2 = (y+2)^2-4$. Substituting these expressions into $(x^2-2x)+(y^2+4y)-4 = 0$. we get $(x-1)^2-1+(y+2)^2-4-4 = 0$ and so $(x-1)^2+(y+2)^2 = 3^2$. Thus the given circle has center $(1, -2)$ and radius 3.

(b) The graph of the equation $7x-5y+23 = 0$ is a straight line. What is its slope? If the point $(a, 2a)$ is on this line, what is a?

Solution: The equation may be rewritten as $5y = 7x+23$ or $y = \frac{7}{5}(x)+\frac{23}{5}$. Thus the slope is $\frac{7}{5}$. If the point $(a, 2a)$ is on the line then $7a-5(2a)+23 = 0$ ad so $-3a+23 = 0$. Thus $a = \frac{23}{3}$.

#5 Suppose that $f(x) = 2x-1$ if $x < 1, f(1) = a$ and $f(x) = 3x+b$ if $x > 1$. Suppose further that $f(x)$ is continuous at $x = 1$. What are a and b?

Solution: If $f(x)$ is continuous at $x = 1$ then $lim_{x\to 1}f(x)$ must exist and so we must have $lim_{x\to 1^-}f(x) = lim_{x\to 1^+}f(x)$. Now $lim_{x\to 1^-}f(x) = lim_{x\to 1^-}2x-1 = 1$ and $lim_{x\to 1^+}f(x) = lim_{x\to 1^+}3x+b = 3+b$. Thus if $f(x)$ is continuous at $x = 1$ we must have $1 = 3+b$ and so $b = -2$. Also, if $f(x)$ is continuous at $x = 1$ we must have $lim_{x\to 1}f(x) = f(1)$. But we have seen that $lim_{x\to 1}f(x) = 1$ and we are given that $f(1) = a$. Thus $a = 1$.

#6 Find each of the following limits or state that the limit does not exist:

(a) $lim_{x\to 2}(x^2+\frac{x}{x-1})$

Solution: This is $lim_{x\to 2}x^2+lim_{x\to 2}\frac{x}{x-1}) = 4+2 = 6$.

(b) $lim_{x\to 2}\frac{x^2+x-6}{x^2-4}$

Solution: This is $lim_{x\to 2}\frac{(x-2)(x+3)}{(x-2)(x+2)} = lim_{x\to 2}\frac{x+3}{x+2} = \frac{5}{4}$.

(c) $lim_{x\to 2^+}\frac{x-2}{|x-2|}$

Solution: If $x > 2$ we have $|x-2| = x-2$ and so $lim_{x\to 2^+}\frac{x-2}{|x-2|} = lim_{x\to 2^+}\frac{x-2}{x-2} = lim_{x\to 2^+}1 = 1$.

(d) $lim_{x\to 2^-}\frac{x-2}{|x-2|}$

Solution: If $x < 2$ we have $|x-2| = -(x-2)$ and so $lim_{x\to 2^-}\frac{x-2}{|x-2|} = lim_{x\to 2^-}\frac{x-2}{-(x-2)} = lim_{x\to 2^-} - 1 = -1$.

(e) $lim_{x\to 2}\frac{x-2}{|x-2|}$

Solution: This limit does not exist since $lim_{x\to 2^-}\frac{x-2}{|x-2|} \neq lim_{x\to 2^+}\frac{x-2}{|x-2|}$.

(f) $lim_{x\to 2^+}\frac{1}{|x-2|}$

Solution: If $x > 2$ then $x-2 > 0$ and if x is close to 2 then $x-2$ is close to 0. Hence $lim_{x\to 2^+}\frac{1}{|x-2|} = +\infty$.

(g) $lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$

Solution: We rationalize the denominator to get

$$lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} = lim_{x\to 4}(\frac{x-4}{\sqrt{x}-2})(\frac{\sqrt{x}+2}{\sqrt{x}+2}) =$$

$$lim_{x\to 4}\frac{(x-4)(\sqrt{x}+2)}{x-4} = lim_{x\to 4}\sqrt{x}+2 = 4.$$

#7 Use the definition of derivative to find

(a) $f'(x)$ if $f(x) = x^2 + x + 1$

Solution:

$$f'(x) = lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} =$$

$$lim_{\Delta x\to 0}\frac{(x+\Delta x)^2 + (x+\Delta x) + 1 - (x^2+x+1)}{\Delta x} =$$

$$lim_{\Delta x\to 0}\frac{x^2 + 2x\Delta x + (\Delta x)^2 + x + \Delta x + 1 - x^2 - x - 1)}{\Delta x} =$$

$$lim_{\Delta x\to 0}\frac{2x\Delta x + (\Delta x)^2 + \Delta x}{\Delta x} =$$

$$lim_{\Delta x\to 0} 2x+\Delta x+1 = 2x+1.$$

(b) $g'(x)$ if $g(x)=\frac{2}{x+1}$

Solution:

$$g'(x) = lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} =$$

$$lim_{\Delta x\to 0}\frac{\frac{2}{x+\Delta x+1}-\frac{2}{x+1}}{\Delta x} =$$

$$lim_{\Delta x\to 0}\frac{\frac{2(x+1)}{(x+\Delta x+1)(x+1)}-\frac{2(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}}{\Delta x} =$$

$$lim_{\Delta x\to 0}\frac{\frac{-2\Delta x}{(x+\Delta x+1)(x+1)}}{\Delta x} =$$

$$lim_{\Delta x\to 0}\frac{-2\Delta x}{(x+\Delta x+1)(x+1)(\Delta x)} =$$

$$= lim_{\Delta x\to 0}\frac{-2}{(x+\Delta x+1)(x+1)} =$$

$$-2(x+1)^{-2}.$$

(c) $h'(x)$ if $h(x)=\sqrt{2x+3}$

Solution:

$$h'(x) = lim_{\Delta x\to 0}\frac{h(x+\Delta x)-h(x)}{\Delta x} =$$

$$lim_{\Delta x\to 0}\frac{\sqrt{2(x+\Delta x)+3}-\sqrt{2x+3}}{\Delta x} =$$

$$lim_{\Delta x\to 0}(\frac{\sqrt{2(x+\Delta x)+3}-\sqrt{2x+3}}{\Delta x})(\frac{\sqrt{2(x+\Delta x)+3}+\sqrt{2x+3}}{\sqrt{2(x+\Delta x)+3}+\sqrt{2x+3}}) =$$

$$lim_{\Delta x\to 0}\frac{2(x+\Delta x)+3-(2x+3)}{(\Delta x)(\sqrt{2(x+\Delta x)+3}+\sqrt{2x+3})} =$$

$$lim_{\Delta x \to 0} \frac{2\Delta x}{(\Delta x)(\sqrt{2(x+\Delta x)+3}+\sqrt{2x+3})} =$$

$$lim_{\Delta x \to 0} \frac{2}{(\sqrt{2(x+\Delta x)+3}+\sqrt{2x+3})} =$$

$$\frac{2}{2\sqrt{2x+3}} = (2x+3)^{\frac{-1}{2}}.$$

#8 In each part, find $f'(x)$ by any method:

(a) $f(x) = x^3 + 2x^2 - x + 3$

Solution: The derivative is the sum of the derivatives of the separate terms. By the rule for powers, this is $3x^2 + 4x - 1$.

(b) $f(x) = x\sqrt{x} + 3\frac{1}{x\sqrt{x}}$

Solution: We can rewrite $f(x)$ as $x^{\frac{3}{2}} + 3x^{\frac{-3}{2}}$. Then, using the rule for powers, the derviative is $f'(x) = (\frac{3}{2})x^{\frac{1}{2}} - (\frac{9}{2})x^{\frac{-5}{2}}$.

(c) $f(x) = sin(2x+3)$

Solution: Using the chain rule, we see that $f'(x) = 2cos(2x+3)$.

(d) $f(x) = e^{(2x+3)}$

Solution: Using the chain rule, we see that $f'(x) = 2e^{(2x+3)}$.

(e) $f(x) = e^{sin(x)}$

Solution: Using the chain rule, we see that $f'(x) = e^{sin(x)}cos(x)$.

(f) $f(x) = \frac{sin(x)}{e^{2x+3}}$

Solution: First use the quotient rule to get

$$f'(x) = \frac{(sin(x))'e^{2x+3} - (sin(x))(e^{2x+3})'}{(e^{2x+3})^2}.$$

Then use the the fact that $(sin(x))' = cos(x)$ and the chain rule to get

$$f'(x) = \frac{cos(x)e^{2x+3} - 2sin(x)e^{2x+3}}{(e^{2x+3})^2}.$$

While this can be simplified, this form is already an acceptable answer.

(g) $f(x) = \sqrt{\frac{x^2+1}{x^2+2}}$

Solution: We have $f(x) = (\frac{x^2+1}{x^2+2})^{\frac{1}{2}}$ and so

$$f'(x) = (\frac{1}{2})(\frac{x^2+1}{x^2+2})^{\frac{-1}{2}}(\frac{x^2+1}{x^2+2})'.$$

By the quotient rule, this is

$$(\frac{1}{2})(\frac{x^2+1}{x^2+2})^{\frac{-1}{2}}(\frac{(2x)(x^2+2)-(2x)(x^2+1)}{(x^2+2)^2}.$$

While this can be simplified, this form is already a satisfactory answer.

(h) $f(x) = (x^3 + 2x)^{17}$

Solution: Using the generalized rule for powers (which is a special case of the chain rule), we get $f'(x) = 17(x^3 + 2x)^{16}(3x^2 + 2)$.

(i) $f(x) = \ln(\sin(2x + 3))$

Solution: $f'(x) = \frac{(\sin(2x+3))'}{\sin(2x+3)} = \frac{2\cos(2x+3)}{(\sin(2x+3)}$.

(j) $f(x) = x^2 \sin(e^{2x} + 3)$

Solution: $f'(x) = (x^2)'(\sin(e^{2x} + 3)) + (x^2)(\sin(e^{2x} + 3))' =$

$$(2x)(\sin(e^{2x} + 3)) + (x^2)(\cos(e^{2x} + 3))(e^{2x} + 3)(2).$$

#9 A straight east-west road goes through the town of Bend. Suppose that at time t (in hours), where $0 \leq t \leq 10$, a car is $20 + 8t - t^2$ miles east of Bend.

(a) What it the velocity of the car at time t?

Solution: The position of the car is given by $s(t) = 20 + 8t - t^2$ and so the velocity is $s'(t) = 8 - 2t$.

(b) What is the speed of the car at time t?

Solution: The speed is the absolute value of the velocity. Hence it is $|8 - 2t|$.

(c) What is the acceleration of the car at time t?

Solution: The acceleration is the second derivative of the position. Hence it is $s''(t) = -2$.

(d) What it is the total distance traveled by the car between $t = 1$ and $t = 7$?

Solution: From $t = 1$ to $t = 4$, $s'(t)$ is ≥ 0 (and so the car is moving east). During this period the total distance traveled is $s(4) - s(1) = 36 - 27 = 9$ miles.

From $t = 4$ to $t = 7$, $s'(t)$ is ≤ 0 (and so the car is moving west). During this period the total distance traveled is $s(4) - s(7) = 36 - 27 = 9$ miles.

Thus the total distance traveled is $9 + 9 = 18$ miles. Note that this can be written as $|s(1) - s(4)| + |s(4) - s(7)|$.

#10 Suppose $f(x)$ and $g(x)$ are two functions which are defined for all real numbers. Suppose that

$f(-2) = 1, f(-1) = 0, f(0) = 2, f(1) = 1, f(2) = -1,$
$g(-2) = -2, g(-1) = 1, g(0) = 0, g(1) = -2, g(2) = 2,$
$f'(-2) = 0, f'(-1) = 3, f'(0) = -3, f'(1) = 2, f'(2) = -1,$
$g'(-2) = 2, g'(-1) = -1, g'(0) = 2, g'(1) = -2,$ and $g'(2) = 3.$

Let $h(x) = f(g(x))$ and $p(x) = g(f(x))$. Find:

(a) $h(2)$

Solution: $h(2) = f(g(2)) = f(2) = -1$;

(b) $h'(2)$

Solution:

Using the chain rule we have $h'(2) = f'(g(2))g'(2) = f'(2)g'(2) = (-1)(3) = -3$;

(c) $p(2)$

$p(2) = g(f(2)) = g(-1) = 1$;

Solution:

(d) $p'(2)$

Using the chain rule we have $p'(2) = g'(f(2))f'(2) = g'(-1)f'(2) = (-1)(-1) = 1$.

#11 Let

$$f(x) = 1 - x^2, \text{if } x < 2;$$

and

$$f(x) = ax + b, \text{if } x \geq 2.$$

Suppose $f(x)$ is differentiable at $x = 2$. What are a and b? Why?

Solution: First of all, since $f(x)$ is differentiable at $x = 2$ it must be continuous at $x = 2$. Thus

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 1 - x^2 = -3$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax + b = 2a + b$$

must be equal. Thus

$$2a + b = -3.$$

Also,

$$f'(2) = \lim_{x\to 2} \frac{f(2) - f(x)}{2 - x}$$

must exist and so

$$\lim_{x\to 2^-} \frac{f(2) - f(x)}{2 - x} = \lim_{x\to 2^-} (1 - 2^2) - (1 - x^2)2 - x =$$

$$\lim_{x\to 2^-} x^2 - 42 - x = \lim_{x\to 2^-} -(x + 2) = -4$$

must be equal to

$$\lim_{x\to 2^+} \frac{f(2) - f(x)}{2 - x} = \lim_{x\to 2^+} (2a + b) - (ax + b)2 - x =$$

$$\lim_{x\to 2} a(2 - x)2 - x = \lim_{x\to 2} a = a.$$

Thus if $f(x)$ is differentiable we must have $a = -4$ and $2a + b = -3$ so $b = 5$.

#12 Let

$$g(x) = x + 3, \text{if } x \leq 1;$$

and

$$g(x) = x^2 + 3, \text{if } x > 1.$$

Is $g(x)$ continuous at $x = 1$? Is $g(x)$ differentiable at $x = 1$? Explain your answers using the definitions.

Solution: $\lim x \to 1^- g(x) = \lim x \to 1^- x + 3 = 4$, $\lim x \to 1^+ g(x) = \lim x \to 1^+ x^2 + 3 = 4$, and $g(1) = 1 + 3 = 4$. Thus $g(x)$ is continuous at $x = 1$.

If $g(x)$ is differentiable at $x = 1$ then

$$g'(1) = \lim_{x\to 1} \frac{g(1) - g(x)}{1 - x}$$

must exist and so

$$\lim_{x\to 1^-} \frac{g(1) - g(x)}{1 - x}$$

and

$$\lim_{x\to 1^+} \frac{g(1)-g(x)}{1-x}$$

must be equal. But

$$\lim_{x\to 1^-} \frac{g(1)-g(x)}{1-x} = \lim_{x\to 1^-} \frac{(1+3)-(x+3}{1-x} =$$

$$\lim_{x\to 1^-} \frac{1-x}{1-x} = 1$$

while

$$\lim_{x\to 1^+} \frac{g(1)-g(x)}{1-x} = \lim_{x\to 1^+} \frac{(1+3)-(x^2+3}{1-x} =$$

$$\lim_{x\to 1^+} \frac{1-x^2}{1-x} = \lim_{x\to 1^+} 1+x = 2.$$

Thus $g(x)$ is not differentiable at $x = 1$.

#13 Show that

$$t^2 + 1 = \frac{10}{3t^2+2}$$

for some t in the interval $[-2, 2]$.

Solution: Let $h(t) = t^2 + 1 - \frac{10}{3t^2+2}$. We need to show that $h(t) = 0$ for some t in the interval $[-2, 2]$. Now $h(t)$ is continuous on the interval $[0, 2]$ (for it is a rational function whose denominator is never 0. Since $h(0) = 1 - \frac{10}{2} = 1 - 5 = -4 < 0$ and $h(2) = 2^2 + 1 - \frac{10}{14} = 5 - \frac{10}{14} > 0$, the Intermediate Value Theorem shows that $h(t) = 0$ for some t in the interval $(0, 2)$ (and, of course, this t is in the interval $[-1, 2]$).

#14 A block of ice in the shape of a cube originally has volume $1,000$ cubic centimeters. It is melting in such a way that it maintains its cubical shape at all times and that the length of each of its edges is decreasing at the rate of 1 centimeter per hour. At what rate is its surface area decreasing at the time its volume is 27 cubic centimeters?

Solution: Let t_0 be the time when the volume of the cube is 27 cubic centimeters. Let $x(t)$ be the length of a side of the cube at time t. Then the volume of the cube at time t is $x(t)^3$. Since the volume of the cube is originally (that is, at time $t = 0$) $1,000$ cubic centimeters, we have $x(0) = 10$. Also, since the volume of the cube at time t_0 is 27 we have $27 = x(t_0)^3$ and so $x(t_0 = 3$. Since $x(t)$ is decreasing at the rate of 1 centimeter per hour, at time t (hours) we have $x(t) = 10 - t$ centimeters. Thus $t_0 = 7$ (hours). Now the cube has six sides and each of these sides has area $x(t)^2 = (10 - t)^2$ at time t (hours). Let $S(t)$ denote the surface area of the cube at time t. Then $S(t) = 6(10 - t)^2$ and so, $s'(t)$

which is the rate of change of surface area is $12(10-t)(-1)$. Thus, $S'(t_0) = S'(7) = -36$. Thus the surface area is decreasing at the rate of 36 square centimeters per hour.

#15 Find an equation for the tangent line to the graph of $y = e^{x^2+3}$ at the point where $x = 1$.

Solution: $\frac{dy}{dx} = e^{x_2+3}(2x)$ and so when $x = 1$ the slope of the tangent line is $2e^4$. Since $y = e^4$ when $x = 1$, the point-slope form of the equation for the tangent line is

$$y - e^4 = (2e^4)(x-1).$$