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Some properties of Koornwinder polynomials

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Dedicated to Prof. R. Askey on his 65th birthday.

Introduction

Let $\mathcal{R}$ be the ring of Laurent polynomials in $x_1, \ldots, x_n$ over a field $\mathbb{F}$, and let $\mathcal{S}$ be the subring consisting of polynomials which are invariant under permutations and inversions of the variables. In [**6**], Koornwinder introduced a basis of $\mathcal{S}$ consisting of certain polynomials P_λ, whose coefficients depend on six parameters, q, t, a, b, c, d, in $\mathbb{F}$, and which are indexed by *partitions* λ,

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \text{ such that } \lambda_1 \geq \cdots \geq \lambda_n \geq 0$$

The Koornwinder polynomials are a generalization of the one-variable Askey-Wilson polynomials [**1**], and they possess several remarkable properties which were conjectured by Macdonald and Koornwinder. In [**3**], van Diejen showed that all of these properties were implied by a single conjecture—the *duality* conjecture. This conjecture was subsequently proved in [**10**] by a suitable generalization of the work of Cherednik to this setting. See also [**8**]. In fact, in [**10**] we introduced certain "nonsymmetric" polynomials E_λ, indexed by λ in $\mathbb{Z}^n$, which form a basis of $\mathcal{R}$. Most properties of the P_λ have natural nonsymmetric analogues for the E_λ, and in [**10**] we state and prove a duality conjecture for the E_λ, as well. In this paper we investigate the E_λ in greater detail. More precisely, we show that they:

- are orthogonal with respect to a natural inner product on $\mathcal{R}$,
- are triangular with respect to a certain partial order on the monomials,
- have positive coefficients for suitable limiting values of the parameters.

In view of the substantial interest and importance attached to the one-variable case, we include a brief, self-contained sketch of our principal results in the setting of Askey-Wilson polynomials.

1. Askey-Wilson polynomials

For the convenience of readers primarily interested in the one variable case, we summarize some of our main results for Askey-Wilson polynomials [**1**]. For the proofs (in the general case) see [**10**] and the later sections of this paper. Let $\mathbb{F}$ be

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the field of rational functions in the square roots of 5 parameters q, t_0, t_1, u_0, u_1:

$$\mathbb{F} = Q\left(q^{1/2}, t_0^{1/2}, t_1^{1/2}, u_0^{1/2}, u_1^{1/2}\right)$$

DEFINITION 1.1. *Let $\mathcal{H}$ be the $\mathbb{F}$-algebra with generators T_0, T_1, U_0, U_1 and relations:*

$$T_0 \sim t_0, T_1 \sim t_1, U_0 \sim u_0, U_1 \sim u_1, \quad T_1T_0U_0U_1 = q^{-1/2}$$

(Here, as elsewhere, $A \sim a$ means $A - A^{-1} = a^{1/2} - a^{-1/2}$.) We define elements X, Y in $\mathcal{H}$ by means of the formulas:

$$X = T_1^{-1}U_1^{-1}, \quad Y = T_1T_0.$$

Let $H = \langle T_0, T_1\rangle$, $H_0 = \langle T_1\rangle$, $\mathcal{R}_X = \langle X, X^{-1}\rangle$, $\mathcal{R}_Y = \langle Y, Y^{-1}\rangle$ be the subalgebras of $\mathcal{H}$ generated by the indicated elements. Then H_0 is two-dimensional (spanned by 1 and T_0), $\mathcal{R}_X$ and $\mathcal{R}_Y$ are isomorphic to the Laurent rings in X and Y, respectively, and we have the following (linear) isomorphisms:

$$\mathcal{H} \approx \mathcal{R}_X \otimes H, \quad H \approx H_0 \otimes \mathcal{R}_Y$$

The map $\chi\colon T_i \mapsto t_i^{1/2}, i = 0, 1$, extends to a character of H, and we consider the induced representation $\mathrm{Ind}_H^{\mathcal{H}}(\chi)$ acting on the quotient space $\mathcal{H}/\mathcal{I}$ where $\mathcal{I}$ is the left ideal generated by the elements $h - \chi(h), h \in H$. Let $\mathcal{R}$ be the Laurent ring in the variable x. Then we can realize the representation on $\mathcal{R}$, via the following formulas

$$\begin{aligned}
Xf(x) &= xf(x)\\
T_0f(x) &= t_0^{1/2}f(x) + t_0^{-1/2}\frac{(1-cx^{-1})(1-dx^{-1})}{(1-qx^{-2})}\left(f(qx^{-1}) - f(x)\right)\\
T_1f(x) &= t_1^{1/2}f(x) + t_1^{-1/2}\frac{(1-ax)(1-bx)}{(1-x^2)}\left(f(x^{-1}) - f(x)\right)
\end{aligned}$$

where we have

$$a = t_1^{1/2}u_1^{1/2}, b = -t_1^{1/2}u_1^{-1/2}, c = q^{1/2}t_0^{1/2}u_0^{1/2}, d = -q^{1/2}t_0^{1/2}u_0^{-1/2}.$$

The action of Y on $\mathcal{R}$ can be diagonalized, and we now describe the eigenvalues and eigenvectors. For this we introduce the "formal" q-logarithms of the parameters as follows:

$$k_0 = \log_q t_0, \quad k_1 = \log_q t_1, \quad l_0 = \log_q u_0, \quad l_1 = \log_q u_1;$$

and for $n \in \mathbb{Z}$, we put

$$\overline{n} = \begin{cases} n + \rho & \text{if } n \geq 0 \\ n - \rho & \text{if } n < 0 \end{cases}, \quad \text{where } \rho = \frac{1}{2}(k_0 + k_1),$$

THEOREM 1.1 ([**10**]). *The action of Y on $\mathcal{R}$ can be diagonalized, and for each $n \in \mathbb{Z}$ there is an eigenvector E_n, unique up to multiple, which satisfies*

$$YE_n = q^{\overline{n}}E_n$$

THEOREM 1.2 (see Theorem 4.1). *The polynomials E_n can be computed recursively by setting $E_0 = 1$, and defining, up to scalar multiple,*

$$\begin{aligned}
E_{-n-1} &= (a_nU_0 + b_n)E_n, \quad \textit{for } n \geq 0\\
E_n &= (c_nT_1 + d_n)E_{-n}, \quad \textit{for } n > 0
\end{aligned}$$

where $U_0 = q^{-1/2}T_0^{-1}X = q^{-1/2}\left(T_0 - t_0^{1/2} + t_0^{-1/2}\right)X$ *and*

$$\begin{aligned} a_n &= q^{\overline{n}} - q^{\overline{-n-1}} \\ b_n &= q^{\overline{n}}(u_0^{-1/2} - u_0^{1/2}) + q^{-1/2}(u_1^{-1/2} - u_1^{1/2}) \\ c_n &= q^{\overline{n}} - q^{\overline{-n}} \\ d_n &= q^{\overline{-n}}(t_1^{-1/2} - t_1^{1/2}) + (t_0^{-1/2} - t_0^{1/2}) \end{aligned}$$

REMARK 1.1. *This defines E_n recursively, along the following sequence:*

(1) $$n = 0, -1, 1, -2, 2, -3, \ldots$$

DEFINITION 1.2. *We also define (up to multiples)*

$$P_n = \left(T_1 + t_1^{-1/2}\right)E_n \quad \text{for } n \geq 0$$

$$Q_n = \left(T_1 - t_1^{1/2}\right)E_n \quad \text{for } n > 0$$

Finally, we normalize E_n, P_n, Q_n so that, in each case, the coefficient of x^n is 1. After this normalization, the other coefficients become rational functions in q, a, b, c, d.

THEOREM 1.3 ([**10**]). *P_n is the Askey-Wilson polynomial.*

THEOREM 1.4 (see Theorem 5.1). *The coefficient of x^m in E_n is nonzero iff m precedes n in the sequence* (1).

For the next result, we treat k_0, k_n, l_0, l_n as the primary parameters, and consider the limits

$$\widetilde{E}_n = \lim_{q\to 1} E_n, \quad \widetilde{P}_n = \lim_{q\to 1} P_n, \quad \widetilde{Q}_n = \lim_{q\to 1} Q_n.$$

For each integer n, define

$$e_n = \prod_{i=1}^{m}(k_0 + k_1 + i), \text{ where } m = \begin{cases} 2n & \text{if } n \geq 0 \\ 2|n| - 1 & \text{if } n < 0 \end{cases}$$

THEOREM 1.5 (see Theorem 6.4). *The coefficients of $e_n\widetilde{E}_n$ and $e_n\widetilde{P}_n$ are polynomials in $k_0 + k_1$ and $l_0 + l_1$ with non-negative integer coefficients.*

For our final result, we now specialize the parameters, assuming that $\mathbb{F} = \mathbb{Q}(q^{1/2})$ and that

$$k_0 = n_1 + n_2, \quad k_1 = n_1 - n_2, \quad l_0 = n_3 + n_4, \quad l_1 = n_3 - n_4$$

for some positive integers $n_i \in \mathbb{N}$. We define an inner product on $\mathcal{R}$ as follows:

First, recall that the Askey-Wilson weight function [**1**] is

$$\Delta(x) := \Delta_+(x)\Delta_+(x^{-1})$$

where

$$\Delta_+(x) := \frac{(x)_\infty(-x)_\infty(q^{1/2}x)_\infty(-q^{1/2}x)_\infty}{(ax)_\infty(bx)_\infty(cx)_\infty(dx)_\infty}$$

and $(u)_\infty = (u;q)_\infty$ denotes the following infinite product (see [**4**]):

$$(1-u)(1-qu)(1-q^2u)\cdots,$$

DEFINITION 1.3. *We define* $\mathcal{C}(x) := \Delta(x)\varphi(x)$, *where*

$$\varphi(x) := \frac{(x-a)(x-b)}{x^2-1}.$$

Next observe that under the present assumptions, $\mathcal{C}(x)$ is a Laurent polynomial.

DEFINITION 1.4. *Let* $\dagger$ *be the involution of* $\mathcal{R}$ *which maps* $q \mapsto q^{-1}, x \mapsto x^{-1}$, *and define an inner product on* $\mathcal{R}$ *by*

$$(f,g) := [fg^{\dagger}\mathcal{C}]_1.$$

where $[\,\cdot\,]_1$ *denotes the constant term of a Laurent polynomial.*

THEOREM 1.6 (see Corollary 3.2).

1. *The polynomials* $\{E_n : n \in \mathbb{Z}\}$ *are an orthogonal basis of* $\mathcal{R}$.
2. *The polynomials* $\{P_0, P_1, \ldots, Q_1, Q_2, \ldots\}$ *are an orthogonal basis of* $\mathcal{R}$.

2. Preliminaries

2.1. The Weyl group. Define $L_0 = \mathbb{Z}^n$, $L = \mathbb{Z}^n \oplus \mathbb{Z}\delta$, and regard L as a space of affine linear functions on L_0, via the pairing

$$(x, y + z\delta) = (x,y) + z, \quad x,y \in \mathbb{Z}^n,\ z \in \mathbb{Z}$$

where the inner product on the right is the usual one on $\mathbb{Z}^n$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the unit vectors in $\mathbb{Z}^n$, then

$$R_0 = \{\pm\varepsilon_i \pm \varepsilon_j, 2\varepsilon_i\} \subset L_0$$

is a root system of type C_n, and

$$R = \{\alpha + z\delta : \alpha \in R_0,\ z \in \mathbb{Z}\} \subset L$$

is an affine root system of type $\widetilde{C}_n$. We fix compatible positive root systems as follows

$$R_0^+ = \{-\varepsilon_i \pm \varepsilon_j, i<j\} \cup \{-2\varepsilon_i\}$$
$$R^+ = \{\alpha + n\delta : n > 0, \alpha \in R_0\} \cup R_0^+$$

Then the corresponding simple roots are

$$\alpha_0 = \delta + 2\varepsilon_1, \alpha_1 = -\varepsilon_1 + \varepsilon_2, dots, \alpha_{n-1} = -\varepsilon_{n-1} + \varepsilon_n, \alpha_n = -2\varepsilon_n.$$

For each α in R, let s_α denote the reflection in L_0 about the hyperplane

$$H_\alpha = \{x \in L_0 : (x,\alpha) = 0\}$$

The Weyl groups $\overset{\circ}{W}$ and W are the groups generated by the reflections from R_0 and R respectively. They are Coxeter groups on generators $s_1, dots, s_n$ and $s_0, \ldots, s_n$ respectively, where $s_i = s_{\alpha_i}$. If $\lambda = (\lambda_1, \ldots, \lambda_n) \in L_0$, then we have the following formulas for the action of the generators:

$$\begin{aligned} s_0 \cdot \lambda &= (-1-\lambda_1, \lambda_2, \ldots, \lambda_n) \\ s_i \cdot \lambda &= (\lambda_1, \lambda_2, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n) \quad i \neq 0, n \\ s_n \cdot \lambda &= (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, -\lambda_n) \end{aligned} \tag{2}$$

Via the pairing, we get a linear action $v \mapsto wv$ of W on L, satisfying

$$(w \cdot v_0, wv) = (v_0, v).$$

For $\lambda + r\delta \in L$, we have

$$(3) \qquad \begin{aligned} s_0(\lambda + r\delta) &= (-\lambda_1, \lambda_2, \ldots, \lambda_n) + (r - \lambda_1)\delta \\ s_i(\lambda + r\delta) &= s_i \cdot \lambda + r\delta, \ i \neq 0 \end{aligned}$$

As before, let $\mathcal{R}$ be the ring of Laurent polynomials in $x_1, \ldots, x_n$ over a field $\mathbb{F}$. Then $\mathcal{R}$ can be regarded as the group algebra of L_0 through the map

$$\lambda = (\lambda_1, \ldots, \lambda_n) \mapsto x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

and via (2) we obtain a representation of W on $\mathcal{R}$, given by the formulas

$$(4) \qquad \begin{aligned} s_0 \cdot f(x) &= x_1^{-1} f\left(x_1^{-1}, x_2, \ldots, x_n\right) \\ s_i \cdot f(x) &= f\left(x_1, x_2, \ldots, x_{i+1}, x_i, \ldots, x_n\right), \ i \neq 0, n \\ s_n \cdot f(x) &= f\left(x_1, x_2, \ldots, x_{n-1}, x_n^{-1}\right). \end{aligned}$$

Now fix an element q in $\mathbb{F}$, and consider the map from L to $\mathcal{R}$, given by

$$\lambda + z\delta \mapsto q^z x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

Using this map, we obtain *another* representation of W on $\mathcal{R}$ corresponding to (3). This is given by the following explicit formulas:

$$(5) \qquad \begin{aligned} s_0 f(x) &= f\left(q x_1^{-1}, x_2, \ldots, x_n\right) \\ s_i f(x) &= f\left(x_1, x_2, \ldots, x_{i+1}, x_i, \ldots, x_n\right), \ i \neq 0, n \\ s_n f(x) &= f\left(x_1, x_2, \ldots, x_{n-1}, x_n^{-1}\right). \end{aligned}$$

In the subsequent discussion, we will need both representations (4) and (5). We will distinguish them from each other by writing them as $f \mapsto w \cdot f$ and $f \mapsto wf$, respectively. (Note that $f \mapsto wf$ is an *algebra* automorphism of $\mathcal{R}$.)

2.2. The Hecke algebra. Let H be the Hecke algebra of W. This is a deformation of the group algebra of W, and depends on three parameters t, t_0, and t_n with square roots in $\mathbb{F}$. We recall (see e.g. [**7**]) that H is generated by elements $T_0, \ldots, T_n$ which satisfy the same braid relations as $s_0, s_1, \ldots, s_n$ (of type $\widetilde{C}_n$), and also satisfy quadratic relations, which we write in the form

$$T_i - T_i^{-1} = t_i^{1/2} - t_i^{-1/2}.$$

where $t_1 = t_2 = \cdots = t_{n-1} = t$. H contains a commutative subalgebra $\mathcal{R}_Y$ isomorphic to the Laurent ring in $Y_1, \ldots, Y_n$, where

$$Y_i = (T_i \ldots T_{n-1})(T_n \ldots T_0)(T_1^{-1} \ldots T_{i-1}^{-1}).$$

Following Noumi [**9**], we can define a representation of H on $\mathcal{R}$ which depends on two additional parameters u_0, u_n with square roots in $\mathbb{F}$, as follows: Put

$$a = t_n^{1/2} u_n^{1/2}, b = -t_n^{1/2} u_n^{-1/2}, c = q^{1/2} t_0^{1/2} u_0^{1/2}, d = -q^{1/2} t_0^{1/2} u_0^{-1/2}$$

and define

$$\begin{aligned} T_0 f &= t_0^{1/2} f + t_0^{-1/2} \frac{(1 - c x_1^{-1})(1 - d x_1^{-1})}{(1 - q x_1^{-2})} (s_0 f - f) \\ T_i f &= t^{1/2} f + t^{-1/2} \frac{(1 - t x_i x_{i+1}^{-1})}{(1 - x_i x_{i+1}^{-1})} (s_i f - f), \ i \neq 0, n \\ T_n f &= t_n^{1/2} f + t_n^{-1/2} \frac{(1 - a x_n)(1 - b x_n)}{(1 - x_n^2)} (s_n f - f) \end{aligned}$$

Then these operators satisfy the quadratic and braid relations, and extend to a representation of H on $\mathcal{R}$. The action of $\mathcal{R}_Y$ can be simultaneously diagonalized, and the nonsymmetric Koornwinder polynomials E_λ (see [**10**]) are the corresponding eigenbasis. The eigenvalues are given as follows: Put

$$k = \log_q t, \quad k_0 = \log_q t_0, \quad k_n = \log_q t_n, \quad l_0 = \log_q u_0, \quad l_n = \log_q u_n.$$

and $\rho = (\rho_1, \dots, \rho_n)$ with

$$\rho_i = \frac{k_0 + k_n}{2} + (n-i)k.$$

DEFINITION 2.1. *For λ in $\mathbb{Z}^n$, let $\overset{o}{w}_\lambda$ be the (unique) shortest element of $\overset{o}{W}$ such that $\overset{o}{w}_\lambda^{-1} \cdot \lambda$ is a partition, and define*

$$\overline{\lambda} = \lambda + \overset{o}{w}_\lambda \cdot \rho$$

PROPOSITION 2.1 (see [**10**]). *The Y_i are simultaneously diagonalizable, and for each λ in $\mathbb{Z}^n$, there is an eigenvector E_λ satisfying*

$$Y_i E_\lambda = q^{\overline{\lambda}_i} E_\lambda, \qquad i = 1, \dots, n. \ \square$$

(The E_λ's are unique up to scalar multiples and can be normalized by requiring that the coefficient of x^λ be 1.)

3. Orthogonality

In this section we specialize parameters, assuming that $\mathbb{F} = \mathbb{Q}\left(q^{1/2}\right)$ and that

$$t = q^{n_0}, a = q^{n_1}, b = -q^{n_2}, c = q^{n_3+1/2}, d = -q^{n_4+1/2}$$

for some integers $n_i \in \mathbb{N}$. Equivalently, we have

$$t = q^{n_0}, t_0 = q^{n_1+n_2}, t_n = q^{n_1-n_2}, u_0 = q^{n_3+n_4}, u_n = q^{n_3-n_4}$$
$$k = n_0, k_0 = n_1 + n_2, k_n = n_1 - n_2, l_0 = n_3 + n_4, l_n = n_3 - n_4$$

We write $(u)_\infty$ for the infinite product

$$(1-u)(1-qu)(1-q^2u)\dots,$$

and following Koornwinder [**6**] we define $\Delta(x) := \Delta_+(x)\Delta_+(x^{-1})$, where

$$\Delta_+(x) := \prod_i \frac{(x_i)_\infty(-x_i)_\infty(q^{1/2}x_i)_\infty(-q^{1/2}x_i)_\infty}{(ax_i)_\infty(bx_i)_\infty(cx_i)_\infty(dx_i)_\infty} \prod_{i<j} \frac{(x_ix_j)_\infty(x_ix_j^{-1})_\infty}{(tx_ix_j)_\infty(tx_ix_j^{-1})_\infty}.$$

Under the present assumptions, Δ_+ and Δ are Laurent polynomials in $\mathcal{R}$ and $\mathcal{S}$, respectively, and Koornwinder [**6**] has shown that the P_λ are mutually orthogonal with respect to the inner product

$$\langle f, g \rangle = [fg\Delta]_1,$$

where $[\,\cdot\,]_1$ denotes the constant term of a Laurent polynomial. In this section we shall prove the nonsymmetric analog of this result.

DEFINITION 3.1. *We define $\mathcal{C}(x) := \Delta(x)\varphi(x)$, where*

$$\varphi(x) := \prod_i \frac{(x_i - a)(x_i - b)}{x_i^2 - 1} \prod_{i<j} \frac{(x_ix_j - t)(x_ix_j^{-1} - t)}{(x_ix_j - 1)(x_ix_j^{-1} - 1)}.$$

Observe that the denominator of φ "occurs" in $\Delta(x^{-1})$, and hence $\mathcal{C}(x)$ is a Laurent polynomial as well.

DEFINITION 3.2. *We define an inner product on $\mathcal{R}$ by*

$$(f,g) := [fg^\dagger \mathcal{C}]_1.$$

where $\dagger$ is the involution of $\mathcal{R}$ which maps $q \mapsto q^{-1}$, and $x_i \mapsto x_i^{-1}$.

Our main result is:

THEOREM 3.1. *For all f, g in $\mathcal{R}$ and $i = 0, \ldots, n$, we have*

$$(T_i f, g) = (f, T_i^{-1} g)\,.$$

PROOF. Writing T_i in the form

$$T_i = f_i + g_i s_i$$

it is easy to check that $g_i^\dagger = g_i$, while

$$f_i - f_i^\dagger = t_i^{1/2} - t_i^{-1/2} = T_i - T_i^{-1}.$$

which implies

$$T_i^{-1} = f_i^\dagger + g_i^\dagger s_i$$

(see also [**10**]). Then we get

$$\begin{aligned}(T_i f, g) - (f, T_i^{-1} g) &= (f_i f + g_i s_i f, g) - (f, f_i^\dagger g + g_i^\dagger s_i g)\\ &= (g_i s_i f, g) - (g_i f, s_i g)\\ &= [(s_i f)\, g^\dagger (g_i \mathcal{C})]_1 - [f\,(s_i g^\dagger)\,(g_i \mathcal{C})]_1\\ &= [(s_i f)\, g^\dagger (g_i \mathcal{C})]_1 - [(s_i f)\, g^\dagger s_i (g_i \mathcal{C})]_1.\end{aligned}$$

(To obtain the last equality, we used the fact that s_i is an algebra homomorphism and does not affect the constant term.)

Thus it suffices to show that $s_i(g_i\mathcal{C}) = g_i\mathcal{C}$ for all i, or equivalently that

$$\frac{s_i(g_i)}{g_i}\frac{s_i(\varphi)}{\varphi}\frac{s_i(\Delta)}{\Delta} = 1 \quad \text{for} \quad i = 0, \ldots, n$$

If $i \neq 0$, then $s_i(\Delta) = \Delta$, and we have

$$g_i = t^{-1/2}\frac{(1 - t x_i x_{i+1}^{-1})}{(1 - x_i x_{i+1}^{-1})} \quad \text{for} \quad 1 \leq i < n$$

$$g_n = t_n^{-1/2}\frac{(1 - a x_n)(1 - b x_n)}{(1 - x_n^2)}.$$

Now by direct computation, we get

$$\frac{\varphi}{s_i(\varphi)} = \frac{(1 - t x_{i+1} x_i^{-1})}{(1 - x_{i+1} x_i^{-1})}\frac{(1 - x_i x_{i+1}^{-1})}{(1 - t x_i x_{i+1}^{-1})} = \frac{s_i(g_i)}{g_i} \quad \text{for} \quad 1 \leq i < n$$

$$\frac{\varphi}{s_n(\varphi)} = \frac{(1 - a x_n^{-1})(1 - b x_n^{-1})}{(1 - x_n^{-2})}\frac{(1 - x_n^2)}{(1 - a x_n)(1 - b x_n)} = \frac{s_n(g_n)}{g_n}$$

which implies the result in these cases.

For $i = 0$, we note that if u is independent of x_1, then

$$\frac{(u x_1)_\infty\,(u x_1^{-1})_\infty}{1 - u x_1^{-1}} = (u x_1)_\infty (u q x_1^{-1})_\infty$$

is s_0-invariant. This implies that

$$\frac{s_0\left((ux_1)_\infty (ux_1^{-1})_\infty\right)}{(ux_1)_\infty (ux_1^{-1})_\infty} = \frac{s_0\left(1-ux_1^{-1}\right)}{1-ux_1^{-1}}$$

Using this we get

$$\frac{s_0(\Delta)}{\Delta} = \frac{s_0(\psi_1)}{\psi_1}$$

where

$$\psi_1 = \frac{(1-x_1^{-1})(1+x_1^{-1})(1-q^{1/2}x_1^{-1})(1+q^{1/2}x_1^{-1})}{(1-ax_1^{-1})(1-bx_1^{-1})(1-cx_1^{-1})(1-dx_1^{-1})} \times \prod_{1<j} \frac{(1-x_1^{-1}x_j)(1-x_1^{-1}x_j^{-1})}{(1-tx_1^{-1}x_j)(1-tx_1^{-1}x_j^{-1})}.$$

On the other hand, by a direct calculation, we see that

$$\frac{s_0(\varphi)}{\varphi} = \frac{s_0(\psi_2)}{\psi_2}$$

where

$$\psi_2 = \frac{(x_1-a)(x_1-b)}{x_1^2-1} \prod_{1<j} \frac{(x_1x_j-t)(x_1x_j^{-1}-t)}{(x_1x_j-1)(x_1x_j^{-1}-1)}.$$

Now, since

$$g_0 = t_0^{-1/2} \frac{(1-cx_1^{-1})(1-bx_1^{-1})}{(1-qx_1^{-2})}$$

it follows that

$$\psi_1\psi_2 g_0 = t_0^{-1/2}.$$

This implies that

$$\frac{s_0(g_0)}{g_0}\frac{s_0(\varphi)}{\varphi}\frac{s_0(\Delta)}{\Delta} = \frac{s_0\left(\psi_1\psi_2 g_0\right)}{\psi_1\psi_2 g_0} = 1,$$

which completes the proof. □

From the theorem it follows that the T_i are †-*unitary* operators. But then so are the Y_i, and since the nonsymmetric Koornwinder polynomials E_λ are simultaneous eigenfunctions of the Y_i, with distinct eigenvalues $q^{\overline{\lambda}_i}$, we deduce the following:

COROLLARY 3.2. *The E_λ are mutually orthogonal with respect to* $(\cdot,\cdot)$.

4. Recursion

In this section we provide explicit recursive formulas for the nonsymmetric Koornwinder polynomials. We work once again with general (unspecialized) parameters q, t, t_0, t_n, u_0,u_n. The recursion is with respect to the W-action (4).

THEOREM 4.1. *Suppose $\lambda = s_i \cdot \mu \neq \mu$ then, up to a scalar multiple,*

$$E_\lambda = \begin{cases} \left[\left(q^{\overline{\mu}_i} - q^{\overline{\lambda}_i}\right) T_i + c_i\right] E_\mu & \text{for } i \neq 0 \\ \left[\left(q^{\overline{\lambda}_1} - q^{\overline{\mu}_1}\right) U_0 + c_0\right] E_\mu & \text{for } i = 0 \end{cases}$$

where $U_0 = q^{-1/2}T_0^{-1}x_1$, *and*

$$c_0 = q^{\overline{\lambda}_1}(u_0^{-1/2} - u_0^{1/2}) + q^{-1/2}(u_n^{-1/2} - u_n^{1/2})$$
$$c_i = q^{\overline{\mu}_i}(t^{-1/2} - t^{1/2}) \qquad \textit{for} \quad 0 < i < n$$
$$c_n = q^{\overline{\mu}_n}(t_n^{-1/2} - t_n^{1/2}) + (t_0^{-1/2} - t_0^{1/2})$$

PROOF. By Theorems 5.3 and 6.1 of [**10**], E_λ is a multiple of $S_i E_\mu$, where

$$S_i = [T_i, Y_i] \quad \text{for} \quad i = 1, \dots, n;$$
$$S_0 = [Y_1, U_n] \quad \text{with} \quad U_n = x_1^{-1} T_0 Y_1^{-1}$$

To deduce the theorem we use the relations

$$T_i \sim t_i, U_0 \sim u_0, U_n \sim u_n$$

from [**10**], where

$$t_1 = \cdots = t_{n-1} = t,$$

and

$$Z \sim z \text{ means } Z - Z^{-1} = z^{1/2} - z^{-1/2}.$$

For $i \neq 0, n$ we have

$$S_i = [T_i^{-1}, Y_i] = T_i^{-1} Y_i - Y_i T_i^{-1}.$$

But $Y_i T_i^{-1} = T_i Y_{i+1}$, so we get

$$S_i = T_i(Y_i - Y_{i+1}) + (t^{-1/2} - t^{1/2})Y_i$$

Since

$$Y_i E_\mu = q^{\overline{\mu}_i} E_\mu$$
$$Y_{i+1} E_\mu = q^{\overline{\mu}_{i+1}} E_\mu = q^{\overline{\lambda}_i} E_\mu,$$

the result follows for $i \neq 0, n$.

For $i = n$, we get

$$S_n = T_n^{-1} Y_n - Y_n T_n^{-1}.$$

But

$$Y_n T_n^{-1} = (T_n \dots T_1) T_0 (T_1^{-1} \dots T_n^{-1}),$$

which is conjugate to T_0. Hence we get

$$Y_n T_n^{-1} \sim t_0,$$

and it follows that

$$S_n = T_n(Y_n - Y_n^{-1}) + (t_n^{-1/2} - t_n^{1/2})Y_n + (t_0^{-1/2} - t_0^{1/2})$$

Since

$$Y_n E_\mu = q^{\overline{\mu}_n} E_\mu$$
$$Y_n^{-1} E_\mu = q^{-\overline{\mu}_n} E_\mu = q^{\overline{\lambda}_n} E_\mu,$$

the result follows in this case.

For $i = 0$, we have

$$S_0 = [Y_1, U_n^{-1}] = [Y_1, Y_1 T_0^{-1} x_1] = q^{1/2} Y_1 S_0'$$

where $S_0' = [Y_1, U_0]$. Since Y_1 is invertible, E_λ is also a multiple of $S_0' E_\mu$. Now

$$Y_1 U_0 = q^{-1/2} U_n^{-1} = q^{-1/2}(U_n + u_n^{-1/2} - u_n^{1/2})$$
$$U_n = q^{-1/2} U_0^{-1} Y_1^{-1} = q^{-1/2}(U_0 + u_0^{-1/2} - u_0^{1/2}) Y_1^{-1}.$$

So we get

$$\begin{aligned} S_0' &= Y_1 U_0 - U_0 Y_1 \\ &= U_0(q^{-1} Y_1^{-1} - Y_1) + (u_0^{-1/2} - u_0^{1/2}) q^{-1} Y_1^{-1} + q^{-1/2}(u_n^{-1/2} - u_n^{1/2}). \end{aligned}$$

The result follows since

$$Y_1 E_\mu = q^{\overline{\mu}_1} E_\mu$$
$$q^{-1} Y_1^{-1} E_\mu = q^{-\overline{\mu}_1 - 1} E_\mu = q^{\overline{\lambda}_1} E_\mu.$$

□

5. Triangularity

In this section and the next we consider the coefficients of the Koornwinder polynomials with respect to the monomial basis. For this we shall need various basic facts about the Bruhat order and Coxeter groups, which can be found in [**5**], for example.

DEFINITION 5.1. *We define w_λ to be the shortest element in W such that $w_\lambda \cdot \lambda = 0$.*

(This conflicts with the notation in [**10**] but that should not cause confusion.)

The element w_λ admits the following alternative description: Let $\geq$ denote the Bruhat order on W with respect to the generators $s_0, \cdots, s_n$. Now W acts transitively on $\mathbb{Z}^n$ via the action (2) and the stabilizer of 0 is $\overset{\circ}{W}$. Thus we can identify

$$\mathbb{Z}^n \approx W/\, \overset{\circ}{W},$$

and w_λ^{-1} is the (unique) coset representative of $\lambda \in \mathbb{Z}^n = W/\, \overset{\circ}{W}$, which is minimal with respect to the Bruhat order. Our second main result is:

THEOREM 5.1. *The coefficient of x^μ in E_λ is nonzero if and only if $w_\lambda \geq w_\mu$.*

The proof is somewhat involved, and in this section we will prove the "only if" implication. For this we need several intermediate results.

LEMMA 5.2. *For $0 \leq i \leq n$ and $w \in W$,*

$$\textit{either} \quad w < w s_i \quad \textit{or} \quad w > w s_i.$$

LEMMA 5.3. *For $0 \leq i \leq n$ and $w, w' \in W$,*

$$\textit{if } w \leq w' \textit{ and } w' \leq w' s_i \textit{ then } w s_i \leq w' s_i.$$

PROOF. These are basic properties of the Bruhat order, see Chapter 5 in [**5**]. □

LEMMA 5.4. *Suppose $\lambda \in \mathbb{Z}^n$ and $s_i \cdot \lambda \neq \lambda$ for some $0 \leq i \leq n$. Then*

$$w_{s_i \cdot \lambda} = w_\lambda s_i.$$

LEMMA 5.5. *Suppose $\lambda \in \mathbb{Z}^n$, and that $\nu \in \mathbb{Z}^n$ is a convex combination of λ and $\mu = s_i \cdot \lambda$. Then,*

$$\textit{either} \quad w_\lambda \leq w_\nu \leq w_\mu \quad \textit{or} \quad w_\mu \leq w_\nu \leq w_\lambda .$$

We shall prove these lemmas in the appendix. In order to use them, we also need the following result:

LEMMA 5.6.

1. *If x^μ occurs in $U_0 x^\lambda$, then μ is a convex combination of λ and $s_0 \cdot \lambda$.*
2. *For $1 \leq i \leq n$, if x^μ occurs in $T_i x^\lambda$, then μ is a convex combination of λ and $s_i \cdot \lambda$.*

PROOF. For the first case, we recall that

$$s_0 \cdot (\lambda_1, \cdots, \lambda_n) = (-\lambda_1 - 1, \cdots, \lambda_n) .$$

Now U_0 commutes with multiplication by $x_2, \cdots, x_n$. Therefore we may assume

$$x^\lambda = x_1^m \text{ for some integer } m.$$

and we need to verify that if

$$x_1^l \text{ occurs in } U_0 x_1^m, \text{ then } \begin{cases} m-1 \leq l \leq m & \text{if } m \geq 0 \\ m \leq l \leq -m-1 & \text{if } m < 0 \end{cases} .$$

Since

$$U_0 x_1^m = q^{-1/2} T_0^{-1} x_1^{m+1}$$

it suffices to establish the following two assertions:

- $T_0^{-1} x_1^k$ is a linear combination of x_1^l for l between k and $-k$.
- If $k > 0$, then x_1^k does not occur in $T_0^{-1} x_1^k$.

To see this we observe that

$$T_0^{-1} x_1^k = t_0^{-1/2} x_1^k + t_0^{-1/2} \frac{(1 - c x_1^{-1})(1 - d x_1^{-1})}{(1 - q x_1^{-2})} \left(\left(q x_1^{-1}\right)^k - x_1^k \right) .$$

If $k \leq 0$, we rewrite this as

$$T_0^{-1} x_1^k = t_0^{-1/2} x_1^k \left(1 + q^{-1}(x_1 - c)(x_1 - d) \frac{\left(q^{-1} x_1^2\right)^{-k} - 1}{q^{-1} x_1^2 - 1} \right) ;$$

and observe that the parenthetical expression is a polynomial of degree $-2k$ in x_1. If $k > 0$, we rewrite the expression in the form

$$T_0^{-1} x_1^k = t_0^{-1/2} x_1^k \left(1 - (1 - c x_1^{-1})(1 - d x_1^{-1}) \frac{\left(q x_1^{-2}\right)^k - 1}{q x_1^{-2} - 1} \right) ;$$

and observe that the parenthetical expression is a polynomial of degree $2k$ in x_1^{-1}, *without* constant term. These considerations imply the two assertions and, thereby, the first part of the lemma. The proof of the second part of the lemma is similar and easier, and we leave the details to the reader. □

We can now prove the first part of Theorem 5.1. More precisely:

LEMMA 5.7. *If x^μ occurs in E_λ then $w_\lambda \geq w_\mu$.*

PROOF. We shall proceed by induction on the Bruhat order of w_λ. If w_λ is the identity, then $\lambda = 0$ and $E_\lambda = 1$, and the result is trivially true. If w_λ is not the identity, then there is some i such that

$$w_\lambda > w_\lambda s_i.$$

By Lemma 5.4, the right side equals w_μ where

$$\mu = s_i \cdot \lambda.$$

By Theorem 4.1 and Lemma 5.6, we conclude that if x^ν occurs in E_λ, then ν is a convex combination of γ and $s_i \cdot \gamma$ for some γ such that x^γ occurs in E_μ. By Lemma 5.5 this implies that

$$w_\nu \leq \max\left(w_\gamma, w_\gamma s_i\right).$$

On the other hand, by the inductive hypothesis, we have

$$w_\mu \geq w_\gamma,$$

and by Lemma 5.4 and Lemma 5.3 we deduce that

$$w_\lambda = w_\mu s_i \geq \max\left(w_\gamma, w_\gamma s_i\right).$$

Combining these, we deduce that $w_\lambda \geq w_\nu$. □

6. Positivity

In this section we treat k, k_0, k_n, l_0, l_n as the primary indeterminates, rather than as formal q-logarithms of t, t_0, t_n, u_0, u_n. Then we have:

$$t = q^k, \quad t_0 = q^{k_0}, \quad t_n = q^{k_n}, \quad u_0 = q^{l_0}, \quad u_n = q^{l_n}.$$

DEFINITION 6.1. *With the above specialization, we define $\widetilde{E}_\lambda$ to be the limit*

$$\widetilde{E}_\lambda = \lim_{q \to 1} E_\lambda$$

A priori, it is not obvious that this limit exists. However we shall deduce this, and more, from a recursion formula for the $\widetilde{E}_\lambda$. Recall the action $f \mapsto s_i \cdot f$ defined in formula 4, then we have

THEOREM 6.1. *Suppose $\lambda = s_i \cdot \mu \neq \mu$; then, up to a scalar multiple,*

$$\widetilde{E}_\lambda = (e s_i + f) \cdot \widetilde{E}_\mu$$

where

$$e,\ f = \begin{cases} \overline{\mu}_1 - \overline{\lambda}_1, & l_0 + l_n \quad \text{for} \quad i = 0 \\ \overline{\lambda}_i - \overline{\mu}_i, & k \quad \text{for} \quad 0 < i < n \\ \overline{\lambda}_n - \overline{\mu}_n, & k_0 + k_n \quad \text{for} \quad i = n \end{cases}.$$

PROOF. From the formula for T_i, it follows that, as $q \to 1$,

$$\begin{aligned} T_i f &\to s_i \cdot f, \quad \text{for} \quad i > 0 \\ U_0 &\to s_0 \cdot f, \quad \text{for} \quad i = 0 \end{aligned}$$

Now, up to scalar multiple, the recursions of 4.1 can be rewritten as

$$T_i + c_i \Big/ \left(q^{\overline{\mu}_i} - q^{\overline{\lambda}_i}\right) \quad \text{and} \quad U_0 + c_0 \Big/ \left(q^{\overline{\lambda}_1} - q^{\overline{\mu}_1}\right).$$

As $q \to 1$, we obtain

$$\begin{aligned}\frac{c_0}{q^{\overline{\lambda}_1} - q^{\overline{\mu}_1}} &= q^{\overline{\lambda}_1} \frac{u_0^{-1/2} - u_0^{1/2}}{q^{\overline{\lambda}_1} - q^{\overline{\mu}_1}} + q^{-1/2} \frac{u_n^{-1/2} - u_n^{1/2}}{q^{\overline{\lambda}_1} - q^{\overline{\mu}_1}} \\ &\longrightarrow \frac{l_0 + l_n}{\overline{\mu}_1 - \overline{\lambda}_1} = \frac{f}{e}.\end{aligned}$$

For $i > 0$, a similar calculation shows that

$$\frac{c_i}{q^{\overline{\mu}_i} - q^{\overline{\lambda}_i}} \longrightarrow \frac{f}{e},$$

and the result follows. $\square$

LEMMA 6.2. *If $\lambda = s_i \cdot \mu \neq \mu$, then $w_\lambda > w_\mu$ iff one of the following conditions is satisfied*

- $i = 0 \quad$ *and* $\quad \mu_1 \geq 0$
- $0 < i < n \quad$ *and* $\quad \mu_i < \mu_{i+1}$
- $i = n \quad$ *and* $\quad \mu_n < 0$.

LEMMA 6.3. *Suppose $\lambda = s_i \cdot \mu \neq \mu$, and one of the conditions of the above lemma is satisfied, then the scalar e in Theorem 6.1 is of the form*

$$e = d_0 + d_1(k_0 + k_n) + d_2 k$$

where d_0 is a positive integer and d_1 and d_2 are non-negative integers.

We shall prove these lemmas in the appendix. Our positivity result for the $\widetilde{E}_\lambda$ is the following:

THEOREM 6.4. *There exists a scalar $c_{\lambda\lambda}$ such that we have*

$$c_{\lambda\lambda}\widetilde{E}_\lambda = \sum_{\mu : w_\mu \leq w_\lambda} c_{\lambda\mu} x^\mu,$$

where each $c_{\lambda\mu}$ is a nonzero polynomial in $k_0 + k_n, k$, and $l_0 + l_n$, with nonnegative integral coefficients.

PROOF. Fix a reduced decomposition of w_λ as follows:

$$w_\lambda = s_{i_1} \cdots s_{i_l}.$$

and for $j = 0, \ldots, l$, define

$$\lambda^{(j)} = s_{i_j} \cdots s_{i_1} \cdot 0$$

Then by Theorem 6.1 we see that there is a scalar $c_{\lambda\lambda}$ such that

$$c_{\lambda\lambda}\widetilde{E}_\lambda = (e_l s_{i_l} + f_{i_l}) \cdots (e_1 s_{i_1} + f_{i_1}) \cdot 1 \tag{6}$$

where,

$$e_j = \begin{cases} \overline{\lambda_1^{(j-1)}} - \overline{\lambda_1^{(j)}} & \text{if} \quad i_j = 0 \\ \overline{\lambda_{i_j}^{(j)}} - \overline{\lambda_{i_j}^{(j-1)}} & \text{if} \quad i_j \neq 0 \end{cases}.$$

Multiplying out the right side of formula 6, we can write it in the form $\sum c_{\lambda\mu} x^\mu$. By Theorem 6.1 and Lemma 6.3 the e_j's and f_{i_j}'s are nonzero polynomials in k_0+k_n, k, and $l_0 + l_n$ with non-negative integral coefficients. Since the coeffcients $c_{\lambda\mu}$ are sums of products of the e_j's and f_{i_j}'s, they too are positive. The monomials which occur in the expansion of formula 6 are those obtained by applying subexpressions of

$$s_{i_l} \dots s_{i_1}$$

to the constant function 1. Now if $w_\mu \le w_\lambda$, then w_μ can be written as a subexpression of

$$w_\lambda = s_{i_1} \dots s_{i_l}.$$

Taking the corresponding subexpressions on the right side of formula 6, we see that the coefficient of x^μ in $c_{\lambda\lambda}\widetilde{E}_\lambda$ is not zero. In conclusion we note that by the minimality of w_λ, no proper subexpression applied to 1 gives x^λ, hence

$$c_{\lambda\lambda} = \prod_{j=1}^{l} e_j.$$

□

7. Appendix: Bruhat order

For each w in W, let $l(w)$ be the length of a reduced (*i.e.*, shortest) expression of w in terms of the s_i. Then we have

$$l(w) = |\Pi(w)|$$

where

$$\Pi(w) = \left\{\alpha \in R^+ : w\alpha \notin R^+\right\}.$$

The Bruhat order on W can be characterized in the following ways

1. For α in R_+ , we have $ws_\alpha < w$ iff α is in $\Pi(w)$.
2. $w' < w$ iff w' can be obtained by omitting some factors in a fixed reduced expression of w.

Similar results hold for $\overset{o}{W}$ and R_0. For λ in $\mathbb{Z}^n$, let w_λ, and $\overset{o}{w}_\lambda$ be as in Definitions 5.1 and 2.1.

LEMMA 7.1. *For λ in $\mathbb{Z}^n$ we have*

1. $\Pi(w_\lambda) = \{\alpha \in R^+ : (\lambda, \alpha) < 0\}$
2. $\Pi\left(\overset{o}{w}{}_\lambda^{-1}\right) = \left\{\alpha \in R_0^+ : (\lambda, \alpha) > 0\right\}$

PROOF. See Theorem 1.4 in [**2**]. □

We can now prove Lemmas 5.4, 6.2, and 5.5.

PROOF (OF LEMMA 5.4). Write $\mu = s_i \cdot \lambda$ and $w = w_\lambda s_i$, then we have

$$w \cdot \mu = w_\lambda s_i \cdot \mu = w_\lambda \cdot \lambda = 0$$

By minimality of w_μ, this implies

$$w > w_\mu.$$

Therefore to prove that $w = w_\mu$, it suffices to show that

$$l(w) \le l(w_\mu).$$

To prove this we first assume that $\alpha_i \in \Pi(w_\lambda)$ and put

$$S = \Pi(w_\lambda) \setminus \{\alpha_i\}$$

Then we have

$$s_i \cdot S \subset R^+$$

Since

$$(\mu, s_i \cdot \alpha) = (s_i \cdot \lambda, s_i \cdot \alpha) = (\lambda, \alpha)$$

It follows from Lemma 7.1 that

$$s_i \cdot S \subset \Pi(w_\mu).$$

This implies that

$$l(w_\lambda) - 1 \leq l(w_\mu).$$

However, $\alpha_i \in \Pi(w_\lambda)$ implies that

$$l(w) = l(w_\lambda s_i) < l(w_\lambda).$$

Thus we get

$$l(w) \leq l(w_\mu)$$

which implies the result in this case. If $\alpha_i \notin \Pi(w_\lambda)$, then from Lemma 7.1 is easy to see that $\alpha_i \in \Pi(w_\mu)$. The result follows by interchanging the role of λ and μ. □

PROOF (OF LEMMA 6.2). By Lemma 5.4 we have

$$w_\lambda = w_\mu s_i$$

Therefore we have

$$w_\lambda > w_\mu \Leftrightarrow \alpha_i \notin \Pi(w_\mu)$$

By Lemma 7.1, we deduce that

$$w_\lambda > w_\mu \Leftrightarrow (\mu, \alpha_i) > 0$$

Now we have

$$(\mu, \alpha_i) = \begin{cases} (\mu, \delta + 2\varepsilon_1) = 2\mu_1 + 1 & \text{if } i = 0 \\ (\mu, -\varepsilon_i + \varepsilon_{i+1}) = -\mu_i + \mu_{i+1} & \text{if } 0 < i < n \\ (\mu, -2\varepsilon_n) = -2\mu_n & \text{if } i = n \end{cases}$$

and the result follows. □

To continue, we recall that

$$\rho = \sum_{i=1}^{n} \left(\frac{k_0 + k_n}{2} + (n-i)k \right) \varepsilon_i$$

Observe that ρ is anti-dominant: that is to say, for all α *not* in R_0^+ we have

$$(\rho, \alpha) = c_1 (k_0 + k_n) + c_2$$

where c_1, c_2 are non-negative integers.

PROOF (OF LEMMA 6.3). We can rewrite e as

$$e = \begin{cases} (\overline{\mu} - \overline{\lambda}, \varepsilon_1) & i = 0 \\ -(\overline{\mu} - \overline{\lambda}, \varepsilon_i) & 0 < i < n \\ -(\overline{\mu} - \overline{\lambda}, \varepsilon_n) & i = n \end{cases}$$

Now by formula (23) of [**10**] we have

$$\overline{\lambda} = s_i \cdot \overline{\mu}.$$

Therefore

$$\overline{\mu}-\overline{\lambda}=\begin{cases}(\overline{\mu},\alpha_i)\,\varepsilon_1, & i=0\\ (\overline{\mu},\alpha_i)\,(\varepsilon_{i+1}-\varepsilon_i)\,, & 0<i<n\\ (\overline{\mu},\alpha_i)\,\varepsilon_n, & i=n\end{cases}$$

Substituting this into the formula for e we get

$$e=(\overline{\mu},\alpha_i)\,.$$

Now if $i\neq 0$, then we get

$$e=(\mu,\alpha_i)+\left(\overset{\circ}{w}_\mu\cdot\rho,\alpha_i\right)=(\mu,\alpha_i)+\left(\rho,\overset{\circ}{w}_\mu^{-1}\cdot\alpha_i\right)$$

By the first part of Lemma 7.1 we get $(\mu,\alpha_i)>0$, and by the second part of the same lemma, we deduce that $\overset{\circ}{w}_\mu^{-1}\cdot\alpha_i$ is a negative root. Since ρ is anti-dominant, we deduce that the second term is positive and this proves the lemma for $i\neq 0$. On the other hand, if $i\neq 0$, then we get

$$e=(\mu,\alpha_0)-(\rho,\overset{\circ}{w}_\mu^{-1}\cdot\theta)$$

where $\theta=-2\varepsilon_1$ is the highest (positive) root. Once again, we have $(\mu,\alpha_0)>0$ by Lemma 7.1. Writing $\alpha_0=\delta-\theta$, we get that

$$(\mu,\theta)<1.$$

Since (μ,θ) is an integer, we deduce in fact that

$$(\mu,\theta)\leq 0.$$

Applying Lemma 7.1 again we conclude that $-\overset{\circ}{w}_\mu^{-1}\cdot\theta$ is a negative root. Thus $-(\rho,\overset{\circ}{w}_\mu^{-1}\cdot\theta)$ is positive, and this proves the lemma for $i\neq 0$. □

Proof (of Lemma 5.5). If $s_i\cdot\lambda=\lambda$, then $\nu=\lambda=\mu$, and the result is trivially true. So we may assume $s_i\cdot\lambda\neq\lambda$. Then by Lemma 5.4 we have $w_\mu=w_\lambda\cdot s_i$, so either $w_\mu<w_\lambda$ or $w_\mu>w_\lambda$, and without loss of generality we may assume that $w_\mu<w_\lambda$. Hence we have $\alpha_i\in\Pi(w_\lambda)$, and by Lemma 7.1 this implies that

$$(\lambda,\alpha_i)<0.$$

Now the reflection corresponding to the affine root $\alpha+k\delta$ is given by

$$s_{\alpha+k\delta}\cdot\lambda=\lambda-(\lambda,\alpha+k\delta)\,\breve{\alpha}=\lambda-(\lambda,\alpha)\,\breve{\alpha}k\breve{\alpha};\ \text{where }\breve{\alpha}=\frac{2}{(\alpha,\alpha)}\alpha.$$

First suppose that $i\neq 0$, then we have

$$\mu=s_i\cdot\lambda=\lambda-(\lambda,\lambda_i)\,\breve{\alpha}_i.$$

If $\nu\in\mathbb{Z}$ is a convex combination of λ and μ, then

$$\nu=\lambda+l\breve{\alpha}_i,\ \text{for some } l \text{ in } \mathbb{Z} \text{ with } 0<l<-(\lambda,\lambda_i)\,.$$

Now let

$$k=-(\lambda,\alpha_i)-l.$$

Then k is positive, and hence $\alpha_i+k\delta$ is a positive affine root. Moreover, then we have

$$s_{\alpha_i+k\delta}\cdot\lambda=\lambda-(\lambda,\alpha_i)\,\breve{\alpha}_i-k\breve{\alpha}_i=\lambda+l\breve{\alpha}_i=\nu$$

Thus we get

$$w_\lambda s_{\alpha_i+k\delta}\cdot\nu=w_\lambda\cdot\lambda=0,$$

and by minimality of w_ν, this implies

$$w_\nu \leq w_\lambda s_{\alpha_i + k\delta}. \tag{7}$$

On the other hand,

$$(\lambda, \alpha_i + k\delta) = (\lambda, \alpha_i) + k = -l < 0$$

so by Lemma 7.1, we get

$$\alpha_i + k\delta \in \Pi(w_\lambda)$$

and so

$$w_\lambda s_{\alpha_i + k\delta} < w_\lambda.$$

Combining this with the inequality 7 we get

$$w_\nu < w_\lambda.$$

A similar calculation shows that

$$w_\mu \leq w_\nu s_{-\alpha_i + l\delta} < w_\nu$$

which completes the proof of the lemma for $i \neq 0$. Now for $i = 0$, we have

$$\mu = s_0 \cdot \lambda = \lambda - (\lambda, \alpha_0)\,\varepsilon_1$$

and

$$\nu = \alpha + l\varepsilon_1 \text{ for some } 0 < l < -(\lambda, \alpha_0).$$

Setting

$$k = -(\lambda, \alpha_0) - l,$$

and arguing as for $i \neq 0$, we deduce that

$$w_\mu \leq w_\nu s_{-\alpha_0 + l\delta} < w_\nu \leq w_\lambda s_{\alpha_0 + k\delta} < w_\lambda.$$

□

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