ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE @ DIRECT•

Journal of Functional Analysis 201 (2003) 430–456

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Explicit Hilbert spaces for certain unipotent representations III

Alexander Dvorsky[a] and Siddhartha Sahi[b,*]

[a] Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA
[b] Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Received 28 March 2002; revised 30 September 2002; accepted 7 October 2002

Communicated by M. Vergne

Abstract

In this paper we construct a family of small unitary representations for real semisimple Lie groups associated with Jordan algebras. These representations are realized on L^2-spaces of certain orbits in the Jordan algebra. The representations are spherical and one of our key results is a precise L^2-estimate for the Fourier transform of the spherical vector. We also consider the tensor products of these representations and describe their decomposition.

Keywords: Jordan algebra; Unipotent representation; Spherical vector

0. Introduction

This paper is the culmination of a series dedicated to the problem of constructing explicit analytic models for small unitary representations of certain semisimple Lie groups (see [S1,S2,S3,DS1,DS2] and also [SS,KS]).

The groups G that we consider arise from real semisimple Jordan algebras via the Tits–Koecher–Kantor construction. Such a G is characterized by the existence of a parabolic subgroup $P = LN$ which is conjugate to its opposite $\overline{P} = L\overline{N}$, and for which N and $\overline{N}$ are abelian. The Lie algebra $\overline{\mathfrak{n}}$ admits a real semisimple Jordan algebra structure and we write n for its *rank*.

In this situation, the Levi component L has a finite number of orbits on $\overline{\mathfrak{n}}$ and each orbit has a *rank* $\leqslant n$. Each non-open orbit $\mathcal{O}$ (rank $< n$) admits an L-equivariant

*Corresponding author.
E-mail addresses: dvorsky@math.miami.edu (A. Dvorsky), sahi@math.rutgers.edu (S. Sahi).

0022-1236/03/$ - see front matter
doi:10.1016/S0022-1236(03)00069-7

measure $d\mu$ which is unique up to scalar multiples. (The open orbits admit one-parameter families of such measures.) By Mackey theory, the Hilbert space $\mathscr{H}_{\mathcal{O}} = \mathscr{L}^2(\mathcal{O}, d\mu)$ carries a natural irreducible unitary representation $\pi_{\mathcal{O}}$ of P, and we consider the following two problems:

- Extend $\pi_{\mathcal{O}}$ to a unitary representation of G.
- Decompose $\pi_{\mathcal{O}^1} \otimes \cdots \otimes \pi_{\mathcal{O}^s}$, for rank $\mathcal{O}^1 + \cdots +$ rank $\mathcal{O}^s \leqslant n$.

For Euclidean Jordan algebras these problems were solved in [S1,S2,DS1]. Thus in this paper we only consider non-Euclidean Jordan algebras. As explained in [DS2], one has to exclude rank 1 orbits in rank 2 Jordan algebras $\mathbb{R}^{p,q}$ $(p \neq q)$—we shall call these orbits *inadmissible* and the remaining non-open orbits *admissible*. In this paper we prove:

Theorem 0.1. *For each admissible orbit $\mathcal{O}$, the representation $\pi_{\mathcal{O}}$ of P extends to an irreducible spherical unitary representation of G on $\mathscr{H}_{\mathcal{O}}$.*

Now suppose $\mathcal{O}^1, \dots, \mathcal{O}^s$ are admissible non-open orbits in $\overline{N}$ such that rank $\mathcal{O}^1 + \cdots +$ rank $\mathcal{O}^s \leqslant n$. In Section 3.2 we define a reductive homogeneous space G'/H', essentially the generic fiber of the addition map from $\mathcal{O}^1 \times \cdots \times \mathcal{O}^s$ to $\overline{N}$, and consider the decomposition of the quasi-regular representation

$$\mathscr{L}^2(G'/H') = \int_{\widehat{G'}}^{\oplus} m(\sigma)\sigma \, d\rho(\sigma),$$

where $m(\sigma)$ is the multiplicity function and $d\rho(\sigma)$ is the Plancherel measure.

Theorem 0.2. *Let $\mathcal{O}^1, \dots, \mathcal{O}^s$ and G', H' be as above; then there is a map θ from the H'-spherical dual of G' to the unitary dual of G such that*

$$\pi_{\mathcal{O}^1} \otimes \cdots \otimes \pi_{\mathcal{O}^s} = \int_{\widehat{G'}}^{\oplus} m(\sigma)\theta(\sigma) \, d\rho(\sigma).$$

Our approach entails three different representation-theoretic techniques. We need to consider:

(a) Harish-Chandra modules for semisimple groups;
(b) operator algebras for parabolic subgroups; and
(c) Fourier analysis for abelian nilradicals.

The algebraic considerations (Harish-Chandra modules) were carried out in [S3]. The necessary operator-algebraic results (C^*-algebras, von Neumann algebras) were obtained in [DS1,DS2]. The missing ingredient, provided by this paper, involves abelian Fourier analysis. The key result (Proposition 2.2) is the proof that a certain function g (eventually, the "spherical" vector in $\pi_{\mathcal{O}}$) belongs to $\mathscr{L}^2(\mathcal{O}, d\mu)$.

For rank 1 orbits this result was obtained in [DS2] by establishing a close connection between this function and a certain one-variable Bessel K-function. The required $\mathscr{L}^2$-estimate then followed from a precise knowledge of the singularity of the Bessel K-function at 0.

For higher rank orbits, we expect that there should exist a similar connection between the spherical vectors and multivariate Bessel K-functions. However in order to exploit this connection one would have to first develop the theory of such functions, possibly along the lines of the theory of the multivariate Bessel J-functions of [Op].

While we feel that the connection with multivariate Bessel K-functions is of interest and should be pursued further, in the present paper we follow a different approach. This approach allows us to obtain the desired estimate directly, obviating the need to first study Bessel functions. The key here is a "stability" result (Lemma 2.12) which transfers the problem from a non-open orbit to a related problem on the open orbit for the smaller group. The open orbit problem turns out to be easier to solve.

This approach was inspired in part by a recent paper of Shimura [Sh]. We thank L. Barchini for drawing our attention to this paper.

1. Preliminaries

In this section we recall basic facts about the Tits–Kantor–Koecher construction. All results of this section are well-known. More details may be found in [KS,DS2] and in the references therein (in particular, [BK,Lo]). This construction associates to a real simple Jordan algebra, a pair (G, P), where G is a real simple Lie group with Cartan involution θ, and maximal compact subgroup K; and $P = LN$ is a parabolic subgroup.

In the context of Lie theory, these pairs can be characterized as follows:

(a) N is abelian,
(b) P is G-conjugate to its opposite parabolic $\overline{P} = \theta(P) = L\overline{N}$.

Conditions (a) and (b) each give rise to a *symmetric* space denoted by K/M and L/H, respectively, and much of the relevant information about the Jordan algebra and the associated pair (G, P) can be described in a simple and coherent manner in terms of these symmetric spaces. This makes it possible to have a uniform discussion for the most part, with only some occasional arguments requiring case-by-case considerations.

We follow the practice of denoting the real Lie algebras of various Lie groups by the corresponding fraktur letters; with the exception of $\mathfrak{p}$ which will denote instead the -1 eigenspace of θ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

1.1. The symmetric space K/M

Condition (a) from the beginning of this section implies that L is a *symmetric* subgroup of G, and $M = K \cap L$ is a symmetric subgroup of K. Let $\mathfrak{t}$ be a Cartan subspace in the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$. The real rank of N as a Jordan algebra is $n = \dim_{\mathbb{R}} \mathfrak{t}$. The roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ *always* form a root system of type C_n, and we fix a basis $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ of $\mathfrak{t}^*$ such that

$$\Sigma(\mathfrak{t}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}) = \{\pm(\gamma_i \pm \gamma_j)/2, \pm\gamma_j\}.$$

For the subsystem $\Sigma = \Sigma(\mathfrak{t}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$, there are *three* possibilities:

$$A_{n-1} = \{\pm(\gamma_i - \gamma_j)/2\}, \quad D_n = \{\pm(\gamma_i \pm \gamma_j)/2\}, \text{ and } C_n.$$

The first of these cases arises precisely when N is a *Euclidean* Jordan algebra. This case was studied in [S1], therefore we restrict our attention to the last two cases. If Σ is C_n, there are two multiplicities, corresponding to the short and long roots, which we denote by d and e, respectively. If Σ is D_n, and $n \neq 2$, then there is a single multiplicity, which we denote by d, so that D_n may be regarded as a special case of C_n, with $e = 0$.

The root system $D_2 \approx A_1 \times A_1$ is reducible and there are two root multiplicities. As mentioned in the introduction, we explicitly exclude the case when these multiplicities are different; this corresponds to $G = O(p, q)$, with $N = \mathbb{R}^{p-1,q-1}$ $(p \neq q)$. When the two multiplicities coincide $(p = q)$, we once again denote the common multiplicity by d.

1.2. S-triples and the Cayley transform

The discussion of the various cases can be made uniform by emphasizing the special role played by a family of n commuting SL_2's or S-triples, together with the associated Cayley transform.

For $\mathfrak{sl}_2(\mathbb{C})$ the Cayley transform is defined by $c = \exp \operatorname{ad} \frac{\pi i}{4}(X + Y) = \exp \operatorname{ad} \frac{\pi i}{4}(x + y)$; and satisfies $c(X) = x, c(Y) = y, c(H) = h$, where

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$X = \frac{1}{2}\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}, \quad Y = \frac{1}{2}\begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix}, \quad H = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Now $\sum(\mathfrak{t}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ is a root system of type C_n, with root multiplicities:

$$\dim \mathfrak{k}_{\pm(\gamma_i \pm \gamma_j)/2} = d, \quad \dim \mathfrak{p}_{\pm(\gamma_i \pm \gamma_j)/2} = d,$$

$$\dim \mathfrak{k}_{\pm\gamma_i} = e, \quad \dim \mathfrak{p}_{\pm\gamma_i} = 1.$$

We fix homomorphisms $\Psi_j : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}_\mathbb{C}$ such that $\Psi_j(X) \in \mathfrak{p}_{\gamma_j}$, and we write

$$X_j = \Psi_j(X), \quad x_j = \Psi_j(x), \quad y_j = \Psi_j(y), \ldots,$$

$$\mathbf{X} = \sum X_j, \quad \mathbf{x} = \sum x_j, \quad \mathbf{y} = \sum y_j, \ldots \ .$$

The *Cayley transform* of $\mathfrak{g}_\mathbb{C}$ is the product

$$\mathbf{c} = \exp \operatorname{ad} \frac{\pi i}{4}(\mathbf{x} + \mathbf{y}) = \exp \operatorname{ad} \frac{\pi i}{4}(\mathbf{X} + \mathbf{Y}).$$

We write $\mathfrak{a} = \mathbf{c}(i\mathfrak{t})$ for the Cayley transform of $i\mathfrak{t}$. This is the abelian subalgebra of $\mathfrak{g}$ spanned by $h_1, \ldots, h_n$.

1.3. The symmetric space L/H

Let $H \subset L$ be the stabilizer of $\mathbf{y} \in \bar{\mathfrak{n}}$, then condition (b) from the beginning of this section implies that L/H is a symmetric space. The involution σ for this symmetric space consists of conjugation by a suitable element of K—corresponding to condition (b).

Example. If $G = O_{2n,2n}$, then $L = \mathrm{GL}_{2n}(\mathbb{R})$ and N is the Jordan algebra of $2n \times 2n$ real skew-symmetric matrices, and $H = \mathrm{Sp}_n(\mathbb{R})$.

In the present situation L/H is always non-Riemannian; and if we consider the Cartan decompositions for θ and σ

$$\mathfrak{l} = \mathfrak{m} + \mathfrak{r}, \quad \mathfrak{l} = \mathfrak{h} + \mathfrak{q};$$

then $\mathfrak{a}$ is a Cartan subspace in $\mathfrak{q} \cap \mathfrak{r}$. Writing $2\varepsilon_i = \gamma_i \circ \mathbf{c}^{-1}$ we have

$$\Sigma(\mathfrak{a}, \mathfrak{g}) = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_j\}, \quad \Sigma(\mathfrak{a}, \mathfrak{l}) = \{\pm(\varepsilon_i - \varepsilon_j)\},$$

$$\Sigma(\mathfrak{a}, \mathfrak{n}) = \{\varepsilon_i + \varepsilon_j, 2\varepsilon_j\}, \quad \Sigma(\mathfrak{a}, \bar{\mathfrak{n}}) = \{-\varepsilon_i - \varepsilon_j, -2\varepsilon_j\}.$$

We observe that for a in $\mathfrak{a}$ we have

$$\operatorname{tr} \operatorname{ad}_{\bar{\mathfrak{n}}}(a) = \left[-2d \sum (\varepsilon_i + \varepsilon_j) - (e+1) \sum 2\varepsilon_j\right](a) = -2r\nu(a),$$

where $\nu = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$, $r = d(n-1) + (e+1)$.

Thus we can define characters ν of $\mathfrak{l}$, and e^ν of L by the formulas

$$\nu(Z) = -\left(\frac{1}{2r}\right) \operatorname{tr} \operatorname{ad}_{\bar{\mathfrak{n}}}(Z); \quad e^\nu(l) = l^\nu = |\det \operatorname{Ad}_{\bar{\mathfrak{n}}} l|^{-1/(2r)}.$$

Extending trivially to N (resp. $\bar{N}$) we obtain positive characters of the groups P (resp. $\bar{P}$), which we write as $g \mapsto e^\nu(g)$, or as $g \mapsto g^\nu$.

To complete the connection with the Jordan structure, we note that the Jordan norm ϕ on $\overline{\mathfrak{n}}$ is a polynomial function which transforms by the character $e^{-2\nu}$ of L. Finally, we observe that the Killing form on $\mathfrak{g}$ gives a pairing between $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ which we *rescale* by setting $\langle x_1, y_1 \rangle = 1$.

1.4. Orbits

As is well known, the orbits of L on $\overline{\mathfrak{n}}$ are parametrized by their rank, with the rank k orbit given by $\mathcal{O} = L \cdot (y_1 + \cdots + y_k)$. If $k = n$, the stabilizer of $y_1 + \cdots + y_k$ is the symmetric subgroup H described previously. We now discuss the remaining orbits; to simplify notation we fix k and write

$$\mathbf{y}^1 = y_1 + \cdots + y_k.$$

In Jordan algebra terms, $\mathbf{y}^1$ is a Peirce idempotent and considering the 1 and 0 Peirce-eigenspaces of $\mathbf{y}^1$, we obtain smaller Jordan algebras $\overline{\mathfrak{n}}_1$ and $\overline{\mathfrak{n}}_0$ with identity elements $\mathbf{y}^1$ and $\mathbf{y}^0 = y_{k+1} + \cdots + y_n$, respectively. The corresponding structure groups L_1 and L_0 are naturally the reductive subgroups of L. Subgroups of L_1 and L_0 will be distinguished by subscripts 1 and 0, respectively. For example,

$$M_1 = M \cap L_1, \quad M_0 = M \cap L_0, \quad H_1 = H \cap L_1, \quad \mathfrak{a}_1 = \mathfrak{a} \cap \mathfrak{l}_1.$$

Thus H_1 is the stabilizer of $\mathbf{y}^1$ in L_1, and the full stabilizer of $\mathbf{y}^1$ in L is given by

$$S = (H_1 \times L_0) \cdot \overline{U}. \tag{1}$$

Here $\overline{U}$ is abelian, and its Lie algebra $\overline{\mathfrak{u}}$ is spanned by the root spaces $\mathfrak{l}^{-\varepsilon_i + \varepsilon_j}$ $(1 \leqslant i \leqslant k < j \leqslant n)$.

Example. Again, take $G = O_{2n,2n}$. One has $L = \mathrm{GL}_{2n}(\mathbb{R})$, $L_1 = \mathrm{GL}_{2k}(\mathbb{R})$ and $L_0 = \mathrm{GL}_{2(n-k)}(\mathbb{R})$. Then $H_1 = H \cap L_1 = \mathrm{Sp}_k(\mathbb{R})$ and $S = (\mathrm{Sp}_k(\mathbb{R}) \times \mathrm{GL}_{2(n-k)}(\mathbb{R})) \cdot \overline{U}$, where $\overline{U}$ is a vector space of $2(n-k) \times 2k$ real matrices.

We have

$$\mathfrak{l} = \mathfrak{s} + (\mathfrak{q}_1 + \mathfrak{u}), \quad \text{where } \mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{l}_1, \ \ \mathfrak{u} = \theta \overline{\mathfrak{u}}.$$

The orbits of L on $\overline{\mathfrak{n}}$ carry equivariant measures, which we now describe. Write e^{ν} for the positive character of L defined in Section 1.3 and let $r = d(n-1) + (e+1)$ be as before. Then we have

Lemma 1.1. (1) *The Lebesgue measure $d\lambda$ on $\overline{\mathfrak{n}}$ is $e^{2r\nu}$-equivariant.*

(2) *The rank k-orbit carries an $e^{2dk\nu}$-equivariant measure $d\mu = d\mu_k$.*

The proof is straightforward. We now describe a "polar coordinates" expression for these equivariant measures. In [Lo] it is shown that the elements

$$\{z_1 y_1 + \cdots + z_k y_k \mid z_1 > z_2 > \cdots > z_k > 0\}$$

give a complete set of orbit representatives for the action of $M = L \cap K$ on the rank k orbit. Accordingly, we write $C_k \subset \mathbb{R}^k$ for the cone

$$C_k = \{z = (z_1, z_2, \dots, z_k) \mid z_1 > z_2 > \cdots > z_k > 0\},$$

and for m in M, z in C_k we write

$$m \cdot z = \mathrm{Ad}\, m(z_1 y_1 + \cdots + z_k y_k) \in \bar{\mathfrak{n}}.$$

For z in C_k we introduce the notation

$$P_k(z) = z_1 \cdots z_k, \quad V_k(z) = \prod_{1 \leqslant i < j \leqslant k} [z_i^2 - z_j^2], \quad d_k^\times z = \prod_{j=1}^{k} \frac{dz_j}{z_j},$$

where each dz_j denotes the Lebesgue measure on $\mathbb{R}$. Then we have:

Proposition 1.2. *Let $d\lambda$ be the Lebesgue measure on $\bar{\mathfrak{n}}$, then*

$$\int_{\bar{\mathfrak{n}}} f d\lambda = c \int_{C_n} \left[\int_M f(m \cdot z)\, dm \right] d_* z, \quad \textit{where } d_* z = [P_n]^{e+1} [V_n]^d \, d_n^\times z.$$

Proposition 1.3. *Let $d\mu$ be the equivariant measure on the rank k orbit $\mathcal{O}$, then*

$$\int_{\mathcal{O}} f d\mu = c \int_{C_k} \left[\int_M f(m \cdot z)\, dm \right] d_k z, \quad \textit{where } d_k z = [P_k^{n-k+1} V_k]^d \, d_k^\times z.$$

The scalars c appearing in the above formulas are independent of f and depend only on the normalization of the measures $d\lambda$ and $d\mu$. These formulas can be obtained by the usual techniques (cf. [Sc, 8.1], also [OS]). For subsequent purposes we also need to consider the Lebesgue measure on $\mathfrak{n}$. For m in M, z in C_n, we write

$$m \circ z = \mathrm{Ad}\, m(z_1 x_1 + \cdots + z_n x_n) \in \mathfrak{n}. \tag{2}$$

Since $\theta : \mathfrak{n} \to \bar{\mathfrak{n}}$ satisfies $\theta(m \circ z) = m \cdot z$, Proposition 1.2 implies

Corollary 1.4. *Let $d\lambda$ be the Lebesgue measure on $\mathfrak{n}$, then*

$$\int_{\mathfrak{n}} f d\lambda = c \int_{C_n} \left[\int_M f(m \circ z)\, dm \right] d_* z, \quad \textit{where } d_* z = [P_n]^{e+1} [V_n]^d \, d_n^\times z.$$

2. Estimates for spherical vectors

We can relate the P-representation $\pi_{\mathcal{O}}$ of Theorem 0.1 to a unitarizable submodule of a certain degenerate principal series for G, which is described as follows: If χ is a character of L, we write $(\pi_\chi, I(\chi))$ for the degenerate principal series representation $\mathrm{Ind}_{\bar{P}}^{G}\chi$ (unnormalized smooth induction); thus

$$I(\chi) = \{ f \in C^\infty(G) \mid f(l\bar{n}g) = \chi(l) f(g) \quad \text{for } l \in L, \bar{n} \in \overline{N}, g \in G \}$$

and the group G acts by right translations. By virtue of the Gelfand–Naimark decomposition $G \approx \overline{P}N$, functions from $I(\chi)$ are determined by their restriction to N. Combining this with the exponential map we can identify $I(\chi)$ with a subspace $E(\chi)$ of $C^\infty(\mathfrak{n})$. We refer to this as the noncompact picture.

For $t \in \mathbb{R}$, we write $I(t)$, $E(t)$ for $I(e^{tv})$, $E(e^{tv})$; more generally, if $\varepsilon : L \to \mathbb{T}$ is a unitary character, we write $I(t, \varepsilon)$, $E(t, \varepsilon)$ for $I(e^{tv} \otimes \varepsilon)$, $E(e^{tv} \otimes \varepsilon)$. These principal series were studied in [S3] via the "Cayley operator" D which is the constant coefficient differential operator on $\mathfrak{n}$, whose symbol is the Jordan norm polynomial ϕ. Powers of D are intertwining operators for the principal series, and their eigenvalues on the various K-isotypic components are given by the Capelli identity of [KS].

$E(t)$ is a spherical representation of G and we write Φ_t for the K-spherical vector. Among the results obtained in [S3] is that for $k = 1, \ldots, n-1$, the space $E(-dk)$ contains a unitarizable spherical submodule. We need to study the Fourier transforms of the corresponding spherical vectors

$$\Phi_{-dk}; \quad k = 1, \ldots, n-1.$$

For this we identify $\bar{\mathfrak{n}}$ with the dual of $\mathfrak{n}^*$ via the normalized Killing form from Section 1.3. Also we fix $k < n$, write Φ for the spherical vector Φ_{-dk}, and write $(\mathcal{O}, d\mu)$ for the rank k orbit in $\bar{\mathfrak{n}}$ together with its equivariant measure described in Lemma 1.1.

The main results of this section are

Proposition 2.1. *The measure $\Phi d\lambda$ is a tempered distribution on $\mathfrak{n}$ and there exists an M-invariant function g in $\mathscr{L}^1(\mathcal{O}, d\mu)$ such that*

$$\Phi d\lambda = \widehat{g\, d\mu}.$$

Proposition 2.2. *For $k < n$, one has $g \in \mathscr{L}^2(\mathcal{O}, d\mu)$.*

We prove these propositions in the next few subsection. The strategy is as follows: Let us write $\Phi_{k,n}$ for the function Φ_{-dk}, in order to emphasize dependence on n as well as k. Now although the above results are *false in general* for the open orbit $(k = n)$, nevertheless, we can prove the desired results by reducing to a slightly weaker estimate for $k = n$, which turns out to be somewhat easier to prove. We establish this result in the next subsection and then outline the reduction procedure in the two following subsection.

2.1. Estimates for the open orbit

As indicated above, we first consider the function

$$\Phi = \Phi_{n,n} = \Phi_{-dn}.$$

We need appropriate $\mathscr{L}^2$-estimates with respect to the Lebesgue measure $d\lambda$ on $\mathfrak{n}$ for the function Ψ and its derivatives. The "straightforward" estimate is actually false for the group $\mathrm{Sp}_n(\mathbb{C})$, but it does work for the other groups G in the table in Appendix A.2. Thus we formulate two results, one for $G \neq \mathrm{Sp}_n(\mathbb{C})$ and the other for all groups:

Proposition 2.3. *For all groups G other than* $\mathrm{Sp}_n(\mathbb{C})$, *we have* $\Phi \in \mathscr{L}^2(\mathfrak{n}, d\lambda)$.

Proposition 2.4. *For all groups G and for all* $m \geqslant 1$, *we have* $D^m\Phi \in \mathscr{L}^2(\mathfrak{n}, d\lambda)$.

For each t, the function Φ_t is M-invariant, and is therefore determined by the restriction to the subspace $\{z_1x_1 + \cdots + z_nx_n\} \subseteq \mathfrak{n}$; we start by giving an explicit formula for the restriction.

Lemma 2.5. *We have* $\Phi_t(m \circ z) = \prod_{i=1}^n (1 + z_i^2)^{\frac{t}{2}}$ *for all m in M.*

Proof. For the group $G = \mathrm{SL}_2(\mathbb{R})$ this is a straightforward calculation which we leave to the reader. In the general case, we view Φ as a function on G which is right K-invariant, and left $\overline{P}$-equivariant with character $e^{t\nu}$. We now restrict Φ to the subgroup $\mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$ corresponding to the S-triples of Appendix A.2. This restriction is right $\mathrm{SO}_2 \times \cdots \times \mathrm{SO}_2$-invariant, and left $\overline{B} \times \cdots \times \overline{B}$-equivariant with character $e^{s\nu} = e^{s\varepsilon_1} \times \cdots \times e^{s\varepsilon_n}$ (here B is the Borel subgroup of SL_2). Thus applying the SL_2-calculation to each factor, we conclude that the restriction to $z_1x_1 + \cdots + z_nx_n$ is given as in the statement of the lemma. □

Combining this with Corollary 1.4 we obtain the following estimate

Lemma 2.6. *For* $t < -[d(n-1) + (e+1)/2]$, *we have* $\Phi_t \in \mathscr{L}^2(\mathfrak{n}, d\lambda)$.

Proof. Combining the previous lemma with Corollary 1.4, we get

$$\int |\Phi_t|^2 d\lambda = \int_{C_n} \prod_{i=1}^n z_i^e (1 + z_i^2)^t \prod_{1 \leqslant i < j \leqslant n} (z_i^2 - z_j^2)^d dz_1 dz_2 \cdots dz_n,$$

Expanding $(z_i^2 - z_j^2)^d$, we can write the integrand as a sum of terms

$$\prod_{i=1}^{n} z_i^{e+k_i}(1+z_i^2)^t, \quad \text{where each } k_i \leqslant 2d(n-1).$$

Each of these integrals is a *product of one-variable* integrals which converge if

$$\int_0^\infty x^{e+2d(n-1)}(1+x^2)^t dx < \infty.$$

This happens if $2t + e + 2d(n-1) < -1$, which proves the lemma. $\square$

Corollary 2.7. *If* $f \in E(t,\varepsilon)$ *for some* $t < -[d(n-1) + (e+1)/2]$ *and* $\mathscr{D}$ *is any constant coefficient differential operator, then we have* $\mathscr{D}f \in \mathscr{L}^2(\mathfrak{n}, d\lambda)$.

Proof. The group G acts on $I(t,\varepsilon)$ by right translations, and in the non-compact picture $E(t,\varepsilon)$ the Lie algebra $\mathfrak{g}$ acts by polynomial coefficient vector fields on $\mathfrak{n}$. The action of $x \in \mathfrak{n}$ is independent of (t,ε) and is simply the directional derivative in the direction x. In particular, the space $E(t,\varepsilon)$ is *invariant* for the action of constant coefficient differential operators.

Thus $f' \equiv \mathscr{D}f$ also belongs to $E(t,\varepsilon)$. Thus f' is the restriction to N of a $\overline{P}$-equivariant smooth function on G. Since $G = \overline{P}K$, any such function is *determined* by its restriction to K. The constant function 1 on K corresponds to the spherical vector Φ_t in $I(t)$. Thus if c is the maximum of $|f'|$ on K, then we have $|f'| \leqslant c\Phi_t$, and the corollary follows from the previous lemma. $\square$

We can now prove Propositions 2.3 and 2.4 (for $G \neq \mathrm{Sp}_n(\mathbb{C})$).

Proof of Propositions 2.3 and 2.4. (For $G \neq \mathrm{Sp}_n(\mathbb{C})$). From the table in Section see that in every case except $G = \mathrm{Sp}_n(\mathbb{C})$, we have $2d > e+1$. Consequently, we get

$$-dn < -[d(n-1) + (e+1)/2].$$

Proposition 2.3 now follows from Lemma 2.6, and Proposition 2.4 follows immediately from Corollary 2.7 for all groups *except* for $G = \mathrm{Sp}_n(\mathbb{C})$. $\square$

Suppose now that G is $\mathrm{Sp}_n(\mathbb{C})$. Then $L = \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{n}$ is the space of $n \times n$ complex symmetric matrices. We write $\mathscr{V}$ for the finite-dimensional space of holomorphic polynomials on $\mathfrak{n}$ spanned by all the minors of the symmetric matrix x, and let ε be the unitary character of L given by $\varepsilon(l) = \dfrac{\det l}{|\det l|}$.

Lemma 2.8. $\mathscr{V}$ *is a* $\mathrm{Sp}_n(\mathbb{C})$*-invariant subspace of* $E(1,\varepsilon)$.

Proof. The character e^{ν} of L is simply $|\det l|$. Therefore, the space $I(1,\varepsilon)$ consists of smooth functions on $G = \mathrm{Sp}_n(\mathbb{C})$ satisfying $f(l\bar{n}g) = \det(l)f(g)$. The group G is generated by the elements $p = \begin{bmatrix} a^{-1} & b \\ 0 & a^t \end{bmatrix} \in P$ and $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ whose action in the noncompact picture $E(1,\varepsilon)$ is as follows:

$$p \cdot f(x) = \det(a^{-1}) f(ab + axa^t),$$

$$w \cdot f(x) = \det(x) f(-x^{-1}).$$

Evidently, transformations of the form $x \mapsto c + axa^t$ take minors of x to linear combinations of (possibly smaller) minors; thus $\mathscr{V}$ is P-invariant. Also each minor of x^{-1} is equal to $\pm \det(x)^{-1}$ times the complementary minor of x; thus $\mathscr{V}$ is w-invariant. Since P is a maximal parabolic subgroup, w and P generate G, and hence the space $\mathscr{V}$ is G-invariant.

Now the functions in $\mathscr{V}$ can be lifted to $\bar{P}$-equivariant functions on the dense open set $\bar{P}N$ in G. The G-invariance of $\mathscr{V}$ implies that these functions transform finitely under right translations by K. Therefore they extend to smooth functions on K, and hence on G. Thus we get $\mathscr{V} \subset E(1,\varepsilon)$. $\square$

Corollary 2.9. *For* $G = \mathrm{Sp}_n(\mathbb{C})$, $\det(x)$ *belongs to the space* $E(1,\varepsilon)$.

We can now finish the proof of Proposition 2.4.

Proof of Proposition 2.4. (For $G = \mathrm{Sp}_n(\mathbb{C})$). For $G = \mathrm{Sp}_n(\mathbb{C})$, we have $d = 1$, $\Psi(x) = \det(1 + x\bar{x})^{-n/2}$, $\phi(x) = \det(x)\det(\bar{x})$, $D = \det(\partial_x)\det(\partial_{\bar{x}})$. Thus

$$D\Psi = \det(\partial_x)\det(\partial_{\bar{x}})\det(1 + x\bar{x})^{-n/2}. \tag{3}$$

Now, it is well known (see e.g. [KS]) that for u a complex symmetric matrix

$$\det(\partial_u)\det(u)^s = \mathrm{const}\det(u)^{s-1}.$$

By a simple change of variables, we deduce that for all complex symmetric w

$$\det(\partial_u)\det(1 + wu)^s = \mathrm{const}\det(w)\det(1 + wu)^{s-1}.$$

Applying this to (3), we obtain

$$D\Psi = \mathrm{const}\det(\partial_x)\det(x)\det(1 + x\bar{x})^{-n/2-1}.$$

The function $\det(1 + x\bar{x})^{-n/2-1}$ is the spherical vector in $E(-n-2)$. Also, by the corollary above, $\det(x)$ belongs to $E(1,\varepsilon)$. Each of these functions extends to a smooth function on G with appropriate $\bar{P}$-equivariance. Considering the equivariance of the product, we deduce

$$\det(x)\det(1+x\overline{x})^{-n/2-1}\in E(-n-1,\varepsilon).$$

Since $D^{m-1}\det(\partial_x)$ is a constant coefficient differential operator, we get

$$D^m\Psi=[\mathrm{const}\,D^{m-1}\det(\partial_x)][\det(x)\det(1+x\overline{x})^{-n/2-1}]\in E(-n-1,\varepsilon).$$

Now in the present case we have $d=1, e=1$, thus we get

$$-[d(n-1)+(e+1)/2]=-n>-n-1$$

and so the result follows from Corollary 2.7. □

2.2. *Proof of the $\mathscr{L}^1$ estimate*

We fix k and denote the spherical vector $\Phi_{k,n}=\Phi_{-dk}$ by simply Φ as before. To prove estimates for Φ, we first relate it to the "rank 1" spherical vector

$$\Upsilon=\Phi_{1,n}=\Phi_{-d}.$$

We now describe the key result in [DS2, Theorem 0.1] concerning the function Υ. Let $\tau=\frac{d-e-1}{2}$ and let K_τ be the corresponding one-variable K-Bessel function; define an M-invariant function υ on the rank 1 orbit $\mathcal{O}_1=L\cdot y_1\subset\overline{\mathfrak{n}}\approx\mathfrak{n}^*$ by the formula

$$\upsilon(z[m\cdot y_1])=\frac{K_\tau(z)}{z^\tau}\quad\text{for } z\in\mathbb{R}^+,\ m\in M.$$

Then writing $d\mu_1$ for the equivariant measure on $\mathcal{O}_1$, we have

$$\widehat{\upsilon d\mu_1}=\Upsilon d\lambda,$$

where $d\lambda$ is the Lebesgue measure on $\mathfrak{n}$, and ^ denotes the Fourier transform of tempered distributions. This result is proved in Propositions 2.1 and 2.2 of [DS2]. For our present purposes, it is crucial that τ depends only on d and e but does *not* depend on n.

An immediate consequence of Lemma 2.5 is the relation

$$\Phi=\Upsilon^k. \tag{4}$$

This in turn implies a relation between the Fourier transforms of Φ and Υ which we now explain. We start with the following abstract situation:

Suppose A is a Lie group, χ is a positive character of A, and $B\supset C$ are subgroups such that each of the homogeneous spaces A/B and A/C admit χ-equivariant measures $dm_{A/B}$ and $dm_{A/C}$.

Lemma 2.10. *The space $Z = B/C$ admits an B-invariant measure dz, and*

$$\mathscr{C}f(aB) = \int_Z f(az)\, dz$$

gives a well-defined operator $\mathscr{C} = \mathscr{C}_{A,B,C} : \mathscr{L}^1(A/C) \to \mathscr{L}^1(A/B)$ satisfying

$$\int_{A/B} [\mathscr{C}f]\, dm_{A/B} = \int_{A/C} f\, dm_{A/C}. \tag{5}$$

Proof. This is completely straightforward. □

We apply the previous result to the situation where

$$A = L, \quad B = S = \mathrm{stab}_L\ \mathbf{y}^1, \quad C = S' = \mathrm{stab}_L\ \mathbf{y}'$$

with $\mathbf{y}^1 = y_1 + y_2 + \cdots + y_k$ as before, and $\mathbf{y}' = (y_1, y_2, \ldots, y_k) \in \mathscr{O}_1 \times \cdots \times \mathscr{O}_1$. The space $\mathscr{O} = L/S$ is the rank k orbit and hence by Lemma 1.1 carries a $e^{2dk\nu}$-equivariant measure. On the other hand, the space $\mathscr{O}_1 \times \cdots \times \mathscr{O}_1$ also carries a $e^{2dk\nu}$-equivariant measure, viz. $d\mu' = d\mu_1 \times \cdots \times d\mu_1$; moreover in this situation, $\mathscr{O}' = L/S'$ is an open subset whose complement has measure 0. Thus $\mathscr{O}'$ also admits an $e^{2dk\nu}$-equivariant measure. Thus by the previous lemma, we obtain a well defined operator $\mathscr{C} = \mathscr{C}_{L,S,S'} : \mathscr{L}^1(\mathscr{O}') \to \mathscr{L}^1(\mathscr{O})$ satisfying formula (5).

Now given a function f on $\mathscr{O}_1$, we define functions $\overline{f}$ on $\mathscr{O}'$, and $\check{f}$ on $\mathscr{O}$ by

$$\overline{f}\,(l \cdot y') = f(l \cdot y_1) \cdots f(l \cdot y_k), \ \check{f} = \mathscr{C}\overline{f}.$$

Then we have the following result:

Lemma 2.11. *For υ as above, put $g = \check{\upsilon} = \mathscr{C}\overline{\upsilon}$, then we have*

$$\widehat{g\, d\mu} = \Phi\, d\lambda.$$

Proof. It suffices to prove

$$\int_{y \in \mathscr{O}} e^{-i\langle x,y \rangle}\, g(y) d\mu(y) = \Phi(x).$$

Using $\langle x, l \cdot y^1 \rangle = \langle x, l \cdot y_1 \rangle + \cdots + \langle x, l \cdot y_k \rangle$, the left side can be rewritten as

$$\int e^{-i\langle x, l \cdot y^1 \rangle} \mathscr{C}\overline{\upsilon}(l \cdot y^1) d\mu(l \cdot y^1) = \int \mathscr{C}\overline{\eta\upsilon}\, d\mu, \tag{6}$$

where $\eta(l \cdot y_1) = \exp(-i\langle x, l \cdot y_1 \rangle)$. By the previous lemma, this becomes

$$\int \overline{\eta\upsilon}\, d\mu' = \prod_{j=1}^{k} \left[\int \upsilon(l \cdot y_j)\eta(l \cdot y_j) d\mu_1(l \cdot y_j) \right] = \Upsilon^k = \Phi. \qquad \square$$

Proof of Proposition 2.1. In view of the previous lemma, it remains only to prove that $g \in \mathscr{L}^1(\mathcal{O}, d\mu)$. In turn, using Lemma 2.10, it suffices to show that $\overline{\upsilon} \in \mathscr{L}^1(\mathcal{O}', d\mu')$, or equivalently that

$$\upsilon \in \mathscr{L}^1(\mathcal{O}_1, d\mu_1).$$

This is essentially contained in Proposition 2.1 of [DS2]. The key point is that by Proposition 1.3 for $k = 1$, we get

$$\int_{\mathcal{O}_1} \upsilon\, d\mu_1 = \int_{\mathbb{R}_+} \frac{K_\tau(z)}{z^\tau} z^{dn-1}\, dz.$$

Since $K_\tau(z)$ has exponential decay at infinity, it suffices to prove that the integral on the right converges at 0. For this we note that $\frac{K_\tau(z)}{z^\tau}$ has a pole of order $2\tau = d - e - 1$ at 0 if $\tau > 0$, and a logarithmic singularity if $\tau = 0$. At any rate $(dn - 1) - 2\tau = d(n-1) + e$ is greater than -1, which guarantees the convergence of the integral. $\square$

2.3. *Proof of the $\mathscr{L}^2$ estimate*

The key to the proof of Proposition 2.2 is a "stability" result for the function g defined in Lemma 2.11. To state this, we temporarily write $g_{k,n}$ and $d\mu_{k,n}$ for g and $d\mu$, in order to emphasize dependence on k (the rank of the orbit) and n (the rank of the Jordan algebra). Thus Lemma 2.11 becomes

$$\widehat{g_{k,n} d\mu_{k,n}} = \Phi_{k,n}\, d\lambda.$$

We now recall the notation $\overline{\mathfrak{n}}_1, \overline{\mathfrak{n}}_0, L_1, M_1$, etc., introduced in Section 1.4. Thus $\overline{\mathfrak{n}}_1$ is a Jordan algebra of rank k (with same values of d and e as $\overline{\mathfrak{n}}$). By applying the considerations of the previous section to $\overline{\mathfrak{n}}_1$ we obtain a family of functions $g_{j,k}$; $j = 1, \dots, k$, defined on the various L_1-orbits in $\overline{\mathfrak{n}}_1$. We are particularly interested in the function

$$\widetilde{g} = g_{k,k}$$

which is defined on the *open* orbit $\widetilde{\mathcal{O}}$ in $\overline{\mathfrak{n}}_1$. Now by definition we have $\overline{\mathfrak{n}}_1 \subset \overline{\mathfrak{n}}$, and moreover we have $\widetilde{\mathcal{O}} \subset \mathcal{O}$, where $\mathcal{O}$ is the rank k orbit in $\overline{\mathfrak{n}}$. Thus we can restrict $g = g_{k,n}$ from $\mathcal{O}$ to $\widetilde{\mathcal{O}}$. The crucial "stability" result is the following:

Lemma 2.12. *With the above notation, we have* $g|_{\widetilde{\mathcal{O}}} = \widetilde{g}$.

Proof. The function $\widetilde{g}$ is also defined by the analogous two-step procedure applied to the Jordan algebra $\overline{\mathfrak{n}}_1$. We start with the M_1-invariant function $\widetilde{\upsilon}$ on the rank 1 orbit $\widetilde{\mathcal{O}}_1 \subset \overline{\mathfrak{n}}_1$ corresponding to the Bessel function K_τ/z^τ. As observed after the definition υ, the parameter $\tau = (d-e-1)/2$ is independent of n. Thus we get

$$\upsilon|_{\widetilde{\mathcal{O}}_1} = \widetilde{\upsilon}, \tag{7}$$

which is the rank 1 version of the present lemma.

Next, we consider the open L_1-orbit $\widetilde{\mathcal{O}}'$ in $\widetilde{\mathcal{O}}_1 \times \cdots \times \widetilde{\mathcal{O}}_1$, and define the analogous function $\overline{\widetilde{\upsilon}}$ by the formula

$$\overline{\widetilde{\upsilon}}(l \cdot y') = \widetilde{\upsilon}(l \cdot y_1) \cdots \widetilde{\upsilon}(l \cdot y_k)$$

for $y' = (y_1, \ldots, y_k) \in \widetilde{\mathcal{O}}'$ and l in L_1. Comparing this with the definition of $\overline{\upsilon}$, and using formula (7) we deduce

$$\overline{\upsilon}|_{\widetilde{\mathcal{O}}'} = \overline{\widetilde{\upsilon}}.$$

Now the functions g and $\widetilde{g}$ are defined by the integrals

$$\begin{aligned} g(l \cdot y^1) &= \int_Z \overline{\upsilon}(l \cdot z)\, dz \quad \text{for } l \text{ in } L, \\ \widetilde{g}(l \cdot y^1) &= \int_{\widetilde{Z}} \overline{\widetilde{\upsilon}}(l \cdot \widetilde{z})\, d\widetilde{z} \quad \text{for } l \text{ in } L_1, \end{aligned} \tag{8}$$

where dz and $d\widetilde{z}$ are the invariant measures on the homogeneous spaces $Z = S/S' \subset L/S' = \mathcal{O}'$ and $\widetilde{Z} = (S \cap L_1)/(S' \cap L_1) \subset L_1/(S' \cap L_1) = \widetilde{\mathcal{O}}'$. However, as in formula (1) we see that

$$S = ((S \cap L_1) \times L_0) \cdot \overline{U},$$

$$S' = ((S' \cap L_1) \times L_0) \cdot \overline{U}.$$

Thus in the imbedding $\widetilde{\mathcal{O}}' \subset \mathcal{O}'$, we have

$$Z = \widetilde{Z}.$$

Moreover, since both measures are L_1-invariant, we have

$$dz = d\widetilde{z}.$$

Thus the integrals in formula (8) *coincide* for l in L_1, and the result follows. $\square$

Let $f \mapsto \check{f}$ denote the inverse Fourier transform which maps functions on $\mathfrak{n}_1$ to functions on $\bar{\mathfrak{n}}_1$. Thus

$$\check{f}(y) = \int_{\mathfrak{n}_1} e^{i\langle x,y\rangle} f(x)\, d\lambda,$$

where $d\lambda$ is the Lebesgue measure on $\mathfrak{n}_1$.

Lemma 2.13. *Writing $\tilde{\phi}$ for the Jordan norm polynomial on $\bar{\mathfrak{n}}_1$, we have*

$$|(\Phi_{k,k})\check{}\,| = \tilde{g}|\tilde{\phi}|^{d-(e+1)} = |\tilde{g}\tilde{\phi}^{d-(e+1)}|.$$

Proof. The Fourier transform of tempered distributions is defined by adjointness from its action on Schwartz functions, and we have the relation

$$\widehat{\check{f}\, d\lambda} = f\, d\lambda.$$

Now by the definition of $\tilde{g}$ we have

$$\widehat{\tilde{g}\, d\mu} = \Phi_{k,k}\, d\lambda,$$

where $d\mu$ is the equivariant measure on the open orbit $\tilde{\mathcal{O}} \subset \bar{\mathfrak{n}}_1$. Propositions 1.2 and 1.3 imply that in polar coordinates, the measures $d\lambda$ and $d\mu$ are given by $P_k^{e+1} V_k^d d_k^\times z$ and $P_k^d V_k^d d_k^\times z$, respectively. Thus, writing $\tilde{\phi}$ for the Jordan norm polynomial on $\bar{\mathfrak{n}}_1$, we get

$$d\mu = |\tilde{\phi}|^{d-(e+1)}\, d\lambda.$$

Combining these formulas we obtain the result. □

Lemma 2.14. *Let $\tilde{D}$ be the Cayley operator on $\mathfrak{n}_1$ then for $l \geqslant 0$ we have*

$$|(\tilde{D}^l \Phi_{k,k})\check{}\,| = |\tilde{g}\tilde{\phi}^{l+d-(e+1)}|.$$

Proof. If $f(x)$ is a Schwartz function on $\mathfrak{n}_1$ and $q(y)$ is a homogeneous polynomial on $\bar{\mathfrak{n}}_1$, then we have (up to a scalar multiple)

$$(\partial_q f)\check{} = q\check{f},$$

where ∂_q is the constant coefficient differential operator on $\mathfrak{n}_1$ with "symbol" q. Thus the proof of the Lemma consists in establishing that the above identity continues to hold when f (like $\Phi_{k,k}$) is a smooth function of polynomial growth such that $\check{f} \in \mathscr{L}^1(\bar{\mathfrak{n}}_1, d\lambda)$. This is fairly standard (e.g. [J, Chapter 7]); indeed by adjointness we have

$$\widehat{q\check{f}d\lambda} = \partial_q(f\,d\lambda),$$

where the derivative on the right is the distributional derivative. Under the assumption on f, the right side equals $(\partial_q f)\,d\lambda$, and the result follows. $\square$

We are now in a position to prove Proposition 2.2.

Proof of Proposition 2.2. The function g is M-invariant, thus by Proposition 1.3, it suffices to prove the convergence of the integral

$$\int_{\mathscr{C}_k} |g(z_1y_1 + \cdots + z_ky_k)|^2 [P_k^{n-k+1} V_k]^d d_k^{\times} z.$$

By the previous lemma, this can be rewritten as

$$\int_{\mathscr{C}_k} |\widetilde{g}(z_1y_1 + \cdots + z_ky_k)|^2 [P_k^{n-k+1} V_k]^d d_k^{\times} z.$$

Using Proposition 1.3 we can further rewrite this as

$$\int_{\overline{\mathfrak{n}}_1} |\widetilde{g}|^2 |\widetilde{\phi}^t| d\lambda \quad \text{where } t = d(n-k+1) - (e+1).$$

Thus it suffices to prove that

$$|\widetilde{g}^2\widetilde{\phi}^t| \in \mathscr{L}^1(\overline{\mathfrak{n}}_1, d\lambda). \tag{9}$$

Now the map $f \mapsto \check{f}$ extends as an isometry from $\mathscr{L}^2(\mathfrak{n}_1, d\lambda)$ to $\mathscr{L}^2(\overline{\mathfrak{n}}_1, d\lambda)$ (after suitable normalizations of the Lebesgue measures). Thus we have

$$f_1, f_2 \in \mathscr{L}^2(\mathfrak{n}_1, d\lambda) \Rightarrow \check{f}_1\check{f}_2 \in \mathscr{L}^1(\overline{\mathfrak{n}}_1, d\lambda), \tag{10}$$

we shall deduce (9) from (10) by a suitable choice of f_1, f_2.

Let us put

$$s = t + 2(e+1-d) = d(n-k-1) + (e+1),$$

since $n > k$, we have $s > 0$. Now if we set

$$f_1 = \widetilde{D}^{l_1}\Phi_{k,k}, \quad f_2 = \widetilde{D}^{l_2}\Phi_{k,k}, \text{ where } l_1 + l_2 = s, \tag{11}$$

then by the previous lemma we have

$$|\check{f}_1\check{f}_2| = \widetilde{g}^2|\widetilde{\phi}|^{l_1+l_2-2(e+1-d)} = |\widetilde{g}^2\widetilde{\phi}^t|. \tag{12}$$

We now consider two cases: if $G \neq \mathrm{Sp}_n(\mathbb{C})$ we set $l_1 = 0$ and $l_2 = s$; if $G = \mathrm{Sp}_n(\mathbb{C})$, we set $l_1 = 1$ and $l_2 = s - 1$. In the former case we have $l_1, l_2 \geqslant 0$; while in the latter case, we have $e = 1$, whence $s \geqslant 2$ and $l_1, l_2 \geqslant 1$. Thus in either case by the open orbit estimates of Propositions 2.3 and 2.4, applied to the Jordan algebra $\mathfrak{n}_1$, we deduce that the functions f_1 and f_2 from formula (11) belong to $\mathscr{L}^2(\mathfrak{n}_1, d_1x)$. Thus formula (9) follows from (12) and (10). □

3. Proof of the main results

We now explain how to deduce Theorems 0.1 and 0.2 from the previous results. As explained in the introduction, the arguments are very similar to those in [S1,DS1,DS2]. Thus, we shall limit ourselves to only sketching the proofs of the various results below.

3.1. Proof of Theorem 0.1

In order to prove Theorem 0.1, we introduce a number of spaces.

First of all, let $E(-dk) \subset C^\infty(\mathfrak{n})$ be the space of smooth vectors in the degenerate principal series defined in Section 2. The representation $\pi = \pi_{-dk\nu}$ of the group G on this space is by "fractional linear transformations", and we have

$$[\pi(l)f](x) = e^{-dk\nu}(l)f(\mathrm{Ad}\,l^{-1}[x]) \quad \text{for } l \text{ in } L,$$

$$[\pi(\exp x')f](x) = f(x + x') \quad \text{for } x' \text{ in } \mathfrak{n}.$$

By Sahi [S3], the space $E(-dk)$ has an irreducible *unitarizable* spherical $(\mathfrak{g}, K)$-submodule V which we also regard as a subspace of $C^\infty(\mathfrak{n})$. Thus by Harish-Chandra theory, the Hilbert space closure $\mathscr{H}$ of V with respect to the $(\mathfrak{g}, K)$-invariant norm carries an irreducible unitary representation of G.

For convenience, we first describe $\mathscr{H}$ as the closure of a G-invariant space. For this we introduce the space $\mathbf{V}$ consisting of those vectors in $I(-dk)$ whose restriction to K, and subsequent expansion in K-isotypic components only involves the K-types of V. Since V is $(\mathfrak{g}, K)$-invariant, the space $\mathbf{V}$ is G-invariant and we have the following result.

Lemma 3.1. *Functions in* $\mathbf{V}$ *have finite* $\mathscr{H}$*-norm and* $\mathscr{H}$ *is the closure of* $\mathbf{V}$.

Proof. This is a consequence of a general result due to Casselman–Wallach on the smooth vectors of a representation. In the present situation, one can also give an alternative proof along the lines of the remark in Section 2.4 of [DS2] as follows.

First of all, the K-types of V have multiplicity 1, and have highest weights of the form

$$m_1\gamma_1 + \cdots + m_k\gamma_k,$$

where $m_1 \geqslant \cdots \geqslant m_k \geqslant 0$ and $\gamma_1, \ldots, \gamma_k$ are as in Section 1.1. Moreover, the $\mathscr{H}$-norm on each K-type is computed explicitly in [S3] and the ratio of the $\mathscr{H}$-norm to the $\mathscr{L}^2(K)$-norm grows at most polynomially in $(m_1, \ldots, m_k)$. On the other hand by the Riemann–Lebesgue lemma for f in $\mathbf{V}$, the $\mathscr{L}^2(K)$-norms of its K-isotypic components decay rapidly. Thus such an f will have finite $\mathscr{H}$-norm. Evidently since $V \subset \mathbf{V}$, the closure of $\mathbf{V}$ is $\mathscr{H}$ as well. $\square$

Next, recall the space $\mathscr{H}_{\mathcal{O}} = \mathscr{L}^2(\mathcal{O}, d\mu)$; by Mackey theory, this space carries a natural irreducible unitary representation $\pi_{\mathcal{O}}$ of P, which is given by the following explicit formulas:

$$[\pi_{\mathcal{O}}(l)\psi](y) = e^{dkv}(l)\psi(\mathrm{Ad}\, l^{-1}[y]) \quad \text{for } l \text{ in } L,$$

$$[\pi_{\mathcal{O}}(\exp x)\psi](y) = e^{-i\langle x,y\rangle}\psi(y) \quad \text{for } x \text{ in } \mathfrak{n},$$

where $\langle x, y\rangle$ is the normalized Killing form of Section 1.3. We shall prove Theorem 0.1 by constructing a unitary P-isomorphism $\mathscr{I}$ between $(\pi|_P, \mathscr{H})$ and $(\pi_{\mathcal{O}}, \mathscr{H}_{\mathcal{O}})$.

We first define $\mathscr{I}$ on a suitable subspace of $\mathscr{H}$. For this, let $\mathscr{C}(G)$ be the convolution algebra of smooth $\mathscr{L}^1$ functions on G. Then by standard arguments, π extends to a representation of $\mathscr{C}(G)$ on $\mathbf{V}$ and we define

$$\mathbf{W} = \pi(\mathscr{C}(G))\Phi \subset \mathbf{V},$$

where $\Phi = \Phi_{-dk}$ is the spherical vector in $I(-dk)$. Since $G = PK$ and Φ is K-fixed, we also have

$$\mathbf{W} = \pi(\mathscr{C}(P))\Phi,$$

and we shall prove the following result:

Lemma 3.2. *For each f in $\mathbf{W}$ there is a unique $\mathscr{I}(f) \in \mathscr{H}_{\mathcal{O}}$ such that*

$$f\, d\lambda = \widehat{\mathscr{I}(f) d\mu},$$

as tempered distributions. Furthermore, for all $F \in \mathscr{C}(P)$ we have

$$\mathscr{I} \circ \pi(F) = \pi_{\mathcal{O}}(F) \circ \mathscr{I}. \tag{13}$$

Proof. By Proposition 2.2, for $\Phi = \Phi_{-dk}$ we have

$$\Phi\, d\lambda = \widehat{\psi d\mu},$$

where $\psi \in \mathscr{H}_{\mathcal{O}}$; or, equivalently,

$$\Phi(x) = \int_{\mathcal{O}} e^{-i\langle x,y\rangle}\psi(y) d\mu(y).$$

Now for l in L, by Lemma 1.1 we have

$$\int_{\mathcal{O}} e^{-i\langle x,y\rangle}[\pi_{\mathcal{O}}(l)\psi](y)\,d\mu(y) = e^{dk\nu}(l)\int_{\mathcal{O}} e^{-i\langle x,y\rangle}\psi(\mathrm{Ad}\,l^{-1}[y])\,d\mu(y)$$
$$= e^{dk\nu}(l)e^{-2dk\nu}(l)\int_{\mathcal{O}} e^{-i\langle x,\mathrm{Ad}\,l[y]\rangle}\psi(y)d\mu(y)$$
$$= e^{-dk\nu}(l)\int_{\mathcal{O}} e^{-i\langle \mathrm{Ad}\,l^{-1}[x],y\rangle}\psi(y)d\mu(y) = \pi(l)f.$$

Similarly for x' in $\mathfrak{n}$, we have

$$\int_{\mathcal{O}} e^{-i\langle x,y\rangle}[\pi_{\mathcal{O}}(\exp x')\psi](y)\,d\mu(y) = \int_{\mathcal{O}} e^{-i\langle x+x',y\rangle}\psi(y)\,d\mu(y) = \pi(\exp x')f.$$

Thus for any $F\in\mathscr{C}(P)$, we have

$$[\pi(F)\Phi]\,d\lambda = ([\pi_{\mathcal{O}}(F)\psi]\,d\mu)^{\widehat{}}$$

and we can define $\mathscr{I}$ by the formula

$$\mathscr{I}(\pi(F)\Phi) = \pi_{\mathcal{O}}(F)\psi.$$

Then $\mathscr{I}$ satisfies the conditions of the lemma. The uniqueness is clear. □

We can now finish the proof of Theorem 0.1. Given the lemma above, the proof of the result proceeds along lines similar to [S1,DS2].

Proof of Theorem 0.1. By the previous lemma, the space $\mathbf{W}_1 = \mathscr{I}(\mathbf{W})$ is a $\mathscr{C}(P)$-invariant subspace of $\mathscr{H}_{\mathcal{O}}$, and moreover we can equip it with a second P-invariant norm, namely that transferred from $\mathscr{H}$ (see the proof of [DS2, Theorem 2.10] for the explicit construction of this second measure).

Now as explained in [S1, 3.3], it follows from [P] that $\mathbf{W}_1$ contains a further $\mathscr{C}(P)$-invariant subspace $\mathbf{W}_2$ on which the two norms coincide (up to a scalar multiple which we normalize to be 1 by rescaling $\mathscr{I}$).

Since $\mathscr{H}_{\mathcal{O}}$ is irreducible, $\mathbf{W}_2$ is dense in $\mathscr{H}_{\mathcal{O}}$ and thus $\mathscr{H}_{\mathcal{O}}$ can be regarded as the closure of $\mathbf{W}_2$ with respect to the $\mathscr{H}$-norm. It follows that the two norms agree on $\mathbf{W}_1$ as well, and thus the map

$$\mathscr{I}^{-1} : \mathbf{W}_1 \to \mathbf{W}$$

extends to an isometric P-invariant imbedding $\mathscr{J}$ of $\mathscr{H}_{\mathcal{O}}$ into $\mathscr{H}$. Now the image of $\mathscr{J}$ is closed, and contains a G-invariant subspace (namely $\mathbf{W}$); thus since $\mathscr{H}$ is an irreducible representation, it follows that $\mathscr{J}$ is surjective as well. Thus $\mathscr{J}$ is a unitary intertwining operator between $(\pi_{\mathcal{O}}, \mathscr{H}_{\mathcal{O}})$ and $(\pi|_P, \mathscr{H})$. The required extension of $(\pi_{\mathcal{O}}, \mathscr{H}_{\mathcal{O}})$ to G is now given by simply transferring the representation from $(\pi, \mathscr{H})$ via $\mathscr{J}^{-1}$. □

3.2. Proof of Theorem 0.2

We now study tensor products of our representations $\pi_{\mathcal{O}}$. The analogous study for conformal groups of *Euclidean* Jordan algebras was conducted in [DS1].

Since the statements and proofs from [DS1] can be transferred to our present (non-Euclidean) setting without substantial changes, we will only sketch some of the arguments below.

Fix $s \geqslant 2$ and a collection of positive integers $k_1, \dots, k_s$ satisfying

$$k = k_1 + \cdots + k_s \leqslant n.$$

For each $i = 1, \dots, s$, let $\mathcal{O}^i$ be the L-orbit on $\bar{\mathfrak{n}}$ of rank k_i, with L-equivariant measure $d\mu^i$. Let $\pi_{\mathcal{O}^i}$ be the unitary representation of G on the space $\mathcal{L}^2(\mathcal{O}^i, d\mu^i)$ as described in Theorem 0.1. We wish to study the tensor product representation

$$\Pi = \pi_{\mathcal{O}^1} \otimes \cdots \otimes \pi_{\mathcal{O}^s}$$

which can be realized explicitly on the space $\mathcal{L}^2(\mathcal{O}^1 \times \cdots \times \mathcal{O}^s, d\mu^1 \times \cdots \times d\mu^s)$.

Let $y_1, \dots, y_n$ be as in Section 1.2, and define

$$v_i = y_{m_i+1} + y_{m_i+2} + \cdots + y_{m_i+k_i}, \quad \text{where } m_i = k_1 + \cdots + k_{i-1}, 1 \leqslant i \leqslant s.$$

Then v_i is an orbit representative for $\mathcal{O}^i$; $v = v_1 + \cdots + v_s$ is an orbit representative for the rank k orbit $\mathcal{O}$; and the L-orbit of

$$v' = (v_1, \dots, v_s)$$

is an open subset of $\mathcal{O}^1 \times \cdots \times \mathcal{O}^s$ with full measure. We denote by S' and S the isotropy subgroups of v' and v, respectively. In the notation of Section 1.4, we have $v = \mathbf{y}^1$, and thus

$$S = (H_1 \times L_0) \cdot U.$$

It is easy to see that S' can then be written as

$$S' = (H_1' \times L_0) \cdot U,$$

where H_1' is a certain reductive subgroup of H_1. We now change the notation slightly and write G' for H_1 and H' for H_1'.

Example. Take $G = E_{7(7)}$, $s = 2$ and $k_1 = 1, k_2 = 2$. Then $k = n = 3$ and $S = G'$ (the stabilizer of the identity element of $\bar{\mathfrak{n}}$, the exceptional Jordan algebra of dimension 27). In this case we have $G' = F_{4(4)}$ and $S' = H' = \mathrm{Spin}_{4,5}$ (cf. [A, p. 119]).

In general, $X = G'/H'$ is a reductive homogeneous space, and we write $\mathrm{Ind}_{H'}^{G'} 1$ for the quasiregular representation of G' on $\mathcal{L}^2(X)$. We decompose this using the Plancherel measure $d\rho$ and the corresponding multiplicity function

$m : \widehat{H} \to \{0, 1, 2, \ldots, \infty\}$, i.e.,

$$\mathrm{Ind}_{H'}^{G'} 1 \simeq \int_{\widehat{G'}}^{\oplus} m(\sigma)\sigma \, d\rho(\sigma).$$

We define a map Θ from irreducible unitary representations of G' to unitary representations of P as follows

$$\Theta(\sigma) = \mathrm{Ind}_{SN}^{P} \, (E\sigma \otimes \chi_v),$$

where $E\sigma$ denotes the trivial extension of σ to $S = (G' \times L_0) \cdot U$, and χ_v is the unitary character of N defined by

$$\chi_v(\exp x) = e^{-i\langle v,x\rangle}.$$

An easy application of Mackey theory shows that all representations $\Theta(\sigma)$ are unitary irreducible representations of P, and $\Theta(\sigma) \simeq \Theta(\sigma')$ if and only if $\sigma \simeq \sigma'$.

Proposition 3.3. *The restriction of Π to P decomposes as follows*:

$$\Pi|_P \simeq \int_{\widehat{G'}}^{\oplus} m(\sigma)\Theta(\sigma) \, d\rho(\sigma), \tag{14}$$

Proof. This is proved as in [DS1, Lemma 2.1]—here is a sketch of the argument. We define an operator F from the space of Π to functions on P by the formula

$$[Ff](ln) = \chi_v(ln\, l^{-1}) f(l \cdot v'), \quad l \in L, \, n \in N.$$

It is an easy exercise to verify that F gives a unitary isomorphism

$$\Pi|_P \simeq \mathrm{Ind}_{S'N}^{P}(1 \otimes \chi_v).$$

Next, using induction by stages we obtain an isomorphism

$$\mathrm{Ind}_{S'N}^{P} \, (1 \otimes \chi_v) \simeq \mathrm{Ind}_{SN}^{P} \, ((\mathrm{Ind}_{S'}^{S} 1) \otimes \chi_v).$$

A final easy calculation shows that

$$\mathrm{Ind}_{S'}^{S} 1 \simeq E(\mathrm{Ind}_{H'}^{G'} 1) \simeq \int_{\widehat{G'}}^{\oplus} m(\sigma)(E\sigma) d\rho(\sigma).$$

Combining the various isomorphisms, we obtain the result. □

Let κ be a unitary representation of G on a Hilbert space $\mathscr{H}$, and R be a subgroup of G. We shall write $\mathscr{A}(\kappa, R)$ for the von Neumann algebra generated by the operators $\{\kappa(g) | g \in R\}$. If G is a type I group [M], then for irreducible κ one has $\mathscr{A}(\kappa, G) = \mathscr{B}(\mathscr{H})$—the full algebra of bounded operators on $\mathscr{H}$. To extend the P-decomposition of Π from formula (12) to the G-decomposition, we require the following

Proposition 3.4. $\mathscr{A}(\Pi, G) = \mathscr{A}(\Pi, P)$.

The analog of Proposition 3.4 for Euclidean Jordan algebras was proved in [DS1, 4.4], by combining the low rank theory of [Li1,Li2] for classical groups and Jordan algebra techniques for the exceptional groups. These arguments extend to the present setting without any significant modifications. For the reader's convenience, we outline the steps of the argument in Appendix A.1.

Proof of Theorem 0.2. Consider the direct integral decomposition of Π

$$\Pi = \int^{\oplus} m(\kappa)\kappa \, d\eta(\kappa)$$

into irreducible representations of G. Then

$$\mathscr{A}(\Pi, P) \subseteq \int^{\oplus} m(\kappa)\mathscr{A}(\kappa, P)\, d\eta(\kappa) \subseteq \int^{\oplus} m(\kappa)\mathscr{A}(\kappa, G)\, d\eta(\kappa) = \mathscr{A}(\Pi, G).$$

The equality of Proposition 3.4 is possible only when the following conditions are satisfied (for almost every κ with respect to $d\eta$):

- $\kappa|_P$ is irreducible (then $\mathscr{A}(\kappa, P) = \mathscr{A}(\kappa, G)$);
- If $\kappa|_P \simeq \kappa'|_P$, then $\kappa \simeq \kappa'$.

Thus in this case (almost) every irreducible representation $\Theta(\sigma)$ from the spectrum of $\Pi|_P$ extends *uniquely* to an irreducible representation of G, which we denote by $\theta(\sigma)$; hence the P-decomposition of formula (14) gives rise to the G-decomposition

$$\Pi = \int^{\oplus}_{\widehat{G'}} m(\sigma)\theta(\sigma)\, d\rho(\sigma)$$

and the theorem follows. □

Example. Again, take $G = E_{7(7)}$, $s = 2$ and $k_1 = 1$, $k_2 = 2$. Then the map $\sigma \mapsto \theta(\sigma)$ establishes a correspondence between the spectrum of Π and the spectrum of the rank 1 reductive symmetric space $F_{4(4)}/\mathrm{Spin}(4,5)$. In other words, we obtain a duality between (some subsets of) the unitary duals of two exceptional groups: split F_4 on one side and split E_7 on the other side. As with Howe's duality correspondence (the usual θ-correspondence), we expect that this new duality will have smooth and global analogues.

Appendix A

A.1. Low rank representations

Let τ be a unitary representation of G. Since N is abelian, the restriction $\tau|_N$ decomposes into a direct integral of unitary characters of N. This decomposition defines a projection valued measure on the dual space N^*, which we identify with $\overline{\mathfrak{n}}$. If this measure is supported on a single non-open orbit $\mathcal{O}_r \subset \overline{N}$, we say that τ is a *low-rank* representation of G and write

$$\mathrm{rank}_N \tau = r.$$

An element x_1 is a primitive idempotent in a Jordan algebra N, and we can consider the associated Peirce decomposition

$$N = N(x_1, 1) + N(x_1, \tfrac{1}{2}) + N(x_1, 0).$$

Observe that the spaces $N(x_1, 1)$ and $N(x_1, 0)$ are the Jordan algebras of ranks 1 and $n-1$, respectively, for the Jordan structure inherited from N.

We will write N_1 for $N(x_1, 1)$ and N_0 for $N(x_1, 0)$. Similarly, we write G_0 for the conformal group of N_0, $P_0 = L_0 N_0$ for the Siegel parabolic subgroup of G_0, etc. Below are the examples of N_0 and G_0 for several different groups G:

- For $G = O_{p+2,p+2}$, $N_0 = \mathbb{R}$ (rank 1 Jordan algebra), and $G_0 = \mathrm{GL}_2(\mathbb{R})$.
- If $G = \mathrm{Sp}_{n,n}$, then $G_0 = \mathrm{Sp}_{n-1,n-1}$.
- If $G = E_{7(7)}$, then $N_0 = \mathbb{R}^{6,6}$ (rank 2 Jordan algebra), and $G_0 = O_{6,6}$.
- If $G = E_7(\mathbb{C})$, then $G_0 = O_{12}(\mathbb{C})$.

Set $\mathfrak{f} = \bigoplus_{i=2}^n \mathfrak{g}^{\varepsilon_1 - \varepsilon_i} \oplus \bigoplus_{i=2}^n \mathfrak{g}^{\varepsilon_1 + \varepsilon_i}$ and $\mathfrak{n}' = \mathfrak{f} + \mathfrak{n}_1$. Then $\mathfrak{n}'$ is a two-step nilpotent subalgebra of $\mathfrak{g}$ with the center $\mathfrak{n}_1$.

Any generic unitary irreducible representation of the group N' is determined by the unitary character of its center N_1. We denote by ρ_t the unitary irreducible representation of N' which restricts to the multiple of the character χ_t on N_1, $t \in N_1^\vee = N_1^* \backslash \{0\}$.

Consider now a subgroup $G_0 N'$ of G. We can view G_0 as a subgroup of a symplectic group $\mathrm{Sp}\,(\mathfrak{f})$ associated with the standard skew-symmetric bilinear form on $\mathfrak{f}$. Hence we can use the oscillator representation of $\mathrm{Sp}\,(\mathfrak{f})$ to extend the representation ρ_t of N' to a representation of $G_0 N'$ which we denote by $\widetilde{\rho}_t$.

Let σ be a unitary representation of G, $\mathrm{rank}_N \sigma = r$, $0 < r < n$. Without loss of generality, we may assume that σ has no N_1-fixed vectors. Then by Mackey theory, we can write down the decomposition

$$\sigma|_{G_0 N'} = \int_{N_1^\vee}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt,$$

where all κ_t are unitary representations of G_0.

Proceeding as in [DS1, 3.1], we verify that all of the representations κ_t are in turn the *low-rank representations of* G_0. More precisely, we have the following.

Lemma A.1. *Let σ be a low-rank representation of G, $\mathrm{rank}_N \sigma = r$, $0<r<n$. Then for any $t \in N_1^\vee$ the N_0-spectrum of the representation κ_t is supported on a single L_0-orbit, and* $\mathrm{rank}_{N_0} \kappa_t = r - 1$.

The next technical lemma is proved exactly as in [DS1, 3.2]:

Lemma A.2. *If for all $t \in N_1^\vee$ one has $\mathscr{A}(\kappa_t, G_0) = \mathscr{A}(\kappa_t, P_0)$, then*

$$\mathscr{A}\left(\int_{N_1^\vee}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt, G_0 \right) \subseteq \mathscr{A}\left(\int_{N_1^\vee}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt, P_0 N' \right).$$

Theorem A.3. *Let σ be a representation of G, $\mathrm{rank}_N \sigma = r$, $0<r<n$. Then*

$$\mathscr{A}(\sigma, G) = \mathscr{A}(\sigma, P).$$

Proof. The proof of the theorem is based on the fact that G_0 and P generate G, so it is enough to verify that $\mathscr{A}(\sigma, G_0) \subseteq \mathscr{A}(\sigma, P)$. Since $P_0 N'$ is a subgroup of P, by the Lemma above the assertion of the theorem is equivalent to the claim that for any $t \in N_1^\vee$

$$\mathscr{A}(\kappa_t, G_0) = \mathscr{A}(\kappa_t, P_0).$$

By Lemma A.1, all κ_t have rank $r-1$. Proceeding in the same manner, we reduce the statement of the theorem to that about rank 0 representations of the certain (classical) group G_{00}. Since all rank 0 representations are the direct integrals of characters, and any character of G_{00} is determined by its restriction to the Siegel parabolic $P_{00} \subset G_{00}$, the theorem follows. □

We now consider the tensor product

$$\Pi = \pi_{\mathcal{O}^1} \otimes \cdots \otimes \pi_{\mathcal{O}^s},$$

for $k = k_1 + k_2 + \cdots + k_s < n$. Then Π is low-rank representation of G and $\mathrm{rank}_N \Pi = k$. Applying the theorem above to Π, we obtain the statement of Proposition 3.4 for $k<n$.

It remains to check Proposition 3.4 for $k = n$. For all groups G except $O_{p+2,p+2}$, $O_{p+4}(\mathbb{C})$ and $E_{7(7)}$, $E_7(\mathbb{C})$ the statement of the proposition follows from the results of [Li2], since all the representations form the spectrum of Π appear in the Howe duality correspondence for appropriate stable range dual pairs (G^1, G). For the exceptional cases listed above, the argument can be constructed along the lines of Section 4 of [DS1].

A.2. Tables of groups and symmetric spaces

In the first table we list the symmetric spaces K/M and L/H; and the root multiplicities d and e in $\Sigma(\mathfrak{t}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$. The rank of the Jordan algebra is n, except for $O_{p+4}(\mathbb{C})$, $O_{p+2,p+2}$ (rank 2), and $E_{7(7)}$, $E_7(\mathbb{C})$ (rank 3). The second table lists the spaces $X = G'/H'$ in the θ-correspondence of Theorem 0.2.

G	K/M	L/H	d	e
$\mathrm{GL}_{2n}(\mathbb{R})$	$O_{2n}/(O_n \times O_n)$	$\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{R})$	1	0
$O_{2n,2n}$	$(O_{2n} \times O_{2n})/O_{2n}$	$\mathrm{GL}_{2n}(\mathbb{R})/\mathrm{Sp}_n(\mathbb{R})$	2	0
$E_{7(7)}$	$\mathrm{SU}_8/\mathrm{Sp}_4$	$\mathbb{R}^* \times E_{6(6)}/F_{4(4)}$	4	0
$O_{p+2,p+2}$	$[O_{p+2}]^2/[O_1 \times O_{p+1}^2]$	$\mathbb{R}^* \times O_{p+1,p+1}/O_{p,p+1}$	p	0
$\mathrm{Sp}_n(\mathbb{C})$	Sp_n/U_n	$\mathrm{GL}_n(\mathbb{C})/O_n(\mathbb{C})$	1	1
$\mathrm{GL}_{2n}(\mathbb{C})$	$U_{2n}/(U_n \times U_n)$	$\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_n(\mathbb{C})$	2	1
$O_{4n}(\mathbb{C})$	O_{4n}/U_{2n}	$\mathrm{GL}_{2n}(\mathbb{C})/\mathrm{Sp}_n(\mathbb{C})$	4	1
$E_7(\mathbb{C})$	$E_7/(E_6 \times U_1)$	$\mathbb{C}^* \times E_6(\mathbb{C})/F_4(\mathbb{C})$	8	1
$O_{p+4}(\mathbb{C})$	$O_{p+4}/(O_{p+2} \times U_1)$	$\mathbb{C}^* \times O_{p+2}(\mathbb{C})/O_{p+1}(\mathbb{C})$	p	1
$\mathrm{Sp}_{n,n}$	$(\mathrm{Sp}_n \times \mathrm{Sp}_n)/\mathrm{Sp}_n$	$\mathrm{GL}_n(\mathbb{H})/O_n^*$	2	2
$\mathrm{GL}_{2n}(\mathbb{H})$	$\mathrm{Sp}_{2n}/(\mathrm{Sp}_n \times \mathrm{Sp}_n)$	$\mathrm{GL}_n(\mathbb{H}) \times \mathrm{GL}_n(\mathbb{H})/\mathrm{GL}_n(\mathbb{H})$	4	3

G	X
$\mathrm{GL}_{2n}(\mathbb{R})$	$\mathrm{GL}_k(\mathbb{R})/[\mathrm{GL}_{k_1}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{k_s}(\mathbb{R})]$
$O_{2n,2n}$	$\mathrm{Sp}_k(\mathbb{R})/[\mathrm{Sp}_{k_1}(\mathbb{R}) \times \cdots \times \mathrm{Sp}_{k_s}(\mathbb{R})]$
$E_{7(7)}$	$\mathrm{Spin}_{4,5}/\mathrm{Spin}_{4,4}$ $(k_1 = 1, k_2 = 1)$
	$F_{4(4)}/\mathrm{Spin}_{4,5}$ $(k_1 = 2, k_2 = 1)$
$O_{p+2,p+2}$	$\mathrm{SO}_{p,p+1}/\mathrm{SO}_{p,p}$ $(k_1 = 1, k_2 = 1)$
$\mathrm{Sp}_n(\mathbb{C})$	$O_k(\mathbb{C})/[O_{k_1}(\mathbb{C}) \times \cdots \times O_{k_s}(\mathbb{C})]$
$\mathrm{GL}_{2n}(\mathbb{C})$	$\mathrm{GL}_k(\mathbb{C})/[\mathrm{GL}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_s}(\mathbb{C})]$
$O_{4n}(\mathbb{C})$	$\mathrm{Sp}_k(\mathbb{C})/[\mathrm{Sp}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{Sp}_{k_s}(\mathbb{C})]$
$E_7(\mathbb{C})$	$\mathrm{Spin}_9(\mathbb{C})/\mathrm{Spin}_8(\mathbb{C})$ $(k_1 = 1, k_2 = 1)$
	$F_4(\mathbb{C})/\mathrm{Spin}_9(\mathbb{C})$ $(k_1 = 2, k_2 = 1)$
$O_{p+4}(\mathbb{C})$	$\mathrm{SO}_{p+1}(\mathbb{C})/\mathrm{SO}_p(\mathbb{C})$ $(k_1 = 1, k_2 = 1)$
$\mathrm{Sp}_{n,n}$	$O_k^*/[O_{k_1}^* \times \cdots \times O_{k_s}^*]$
$\mathrm{GL}_{2n}(\mathbb{H})$	$\mathrm{GL}_k(\mathbb{H})/[\mathrm{GL}_{k_1}(\mathbb{H}) \times \cdots \times \mathrm{GL}_{k_s}(\mathbb{H})]$

References

[A] J.F. Adams, Lectures on Exceptional Lie Groups, University of Chicago Press, Chicago, 1996.

[BK] H. Braun, M., Koecher, Jordan-Algebren, Springer, Berlin, New York, 1966.

[DS1] A. Dvorsky, S. Sahi, Tensor products of singular representations and an extension of the θ-correspondence, Selecta Math. 4 (1998) 11–29.

[DS2] A. Dvorsky, S. Sahi, Explicit Hilbert spaces for certain unipotent representations II, Invent. Math. 138 (1999) 203–224.

[J] D.S. Jones, Generalised Functions, McGraw-Hill, New York, Toronto, 1966.

[KS] B. Kostant, S. Sahi, Jordan algebras and Capelli identities, Invent. Math. 112 (1993) 657–664.

[Li1] J.-S. Li, Singular unitary representations of classical groups, Invent. Math. 97 (1989) 237–255.

[Li2] J.-S. Li, On the classification of irreducible low rank unitary representations of classical groups, Compositio Math 71 (1989) 29–48.

[Lo] O. Loos, Bounded Symmetric Domains and Jordan Pairs, Mathematical Lectures, University of California, Irvine, 1977.

[M] G. Mackey, The Theory of Unitary Group Representations, University of Chicago Press, Chicago, 1976.

[Op] E. Opdam, Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group, Compositio Math. 85 (1993) 333–373.

[OS] T. Oshima, J. Sekiguchi, The restricted root system of a semisimple symmetric pair, in: Advanced Studies in Pure Mathematics, Vol. 4, North-Holland, Amsterdam, 1984, pp. 433–497.

[P] D. Poguntke, Unitary representations of Lie groups and operators of finite rank, Ann. of Math. (2) 140 (1994) 503–556.

[S1] S. Sahi, Explicit Hilbert spaces for certain unipotent representations, Invent. Math. 110 (1992) 409–418.

[S2] S. Sahi, Unitary representations on the Shilov boundary of a symmetric tube domain, Contemp. Math. 145 (1993) 275–286.

[S3] S. Sahi, Jordan algebras and degenerate principal series, J. Reine Angew. Math. 462 (1995) 1–18.

[SS] S. Sahi, E. Stein, Analysis in matrix space and Speh's representation, Invent. Math. 101 (1990) 379–393.

[Sc] H. Schlichtkrull, Hyperfunctions and Harmonic Analysis on Symmetric Spaces, Birkhäuser, Boston, 1984.

[Sh] G. Shimura, Generalized Bessel functions on symmetric spaces, J. Reine Angew. Math. 509 (1999) 35–66.