

Triple groups and Cherednik algebras

Bogdan Ion and Siddhartha Sahi

ABSTRACT. The goal of this paper is to define a new class of objects which we call triple groups and to relate them with Cherednik's double affine Hecke algebras. This has as immediate consequences new descriptions of double affine Weyl and Artin groups, the double affine Hecke algebras as well as the corresponding elliptic objects. From the new descriptions we recover results of Cherednik on automorphisms of double affine Hecke algebras.

Introduction

The classical theory of root systems associated with finite dimensional semisimple Lie algebras admits two generalizations. The first one, the Kac–Moody theory, originated in the effort to extend the classical theory to include certain classes of infinite dimensional Lie algebras as, for example, the simple infinite dimensional Lie algebras of vector fields on finite dimensional spaces as classified by Cartan at the beginning of the 20th century. The theory, which was initiated in the mid 1960's by V. Kac and R. Moody, grew to be extremely rich and with abundant connections with diverse areas of mathematics. The second generalization, axiomatized by K. Saito, emerged from the theory of semi-universal deformations of simply elliptic singularities and the structure of Milnor lattices attached to these singularities. This second theory is still in the very early stages of development from a representation–theoretical point of view, at present extending only the classical theory of root systems. These root systems were called by Saito 2–extended root systems or elliptic root systems. Saito defined in fact more general classes of root systems, k –extended root systems (k being any positive integer), but for values of k higher than three these do not have any geometrical or representation–theoretical relevance yet. The problem of attaching corresponding notions of Lie algebras and groups to k –extended root systems is wide open although in the elliptic case some constructions exist in the literature.

To briefly compare the two theories note that they include the classical theory and they agree on one other class of root systems: the 1–extended root systems coincide with affine root systems in Kac–Moody theory. With this exception, they produce quite different outcomes. To mention a few differences note that the Weyl groups associated to Kac–Moody root systems are always Coxeter groups, but the

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corresponding groups associated to k -extended root systems are not. Also, the Weyl group of a k -extended root system does not act properly discontinuously anywhere in its reflection representation and as a consequence we do not have a Tits cone, and no natural notion of positive root system or Dynkin diagram. Saito was nevertheless able to classify the irreducible elliptic root systems parameterizing them by what he called “elliptic Dynkin diagrams”.

Since, as mentioned above, the Weyl groups associated to elliptic root systems (also called elliptic Weyl groups) are not Coxeter groups, a presentation of them in terms of elliptic Dynkin diagrams was one of the first questions to be asked. A candidate for such a presentation was exposed in [14]. The presentation is complicated and does not bring any light into the structure of the elliptic Weyl group. Another problem along these lines which was asked in [14] is the description of the fundamental group of the complement of the discriminant of the semi-universal deformation of simply-elliptic singularities: the elliptic Artin group. In fact, the elliptic Weyl groups appeared before of the formal definition of elliptic root systems in the work of Looijenga [10] who considered extended Coxeter groups. Under this terminology the elliptic Weyl groups would be called extended affine Weyl groups. Also, the elliptic Artin groups were described (in the non-exceptional cases) by generators and relations by H. van der Lek [9] under the name of extended Artin groups. The elliptic Artin group was recognized by Cherednik [1] as being closely related to his double affine Artin groups and Hecke algebras.

At this point a broad picture emerges since Cherednik’s work has deep connections with quantum many-body problems, conformal field theory and also with representation theory via Macdonald theory, which encloses information about real and p -adic groups and also about affine Kac-Moody groups. There are also connections with harmonic analysis, q -Riemann zeta functions, Gauss-Selberg sums and more. The area is in full development and interesting connections are still to be made. In all the applications the structure of double affine Artin groups and Hecke algebras is especially important.

The first results on the structure of double affine Hecke algebras appeared in Cherednik’s work on Macdonald’s conjectures. For the proof of the constant term conjecture only basic properties of Hecke algebras are used, but for the proof of the evaluation-duality conjecture a deeper result was needed: the existence of a special involution on double affine Hecke algebras called the duality involution. The existence of this involution is certainly non-obvious, see for example [5], [12, Chapter 3] for a proof. Cherednik discovered that in fact this is part of an action of $GL(2, \mathbb{Z})$ as outer automorphisms of double affine Artin groups. This allows, for example, representations of $PSL(2, \mathbb{Z})$ for which the matrix coefficients are expressed in terms of special values of Macdonald polynomials at roots of unity. Let us mention that in the case of a root system of type A_n such actions appeared before in the work of A. Kirillov, Jr [8] on modular tensor categories and quantum groups at roots of unity. In this case the two approaches are most probably connected via the equivalence (see [7]) between certain categories of representations of quantum groups and affine Kac-Moody algebras. For the root system of type A_n the picture seems to be even richer since $PSL(2, \mathbb{Z})$ (which is the mapping class group of the torus with one marked point) can be replaced with the mapping class group of any two dimensional surface with marked points.

The main goal of this paper is to give a completely new description of double affine Hecke algebras, Artin groups and double affine Weyl groups, and consequently of the corresponding elliptic objects. Our description is much simpler than the existing ones and has the virtue of making the existence of the difference Fourier transform and of the $\mathrm{PSL}(2, \mathbb{Z})$ action mentioned above simple consequences. All our proofs are at the level of Artin groups, since any other result is then obtained by passing to various quotients. It turns out that in fact all these symmetries of double affine Artin groups are descending from those of a new object, which we call triple group, of which the double affine Artin group is a quotient. A brief description of our results follows. The reader should refer to Section 1 for notation and conventions.

DEFINITION 1. *Let A be an irreducible affine Cartan matrix subject to our restriction and $S(A)$ its Dynkin diagram. The triple group \mathcal{A} is given by generators and relations as follows:*

Generators: one generator \mathcal{T}_i for each node, with the exception of the affine node for which we have three generators \mathcal{T}_{01} , \mathcal{T}_{02} and \mathcal{T}_{03} . We use the notation $\mathcal{D} = \mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_{03}$.

Relations: a) Braid relations for each pair of generators associated to any pair of distinct nodes (note that there are three generators associated to the affine node).

b) If the affine node is connected with the node α by a single lace, the elements \mathcal{T}_{01} , \mathcal{T}_{02} , \mathcal{T}_{03} , $\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_{01}^{-1}$ and $\mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03}$ satisfy the single lace Coxeter relation with $\mathcal{D}^k\mathcal{T}_\alpha\mathcal{D}^{-k}$ for all integers k .

c) If there are double laces connecting the affine node with the node α the following relation holds for any $1 \leq i < j \leq 3$

$$\mathcal{T}_{0i}\mathcal{T}_\alpha^{-1}\mathcal{T}_{0j}\mathcal{T}_\alpha = \mathcal{T}_\alpha^{-1}\mathcal{T}_{0j}\mathcal{T}_\alpha\mathcal{T}_{0i} \quad (0.1)$$

Of course we can define the corresponding triple Weyl group \mathcal{W} by further asking that the generators have order two. All the facts we discuss will have as an immediate consequence corresponding results for the triple Weyl group.

A few comments on this definition may be useful at this stage. We would like to stress that the above definition is not symmetric in the generators \mathcal{T}_{0i} which means that the symmetric group on three letters will not act on the triple group. However, we will prove in Theorem 2.6 and Theorem 4.10 that the braid group on three letters acts faithfully as automorphisms of the triple group. What is completely obvious from this Definition is the existence of an *anti-involution* which fixes all the generators except \mathcal{T}_{01} and \mathcal{T}_{03} which are interchanged. Also, we like to note that in Definition 1, b) we ask the relations for the element $\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_{01}^{-1}$ merely for symmetry. They follow from the ones for $\mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03}$ by observing that

$$\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_{01}^{-1} = \mathcal{D}\mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03}\mathcal{D}^{-1}.$$

Our main result, Theorem 3.10, shows that the double affine Artin group is a quotient of the triple group. In fact, in this quotient the relations in Definition 1, b) will become redundant, allowing us to give a new description of double affine Artin groups and Hecke algebras.

For double affine Artin groups the descent of the above anti-involution, whose existence is a trivial consequence of our new description, gives (by composition with the anti-involution which sends all the generators to their inverses) the duality involution responsible for the difference Fourier transform. Note that previous results

explaining in an uniform way the existence of this involution required topological arguments [1], [5]. There is also an algebraic proof of this result [12]. This proof proceeds by carefully checking of all the necessary relations between the generators in the Cherednik presentation and requires special considerations for some types of root systems.

Also, the above action of the braid group on three letters descends to double affine Artin groups and Hecke algebras. Here we rediscover results of Cherednik which explain that there is a morphism from the modular group to the group of outer automorphisms of an double affine Artin group. These results are described in Theorem 4.4, Theorem 4.5 and Corollary 4.8. As another consequence we obtain descriptions of elliptic Weyl groups and elliptic Artin groups for certain types of elliptic root systems.

There are interesting quotients of the triple groups of which the double affine Artin groups are quotients but which, unlike the triple groups, are described by a finite number of relations. These quotients also inherit the anti-involution and the action of the braid group on three letters. It is not yet clear if any of these quotients (or the triple group itself) have a topological interpretation.

1. Preliminaries

1.1. Notation and conventions. For the most part we adhere to the notation in [6]. Let $A = (a_{jk})_{0 \leq j, k \leq n}$ be an irreducible *affine* Cartan matrix, $S(A)$ the Dynkin diagram and (a_0, \dots, a_n) the numerical labels of $S(A)$ in Table Aff from [6, p.48-49]. Note that we consider that the nodes i and j are connected in the Dynkin diagram by $a_{ij}a_{ji}$ laces. Unless $A = A_1^{(1)}$ this produces the same diagrams as in [6]. We denote by $(a_0^\vee, \dots, a_n^\vee)$ the labels of the Dynkin diagram $S(A^t)$ of the dual algebra which is obtained from $S(A)$ by reversing the direction of all arrows and keeping the same enumeration of the vertices.

Let $(\mathfrak{h}, R, R^\vee)$ be a realization of the Cartan matrix A and let $(\mathring{\mathfrak{h}}, \mathring{R}, \mathring{R}^\vee)$ be the associated finite root system (which is a realization of the Cartan matrix $\mathring{A} = (a_{jk})_{1 \leq j, k \leq n}$). If we denote by $\{\alpha_j\}_{0 \leq j \leq n}$ a basis of R such that $\{\alpha_j\}_{1 \leq j \leq n}$ is a basis of \mathring{R} we have the following description

$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* + \mathbb{R}\delta + \mathbb{R}\Lambda_0 ,$$

where $\delta = \sum_{j=0}^n a_j \alpha_j$. The vector space \mathfrak{h}^* has a canonical scalar product defined as follows

$$(\alpha_j, \alpha_k) := d_j^{-1} a_{jk} , \quad (\Lambda_0, \alpha_j) := \delta_{j,0} a_0^{-1} \quad \text{and} \quad (\Lambda_0, \Lambda_0) := 0 ,$$

with $d_j := a_j a_j^\vee^{-1}$ and $\delta_{j,0}$ Kronecker's delta. The simple coroots are $\{\alpha_j^\vee := d_j \alpha_j\}_{0 \leq j \leq n}$. The lattice $\mathring{Q} = \bigoplus_{j=1}^n \mathbb{Z} \alpha_j$ is the root lattice of \mathring{R} and if the affine Cartan matrix is not $A_{2n}^{(2)}$ the lattice $Q = \bigoplus_{j=0}^n \mathbb{Z} \alpha_j = \mathring{Q} \oplus \mathbb{Z} \delta$ is the root lattice of R . For the affine Cartan matrix $A_{2n}^{(2)}$ we will have to replace the root lattice \mathring{Q} by the weight lattice P and to keep in mind that the root lattice of R is $Q = \bigoplus_{j=0}^n \mathbb{Z} \alpha_j = P \oplus \frac{1}{2} \mathbb{Z} \delta$.

To keep the notation as simple as possible we agree to use the same symbol \mathring{Q} to refer to the root lattice or the weight lattice depending on the affine root system as explained above.

Although the ideas we will present here apply in general, the objects associated to the irreducible affine Cartan matrices $B_n^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$ require special treatment and we will not study them here. Their special behavior is a consequence of the fact that they lack certain types of symmetries which are otherwise abundant.

For the rest of the paper we assume our irreducible affine root system to be such that the affine simple root α_0 is short (this includes of course the case when all the roots have the same length).

The exceptions to this condition are: $B_n^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$.

For any root system denote by α the simple root corresponding to the node in the Dynkin diagram which is connected to node associated to the affine simple root α_0 . If there are two such nodes (which is the case for $S(A_n^{(1)})$) we choose one of them. By l_0 we denote the number of laces by which the nodes corresponding to α_0 and α are connected. Note that l_0 is always 1, except for $A = B_n^{(2)}$, $A_{2n}^{(2)}$ for which it takes the value 2.

1.2. Affine and double affine Weyl groups. Given $\alpha \in R$, $x \in \mathfrak{h}^*$ let

$$s_\alpha(x) := x - \frac{2(x, \alpha)}{(\alpha, \alpha)} \alpha.$$

The *affine Weyl group* W is the subgroup of $GL(\mathfrak{h}^*)$ generated by all s_α (the simple reflections $s_j = s_{\alpha_j}$ are enough). The *finite Weyl group* \dot{W} is the subgroup generated by s_1, \dots, s_n . Both the finite and the affine Weyl group are Coxeter groups and they can be abstractly defined as generated by s_1, \dots, s_n , respectively s_0, \dots, s_n , and some relations. These relations are called Coxeter relations and they are of two types:

- a) reflection relations: $s_j^2 = 1$;
- b) braid relations: $s_i s_j \cdots = s_j s_i \cdots$ (there are m_{ij} factors on each side, m_{ij} being equal to 2, 3, 4, 6 if the number of laces connecting the corresponding nodes in the Dynkin diagram is 0, 1, 2, 3 respectively).

The *double affine Weyl group* \tilde{W} is defined to be the semidirect product $W \rtimes Q$ of the affine Weyl group and the lattice Q (regarded as an abelian group with elements τ_β , where β is a root), the affine Weyl group acting on the root lattice as follows

$$w\tau_\beta w^{-1} = \tau_{w(\beta)}.$$

This group is the hyperbolic extension of an elliptic Weyl group which is by definition the Weyl group associated with an elliptic root system (see [14] for definitions). The cases we consider cover all elliptic root systems with equal labels (p, p) (some of these will have isomorphic elliptic Weyl groups). It also has a presentation with generators and relations (called elliptic Coxeter relations). We refer the reader to [14] for the details. We only note here that the above mentioned presentation is considerably more complex than the one for Coxeter groups: for the infinite series it involves at least $2n - 2$ generators and the genuine elliptic Coxeter relations (i.e. which are not Coxeter relations) are relations among groups of three or four of the generators.

The affine Weyl group W can also be presented as a semidirect product in the following way: it is the semidirect product of \mathring{W} and the lattice \mathring{Q} (regarded as an abelian group with elements λ_μ , where μ is in \mathring{Q}), the finite Weyl group acting on the root lattice as follows

$$\mathring{w}\lambda_\mu\mathring{w}^{-1} = \lambda_{\mathring{w}(\mu)}.$$

Using this description we immediately see that the double affine Weyl group could be described as follows.

PROPOSITION 1.1. *The double affine Weyl group is the group generated by the finite Weyl group \mathring{W} , two lattices $\{\lambda_\mu\}_{\mu \in \mathring{Q}}$, $\{\tau_\beta\}_{\beta \in \mathring{Q}}$ and an element $\tau_{a_0^{-1}\delta}$ with the following relations:*

- (i) $\mathring{w}\lambda_\mu\mathring{w}^{-1} = \lambda_{\mathring{w}(\mu)}$ and $\mathring{w}\tau_\beta\mathring{w}^{-1} = \tau_{\mathring{w}(\beta)}$ for any \mathring{w} in the finite Weyl group and any μ, β in the root lattice;
- (ii) $\lambda_\mu\tau_\beta = \tau_\beta\lambda_\mu\tau_{-(\beta, \mu)\delta}$;
- (iii) $\tau_{a_0^{-1}\delta}$ is central.

To elucidate the alluded relation between the affine Weyl group and the elliptic Weyl group we give the following definition.

DEFINITION 1.2. *The elliptic Weyl group is the factor group of the double affine Weyl group by the group generated by $\tau_{a_0^{-1}\delta}$.*

For r a real number, $\mathfrak{h}_r^* = \{x \in \mathfrak{h} ; (x, \delta) = r\}$ is the level r of \mathfrak{h}^* . We have

$$\mathfrak{h}_r^* = \mathfrak{h}_0^* + r\Lambda_0 = \mathring{\mathfrak{h}}^* + \mathbb{R}\delta + r\Lambda_0 .$$

The action of W preserves each \mathfrak{h}_r^* and we can identify each \mathfrak{h}_r^* canonically with \mathfrak{h}_0^* and obtain an (affine) action of W on \mathfrak{h}_0^* . For example, the level zero action of s_θ and λ_μ on \mathfrak{h}_0^* is

$$\begin{aligned} s_\theta(x) &= s_\theta(x) + (x, \theta)a_0^{-1}\delta , \\ \lambda_\mu(x) &= x - (x, \mu)\delta , \end{aligned}$$

and the full action of the same elements on \mathfrak{h}^* is

$$\begin{aligned} s_\theta\langle x \rangle &= s_\theta(x) + (x, \theta)a_0^{-1}\delta - (x, \delta)\alpha_0 , \\ \lambda_\mu\langle x \rangle &= x - (x, \mu)\delta + (x, \delta)(\mu - \frac{1}{2}|\mu|^2\delta) , \end{aligned}$$

where we denoted by $\theta = \delta - a_0\alpha_0$.

1.3. Artin groups and Hecke algebras. To any Coxeter group we can associate its Artin group, the group defined with the same generators which satisfy only the *braid relations* (that is, forgetting the reflection relations). The finite and affine Weyl groups are Coxeter groups; we will make precise the definition of the Artin groups in these cases.

DEFINITION 1.3. *With the notation above define*

- (i) *the finite Artin group $\mathcal{A}_{\mathring{W}}$ as the group generated by elements*

$$T_1, \dots, T_n$$

satisfying the same braid relations as the reflections s_1, \dots, s_n ;

(ii) the affine Artin group \mathcal{A}_W as the group generated by the elements

$$T_0, \dots, T_n$$

satisfying the same braid relations as the reflections s_0, \dots, s_n .

From the definition it is clear that the finite Artin group can be realized as a subgroup inside the affine Artin group. For further use we also introduce the following lattices $\mathcal{Q}_Y := \{Y_\mu; \mu \in \check{Q}\}$ and $\mathcal{Q}_X := \{X_\beta; \beta \in \check{Q}\}$. Recall that the affine Weyl group also has a second presentation, as a semidirect product. There is a corresponding description of the affine Artin group due to van der Lek [9] and independently obtained by Lusztig [11] and Bernstein (unpublished). To be more precise, Lusztig and Bernstein give a proof for the corresponding description of the extended Hecke algebra (the proof also works for the extended Artin group), which is easier and purely algebraic. H. van der Lek's result is more difficult and the proof uses the topological construction on the affine Artin group.

PROPOSITION 1.4. *The affine Artin group \mathcal{A}_W is generated by the finite Artin group and the lattice \mathcal{Q}_Y such that the following relations are satisfied for all $1 \leq i \leq n$*

- (i) $T_i Y_\mu = Y_\mu T_i$ if $(\mu, \alpha_i^\vee) = 0$,
- (ii) $T_i Y_\mu T_i = Y_{s_i(\mu)}$ if $(\mu, \alpha_i^\vee) = 1$.

REMARK 1.5. In this description $Y_\mu = T_{\lambda_\mu}$ for μ any anti-dominant element of the root lattice. For example $Y_{-\alpha_0^{-1}\theta} = T_{s_\theta} T_0$. In fact, the above Proposition implies that the element $T_{s_\theta}^{-1} Y_{-\alpha_0^{-1}\theta} (= T_0)$ satisfies the predicted braid relations with the generators T_i .

The special form of the Coxeter relations makes clear that the affine Artin group admits an anti-involution ι which fixes all its generators. From Proposition 1.4 and the above Remark it follows that the element $T_{s_\theta}^{-1} \iota(Y_{-\alpha_0^{-1}\theta}) = T_{s_\theta}^{-1} T_0 T_{s_\theta}$, and as a consequence also $\iota(T_{s_\theta}^{-1} T_0 T_{s_\theta}) = T_{s_\theta} T_0 T_{s_\theta}^{-1}$, satisfies with the generators T_i the same braid relations as T_0 . This also follows from the topological description of the affine Artin group.

PROPOSITION 1.6. *The cyclic group of infinite order \mathbb{Z} acts on the affine Artin group as automorphisms by conjugation with T_{s_θ} on T_0 and by fixing the rest of the generators.*

In fact van der Lek's description is even more precise; he identifies a finite set of relations which should be imposed.

PROPOSITION 1.7. *The affine Artin group \mathcal{A}_W is generated by the finite Artin group and the lattice \mathcal{Q}_Y such that the following relations are satisfied for $1 \leq i, j \leq n$*

- (i) For any pair of indices (i, j) such that $2r_{ji} = -(\alpha_j, \alpha_i^\vee)$, with r_{ji} a non-negative integer we have

$$T_i Y_{\mu_j} = Y_{\mu_j} T_i \tag{1.1}$$

where $\mu_j = \alpha_j + r_{ji} \alpha_i$; note that $(\mu_j, \alpha_i^\vee) = 0$,

- (ii) For any pair of indices (i, j) such that $2r_{ji} - 1 = -(\alpha_j, \alpha_i^\vee)$, with r_{ji} a non-negative integer we have

$$T_i Y_{\mu_j} T_i = Y_{s_i(\mu_j)} \tag{1.2}$$

where $\mu_j = \alpha_j + r_{ji}\alpha_i$; note that $(\mu_j, \alpha_i^\vee) = 1$.

If we are in the $A_{2n}^{(2)}$ case the relations for the pair (i, n) are those obtained by considering $a_0^{-1}\alpha_n$ instead of α_n in the above formulas.

Cherednik [2] (in the case of a reduced root system) and Sahi [13] (for a non-reduced root system) defined the so-called double affine Hecke algebra. We extract here from their definition only the definition of the double affine Artin group.

DEFINITION 1.8. *The double affine Artin group $\mathcal{A}_{\tilde{W}}$ is generated by the affine Artin group \mathcal{A}_W , the lattice \mathcal{Q}_X and the element $X_{a_0^{-1}\delta}$ such the following relations are satisfied for all $0 \leq i \leq n$*

- (i) $X_{a_0^{-1}\delta}$ is central ,
- (ii) $T_i X_\beta = X_\beta T_i$ if $(\beta, \alpha_i^\vee) = 0$,
- (iii) $T_i X_\beta T_i = X_{s_i(\beta)}$ if $(\beta, \alpha_i^\vee) = -1$.

To avoid cumbersome notation we will make use of the convention $Y_{a_0^{-1}\delta} := X_{-a_0^{-1}\delta}$. The following fact exploits the similarity of the above definition with presentation of the affine Artin group given in Proposition 1.4.

REMARK 1.9. The elements $X_{a_0^{-1}\theta} T_{s_\theta}^{-1}$ and T_0 satisfy the same braid relations with the generators T_i . In fact, subgroup of the double affine Artin group generated by $T_i, i \neq 0$ and the lattice \mathcal{Q}_X could be described as follows. It is the group with generators $T_i, i \neq 0$ and $X_{a_0^{-1}\theta}$ and such that the elements $T_i, X_{a_0^{-1}\theta} T_{s_\theta}^{-1}$ satisfy the same braid relations as $s_i, i \neq 0$ and s_0 . The relations (ii) and (iii) in the above Proposition can be used to define generators for the commutative lattice \mathcal{Q}_X .

The double affine Weyl group is not a Coxeter group, but a generalized Coxeter group (in the sense of Saito and Takebayashi, see [14]) and we can define the associated Artin group in the same way as for a Coxeter group (i.e. by keeping the generalized braid relations and forgetting the reflection relations). The equivalence between the two definitions has been recently established in [15].

DEFINITION 1.10. *The elliptic Artin group is the factor group of the double affine Artin group by the group generated by $X_{a_0^{-1}\delta}$.*

As before, let us give a more refined description of the double affine Artin group. It is based, as in the case of the affine Artin group, on the topological description of the double affine Artin group [5].

PROPOSITION 1.11. *The double affine Artin group $\mathcal{A}_{\tilde{W}}$ is generated by the affine Artin group and the lattice \mathcal{Q}_X such that the following relations are satisfied for $0 \leq i \leq n$ and $1 \leq j \leq n$*

- (i) *For any pair of indices (i, j) such that $2r_{ji} = -(\alpha_j, \alpha_i^\vee)$, with r_{ji} a non-negative integer we have*

$$T_i X_{\mu_j} = X_{\mu_j} T_i \tag{1.3}$$

where $\mu_j = \alpha_j + r_{ji}\alpha_i$; note that $(\mu_j, \alpha_i^\vee) = 0$,

- (ii) *For any pair of indices (i, j) such that $2r_{ji} + 1 = -(\alpha_j, \alpha_i^\vee)$, with r_{ji} a non-negative integer we have*

$$T_i X_{\mu_j} T_i = X_{s_i(\mu_j)} \tag{1.4}$$

where $\mu_j = \alpha_j + r_{ji}\alpha_i$; note that $(\mu_j, \alpha_i^\vee) = -1$,

(iii) $X_{\alpha_0^{-1}\delta}$ is central.

If we are in the $A_{2n}^{(2)}$ case the relations for the pair (i, n) are those obtained by considering $a_0^{-1}\alpha_n$ instead of α_n in the above formulas.

To define the Hecke algebras, we introduce a field \mathbb{F} (of parameters) as follows: fix indeterminates q and t_0, \dots, t_n such that $t_j = t_k$ if and only if $d_j = d_k$; let m be the lowest common denominator of the rational numbers $\{(\alpha_j, \lambda_k) \mid 1 \leq j, k \leq n\}$, and let \mathbb{F} denote the field of rational functions in $q^{\frac{1}{m}}$ and $t_j^{\frac{1}{2}}$. For the Cartan matrix $A_{2n}^{(2)}$ we need in fact more parameters: in this case two more indeterminates are added to our field t_{03} and t_{02} . To keep a uniform notation we use t_{01} to refer to t_0 in this case.

DEFINITION 1.12.

(i) The finite Hecke algebra $\mathcal{H}_{\tilde{W}}$ is the quotient of the group \mathbb{F} -algebra of the finite Artin group by the relations

$$T_j - T_j^{-1} = t_j^{\frac{1}{2}} - t_j^{-\frac{1}{2}}, \tag{1.5}$$

for $1 \leq j \leq n$.

(ii) The affine Hecke algebra \mathcal{H}_W is the quotient of the group \mathbb{F} -algebra of the affine Artin group by the relations

$$T_j - T_j^{-1} = t_j^{\frac{1}{2}} - t_j^{-\frac{1}{2}}, \tag{1.6}$$

for all $0 \leq j \leq n$.

Note here that by [5, Remark 1.7] the relation (1.6) for $j = 0$ needn't be imposed if the root system is reduced, since it is a consequence of the other relations. However it is absolutely necessary to impose it for $A_{2n}^{(2)}$. The definition of the double affine Hecke algebra makes use of this fact.

DEFINITION 1.13. The double affine Hecke algebra $\mathcal{H}_{\tilde{W}}$ is the quotient of the group \mathbb{F} -algebra of the double affine Artin group by the relations

$$T_j - T_j^{-1} = t_j^{\frac{1}{2}} - t_j^{-\frac{1}{2}}, \quad \text{for all } 1 \leq j \leq n, \tag{1.7}$$

and

$$X_\delta = q^{-1}. \tag{1.8}$$

If the root system is non-reduced the following relations are also imposed

$$T_0 - T_0^{-1} = t_{01}^{\frac{1}{2}} - t_{01}^{-\frac{1}{2}}, \tag{1.9}$$

$$X_{\alpha_0^{-1}\theta} T_{s_\theta}^{-1} - T_{s_\theta} X_{-\alpha_0^{-1}\theta} = t_{03}^{\frac{1}{2}} - t_{03}^{-\frac{1}{2}}, \tag{1.10}$$

$$T_0^{-1} X_{\alpha_0} - X_{-\alpha_0} T_0 = t_{02}^{\frac{1}{2}} - t_{02}^{-\frac{1}{2}}. \tag{1.11}$$

The above definition of the double affine Hecke algebra for $A_{2n}^{(2)}$ coincides indeed with the definition given by Sahi [13] after setting $t_{03} = u_n$, $t_{02} = u_0$ and making use of Lemma 4.1 in [13].

1.4. The modular group and the braid group on three letters. Let us consider the modular group $SL(2, \mathbb{Z})$ of two-by-two matrices with integer entries and determinant one. By B_3 we denote the braid group acting on three letters. With generators and relations it has the following description: B_3 is the group generated by \mathbf{a}, \mathbf{b} satisfying the relation

$$\mathbf{aba} = \mathbf{bab}. \tag{1.12}$$

Inside $SL(2, \mathbb{Z})$ consider the following elements

$$u_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad u_{21} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

LEMMA 1.14. *There exists a surjective morphism of groups*

$$\pi : B_3 \rightarrow SL(2, \mathbb{Z})$$

defined by $\pi(\mathbf{a}) = u_{12}$, $\pi(\mathbf{b}) = u_{21}$. The kernel of this morphism is the subgroup of the braid group spanned by \mathbf{c}^2 , with $\mathbf{c} = (\mathbf{aba})^2$ the generator of the center of B_3 .

PROOF. Indeed, a simple computation shows that u_{12} and u_{21} satisfy the braid relation and also $(u_{12}u_{21})^6 = I$, where I is the identity matrix. It is a straightforward check that imposing the above two relations on the two generators u_{12} and u_{21} constitutes an abstract description of the modular group. \square

2. Automorphisms of triple groups

2.1. A few remarks. We start by a few comments on the definition of the triple group. There are three copies of the affine Artin group imbedded inside the triple group and their interaction needs to be elucidated. Each of them contains a commutative subgroup isomorphic with the lattice \mathring{Q} , whose elements will be denoted by \mathcal{X}_μ^{0i} .

We shall see how slightly more complicated relations also hold. We approach first the case of a single affine bond. The following Lemma will be very useful.

LEMMA 2.1. *Assume that two elements p and q satisfy the single lace Coxeter relations with a third element x . Then the following are equivalent*

- (i) *The element qp satisfies the double lace Coxeter relation with x ;*
- (ii) *The element $p^{-1}qp$ satisfies the single lace Coxeter relation with x .*

Furthermore, if the above equivalent conditions also hold the element $(qp)^{-1}p(qp)$ satisfies the single lace Coxeter relation with x .

PROOF. To prove the equivalence let us note that

$$\begin{aligned} p^{-1}qp \cdot x \cdot p^{-1}qp &= p^{-1}qx^{-1}p \cdot x \cdot qp \\ &= p^{-1}qx^{-1}q^{-1} \cdot qp \cdot x \cdot qp \\ &= p^{-1}x^{-1}q^{-1} \cdot x \cdot qp \cdot x \cdot qp \end{aligned}$$

and also

$$\begin{aligned} x \cdot p^{-1}qp \cdot x &= xp^{-1}x^{-1} \cdot x \cdot qp \cdot x \\ &= p^{-1}x^{-1}p \cdot x \cdot qp \cdot x \\ &= p^{-1}x^{-1}q^{-1} \cdot qp \cdot x \cdot qp \cdot x \end{aligned}$$

In consequence the left hand sides are equal if and only if the right hand sides are equal. The equivalence is now clear.

Suppose now that the equivalent conditions also hold. Since

$$p \cdot p^{-1}qp = qp$$

and $p, p^{-1}qp$ satisfy the single lace Coxeter relation with x and qp satisfies the double lace Coxeter relation, by the first part of the Lemma we obtain that

$$(p^{-1}qp)^{-1}p(p^{-1}qp) = (qp)^{-1}p(qp)$$

satisfies the single lace Coxeter relation with x . □

An immediate consequence of this principle is the following

PROPOSITION 2.2. *If $l_0 = 1$ the elements*

$$\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02} \quad \text{and} \quad \mathcal{T}_{03}^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_{03}\mathcal{T}_{02}\mathcal{T}_{03}$$

satisfy the single lace Coxeter relation with $\mathcal{D}^k\mathcal{T}_\alpha\mathcal{D}^{-k}$, for any integer k .

PROOF. Denote $\mathcal{D}^k\mathcal{T}_\alpha\mathcal{D}^{-k}$ by x . For the first element we apply the second part of previous Lemma for $p = \mathcal{T}_{01}$ and $q = \mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_{01}^{-1}$ and for the second element we apply the Lemma for $p = \mathcal{T}_{03}$ and $q = \mathcal{T}_{02}$. The verification of the hypothesis is straightforward. □

Next we shift to the case of an affine double bond.

LEMMA 2.3. *If the affine node is connected to some other node with two laces then the elements \mathcal{T}_{0i} and \mathcal{T}_{0j} commute with $\mathcal{T}_\alpha\mathcal{T}_{0i}\mathcal{T}_{0j}\mathcal{T}_\alpha$ for any $1 \leq i < j \leq 3$.*

PROOF. We prove our claim only for \mathcal{T}_{02} .

Now, \mathcal{T}_{02} commutes with $\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_\alpha^{-1}$ (by (0.1)) and with $\mathcal{T}_\alpha\mathcal{T}_{02}\mathcal{T}_\alpha$ (by braid relations), therefore commutes with their product.

Hence, we have proved that

$$\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{02} = \mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha. \tag{2.1}$$

A similar argument shows that the rest of our statement is true. □

The following result uses these observations.

PROPOSITION 2.4. *If $l_0 = 2$ the elements*

$$\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02} \quad \text{and} \quad \mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03}.$$

satisfy with the \mathcal{T}_i ($i \neq 0$) the same braid relations as \mathcal{T}_{01} does.

PROOF. For the nodes not connected with the affine node the braid relations are clear, following directly from the ones with the \mathcal{T}_{0i} 's. The only thing to be checked is the double lace Coxeter relation with \mathcal{T}_α (recall that we denote by α the simple root corresponding to the node connected to the affine node).

$$\begin{aligned} \mathcal{T}_\alpha\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02} &= \mathcal{T}_\alpha\mathcal{T}_{02}^{-1}\mathcal{T}_\alpha^{-1}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02} \\ &= \mathcal{T}_\alpha\mathcal{T}_{02}^{-1}\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha \quad \text{by (2.1)} \\ &= \mathcal{T}_{02}^{-1}\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha \\ &= \mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha \quad \text{by (2.1)}. \end{aligned}$$

The result for $\mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03}$ follows along the same lines. □

Similar things happen with respect to relation (0.1).

PROPOSITION 2.5. *If $l_0 = 2$, the relation (0.1) holds also for the following pairs of elements*

$$(\mathcal{T}_{02}, \mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}), (\mathcal{T}_{01}, \mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03}), (\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}, \mathcal{T}_{03}) \text{ and } (\mathcal{T}_{03}, \mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03})$$

PROOF. As before we show this only for one of the above pairs. The argument can be easily repeated to explain the result for any of the remaining pairs.

The element \mathcal{T}_{02} commutes with $\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_\alpha^{-1}$ (by the braid relations) and with $\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha$ (by (2.1)). Therefore it commutes with their product. We have showed that

$$\mathcal{T}_{02}\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha = \mathcal{T}_\alpha^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{02}$$

which is precisely the desired relation. □

2.2. An action of the braid group on three letters. Next we will explain how the braid group on three letters act on the triple group as automorphisms which fix all the generators corresponding to the non-affine nodes. Before defining this action let us construct an *anti-automorphism* which satisfies the same restrictions. Of course we only need to define it only on \mathcal{T}_{01} , \mathcal{T}_{03} and \mathcal{T}_{02} . Let ϵ be the anti-involution which fixes \mathcal{T}_{02} and interchanges \mathcal{T}_{01} and \mathcal{T}_{03} . This is of course possible since the defining relations are invariant under such a change. It clear that this is an involution, its square being the identity morphism.

Let us define next two special elements morphisms of the triple group. The first one will be called $\underline{\mathfrak{a}}$ and it will fix \mathcal{T}_{03} and it will send \mathcal{T}_{01} to \mathcal{T}_{02} and \mathcal{T}_{02} to $\mathcal{T}_{02}^{-1}\mathcal{T}_{01}\mathcal{T}_{02}$, and the second one will be called $\underline{\mathfrak{b}}$ and it will fix \mathcal{T}_{01} and it will send \mathcal{T}_{02} to \mathcal{T}_{03} and \mathcal{T}_{03} to $\mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03}$.

THEOREM 2.6. *The above maps define indeed automorphisms of the triple group. Moreover, the map sending \mathfrak{a} and \mathfrak{b} to $\underline{\mathfrak{a}}$ and $\underline{\mathfrak{b}}$, respectively, defines a group morphism*

$$\Upsilon : B_3 \rightarrow \text{Aut}(\mathcal{A}).$$

PROOF. Let us first argue that $\underline{\mathfrak{a}}$ defines indeed a endomorphism of the double affine Artin group. Indeed, by Proposition 2.2 and Proposition 2.4 the images of the generators \mathcal{T}_{0i} satisfy indeed the braid relations.

In the case of a single affine bond we have to check that the images of $\mathcal{T}_{03}^{-1}\mathcal{T}_{02}\mathcal{T}_{03}$ satisfy the required braid relations. The image by $\underline{\mathfrak{a}}$ is $\mathcal{D}^{-1}\mathcal{T}_{01}\mathcal{D}$ and there is nothing to prove and the image by $\underline{\mathfrak{b}}$ is $\mathcal{T}_{03}^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_{03}\mathcal{T}_{02}\mathcal{T}_{03}$ and the check it was done in Proposition 2.2.

In the case of the double affine bond we need to also check that the images of the relations (0.1) hold. This was proved in Proposition 2.5. Therefore $\underline{\mathfrak{a}}$ and $\underline{\mathfrak{b}}$ are indeed endomorphisms of the triple group.

They are indeed isomorphisms since a simple check shows that

$$\underline{\mathfrak{a}}\underline{\mathfrak{b}} = \underline{\mathfrak{b}}^{-1}.$$

As for the second part of our statement, this is again a straightforward check. □

We will later prove that this morphism is injective.

3. The triple group and the double affine Artin group

3.1. A refinement of Cherednik’s presentation. We start by analyzing the presentation of the double affine Artin group in Proposition 1.11. We first focus on the relations (1.3) and (1.4) for the pairs of type $(0, j)$.

Type I: These are relations associated to $1 \leq j \leq n$ such that $2r_{j0} = -(\alpha_j, \alpha_0^\vee) = (\alpha_j, \theta)$, with r_{j0} a non-negative integer. For such a j if we set $\mu_j = \alpha_j + r_{j0}\alpha_0$ the following relation holds

$$T_0 X_{\mu_j} = X_{\mu_j} T_0.$$

Since $X_{a_0^{-1}\delta}$ is central, we can certainly replace μ_j in the above relation by $\alpha_j - r_{j0}a_0^{-1}\theta$. The only case in which the scalar product (α_j, θ) is even and nonzero is for $A = B_n^{(2)}, A_{2n}^{(2)}$ when it could be equal to 2. Therefore, the relations of this type are

$$T_0 X_{\alpha_j} = X_{\alpha_j} T_0 \quad \text{if } (\alpha_j, \theta) = 0 \tag{3.1}$$

$$T_0 X_{\alpha_j - a_0^{-1}\theta} = X_{\alpha_j - a_0^{-1}\theta} T_0 \quad \text{if } (\alpha_j, \theta) = 2. \tag{3.2}$$

Note the the relation (3.2) is present only if $A = B_n^{(2)}, A_{2n}^{(2)}$.

Type II: These are relations associated to $1 \leq j \leq n$ such that $2r_{j0} + 1 = -(\alpha_j, \alpha_0^\vee) = (\alpha_j, \theta)$, with r_{j0} a non-negative integer. For such a j if we set $\mu_j = \alpha_j + r_{j0}\alpha_0$ the following relation holds

$$T_i X_{\mu_j} T_i = X_{s_i(\mu_j)}$$

The only odd value the scalar product (α_j, θ) could take is 1, and this happens only if $A \neq B_n^{(2)}, A_{2n}^{(2)}$. In the two excepted cases the scalar product takes only even values. Therefore, the relations of this type are

$$T_0 X_{\alpha_j} T_0 = X_{\alpha_j + \alpha_0} \quad \text{if } (\alpha_j, \theta) = 1. \tag{3.3}$$

Note the the relation (3.3) is not present if $A = B_n^{(2)}, A_{2n}^{(2)}$.

PROPOSITION 3.1. *Assuming the notation above, we can reduce the number of relations in the presentation of the double affine Artin group by keeping from all relations of type I and II described above only the following one:*

- (i) $T_0 X_\alpha T_0 = X_{\alpha + \alpha_0}$ if $l_0 = 1$,
- (ii) $T_0 X_{\alpha - a_0^{-1}\theta} = X_{\alpha - a_0^{-1}\theta} T_0$ if $l_0 = 2$.

PROOF. Assume that β and γ are simple roots whose nodes in the Dynkin diagram are connected, but none of them is connected to the node of α_0 . We also assume that β is shorter than γ or they have the same length. This will imply that $(\beta, \gamma^\vee) = -1$ always and consequently

$$s_\gamma(\beta) = \beta + \gamma.$$

From equation (1.4) we know that

$$T_\gamma X_\beta T_\gamma = X_{\alpha + \beta}$$

or equivalently $X_\gamma = T_\gamma X_\beta T_\gamma X_{-\beta}$. From this expression it is clear that if we know that T_0 satisfies the braid relations and it commutes with X_β it will follow that T_0 commutes with X_γ .

Let $l_0 = 1$ and α_j a simple root for which $(\alpha_j, \theta) = 0$ (in other words the nodes of α_0 and α_j are not connected in the Dynkin diagram). Using the above remark several times if necessary we see that the relation (3.1) is implied by the knowledge of the braid relations and of the commutation of T_0 with X_β , where β

is any neighbor of α . The commutation of T_0 and X_β holds indeed since α is short (remember that $l_0 = 1$) and therefore as explained above

$$X_\beta = T_\beta X_\alpha T_\beta X_{-\alpha}.$$

Now,

$$\begin{aligned} T_0 X_\beta T_0^{-1} &= T_0 T_\beta X_\alpha T_\beta X_{-\alpha} T_0^{-1} \\ &= T_\beta T_0 X_\alpha T_0 T_\beta T_0^{-1} X_{-\alpha} T_0^{-1} && \text{by the braid relations for } T_0 \text{ and } T_\beta \\ &= T_\beta X_{\alpha+\alpha_0} T_\beta X_{-\alpha-\alpha_0} && \text{by the hypothesis} \\ &= T_\beta X_\alpha T_\beta X_{-\alpha} && \text{by the commutation of } T_\beta \text{ and } X_{\alpha_0} \\ &= X_\beta. \end{aligned}$$

This computation shows that if we impose on T_0 the relation stated in the hypothesis (besides the braid relations) then all relations of type I automatically hold. In the case $A_n^{(1)}$ there are two type II relations: one which we have by hypothesis and another one, associated to the second neighbor (let us call it α') of the affine simple root in the Dynkin diagram. A straightforward computation, which exploits the fact that T_0 commutes with $X_{\alpha+\alpha'+\alpha_0}$, (fact which is a consequence of the commuting relations proved above) will show that the type II relation for α' holds. For completeness, let us explain the details:

$$\begin{aligned} T_0 X_{\alpha'} T_0 &= T_0 X_{-\alpha-\alpha_0} X_{\alpha+\alpha'+\alpha_0} T_0 \\ &= T_0 X_{-\alpha-\alpha_0} T_0 X_{\alpha+\alpha'+\alpha_0} && \text{by the commuting relations} \\ &= X_{-\alpha} X_{\alpha+\alpha'+\alpha_0} && \text{by the hypothesis} \\ &= X_{\alpha'+\alpha_0}. \end{aligned}$$

The proof of our result in the case $l_0 = 1$ is now completed. The case $l_0 = 2$ is treated completely similarly. \square

3.2. Some relations. We present here some relations which hold inside the double affine Artin group. The relations will be useful later.

PROPOSITION 3.2. *The elements T_i ($i \neq 0$) and $T_0^{-1} X_{\alpha_0}$ satisfy inside the double affine Artin group the same braid relations as T_i and T_0 .*

PROOF. The claim is obvious if $(\theta, \alpha_i) = 0$, or equivalently if the nodes in question are not connected by laces in the Dynkin diagram. The other possible values for the scalar product are $(\theta, \alpha_i) = 1$ (if there is a single lace connecting the α_0 and α_i ; consequently α_i is short) and $(\theta, \alpha_i) = 2$ (if there are two laces connecting the α_0 and α_i ; consequently α_i is not short). We will consider them separately. In what follows we use our convention to denote by α the simple root whose node is connected to the affine node in the Dynkin diagram.

First, if $(\theta, \alpha) = 1$ by Definition 1.8 we know that

$$X_{\alpha_0} T_\alpha = T_\alpha^{-1} X_{\alpha_0+\alpha} \quad \text{and} \tag{3.4}$$

$$X_{\alpha+\alpha_0} T_0^{-1} = T_0 X_\alpha. \tag{3.5}$$

Now,

$$\begin{aligned}
 T_0^{-1}X_{\alpha_0}T_\alpha T_0^{-1}X_{\alpha_0} &= T_0^{-1}T_\alpha^{-1}X_{\alpha_0+\alpha}T_0^{-1}X_{\alpha_0} && \text{by (3.4)} \\
 &= T_0^{-1}T_\alpha^{-1}T_0X_\alpha X_{\alpha_0} && \text{by (3.5)} \\
 &= T_\alpha T_0^{-1}T_\alpha^{-1}X_{\alpha_0+\alpha} && \text{by the braid relation} \\
 &= T_\alpha T_0^{-1}X_{\alpha_0}T_\alpha && \text{by (3.4)}.
 \end{aligned}$$

We proved the desired braid relation (recall that in this case the relevant nodes in the Dynkin diagram are connected by a single lace):

$$T_0^{-1}X_{\alpha_0}T_\alpha T_0^{-1}X_{\alpha_0} = T_\alpha T_0^{-1}X_{\alpha_0}T_\alpha.$$

Second, if $(\theta, \alpha) = 2$ from Definition 1.8 we know that

$$X_{\alpha_0}T_\alpha = T_\alpha^{-1}X_{\alpha_0+\alpha} \tag{3.6}$$

$$X_{\alpha+\alpha_0}T_0^{-1} = T_0^{-1}X_{\alpha+\alpha_0} \tag{3.7}$$

$$X_{\alpha+2\alpha_0}T_\alpha = T_\alpha X_{\alpha+2\alpha_0}. \tag{3.8}$$

In the same manner,

$$\begin{aligned}
 T_0^{-1}X_{\alpha_0}T_\alpha T_0^{-1}X_{\alpha_0}T_\alpha &= T_0^{-1}T_\alpha^{-1}X_{\alpha_0+\alpha}T_0^{-1}X_{\alpha_0}T_\alpha && \text{by (3.6)} \\
 &= T_0^{-1}T_\alpha^{-1}T_0^{-1}X_{\alpha_0+\alpha}X_{\alpha_0}T_\alpha && \text{by (3.7)} \\
 &= T_0^{-1}T_\alpha^{-1}T_0^{-1}T_\alpha X_{2\alpha_0+\alpha} && \text{by (3.8)} \\
 &= T_\alpha T_0^{-1}T_\alpha^{-1}T_0^{-1}X_{2\alpha_0+\alpha} && \text{by the braid relation} \\
 &= T_\alpha T_0^{-1}T_\alpha^{-1}X_{\alpha_0+\alpha}T_0^{-1}X_{\alpha_0} && \text{by (3.7)} \\
 &= T_\alpha T_0^{-1}X_{\alpha_0}T_\alpha T_0^{-1}X_{\alpha_0} && \text{by (3.6)}.
 \end{aligned}$$

We proved the desired braid relation (recall that in this case the relevant nodes in the Dynkin diagram are connected by two laces):

$$T_0^{-1}X_{\alpha_0}T_\alpha T_0^{-1}X_{\alpha_0}T_\alpha = T_\alpha T_0^{-1}X_{\alpha_0}T_\alpha T_0^{-1}X_{\alpha_0}.$$

The proof is completed. □

If $l_0 = 2$ another important relation holds.

LEMMA 3.3. *With the notation above, if the affine node is connected by a double lace with the node corresponding to the simple root α , the elements T_0 and $T_\alpha^{-1}T_0^{-1}X_{\alpha_0}T_\alpha$ commute inside the double affine Artin group.*

PROOF. Indeed,

$$\begin{aligned}
 T_0T_\alpha^{-1}T_0^{-1}X_{\alpha_0}T_\alpha &= T_0T_\alpha^{-1}T_0^{-1}T_\alpha^{-1}X_{\alpha_0+\alpha} && \text{by (3.6)} \\
 &= T_\alpha^{-1}T_0^{-1}T_\alpha^{-1}T_0X_{\alpha_0+\alpha} && \text{by the braid relation} \\
 &= T_\alpha^{-1}T_0^{-1}T_\alpha^{-1}X_{\alpha_0+\alpha}T_0 && \text{by (3.7)} \\
 &= T_\alpha^{-1}T_0^{-1}X_{\alpha_0}T_\alpha T_0 && \text{by (3.6)}
 \end{aligned}$$

□

3.3. A quotient of the triple group. The following element will play a central role. If, with the usual notation, $s_\theta = s_{j_1} \cdots s_{j_m}$ is a reduced decomposition of the reflection s_θ in terms of simple reflections, we denote by \mathcal{T}_{s_θ} the product $\mathcal{T}_{j_1} \cdots \mathcal{T}_{j_m}$. Since we imposed the braid relations the definition of \mathcal{T}_θ will not depend on the reduced decomposition chosen. Define

$$C := \mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_{03}\mathcal{T}_{s_\theta}.$$

Consider next the following quotient of the triple group.

DEFINITION 3.4. *Let A be an irreducible affine Cartan matrix subject to our restriction and $S(A)$ its Dynkin diagram. The group \tilde{A} is the quotient of the triple group A by the following relations*

$$C := \mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_{03}\mathcal{T}_{s_\theta} \text{ is central} \tag{3.9}$$

From now on we will consider only the group \tilde{A} , therefore no confusion will arise if we denote the images of the triple group elements by the quotient map by the same symbols.

Let us collect a few facts which immediately follow from the above definition.

LEMMA 3.5. *Using the notation above, if $l_0 = 1$, \mathcal{T}_{02} could be expressed in terms of the other generators.*

PROOF. Let us start with an immediate consequence of (3.9). Since from a straightforward check

$$\mathcal{T}_{02} = \mathcal{T}_\alpha \mathcal{T}_{01}^{-1} \mathcal{D} \mathcal{T}_{03}^{-1} \mathcal{T}_\alpha \mathcal{T}_{03} \mathcal{D}^{-1} \mathcal{T}_{01} \mathcal{T}_\alpha^{-1},$$

using the centrality of C we get

$$\mathcal{T}_{02} = \mathcal{T}_\alpha \mathcal{T}_{01}^{-1} C^{-1} \mathcal{D} \mathcal{T}_{03}^{-1} \mathcal{T}_\alpha \mathcal{T}_{03} C \mathcal{D}^{-1} \mathcal{T}_{01} \mathcal{T}_\alpha^{-1},$$

and using the relation (3.9) we obtain the following formula

$$\mathcal{T}_{02} = \mathcal{T}_{01}^{-1} \mathcal{T}_\alpha^{-1} \mathcal{T}_{01} \mathcal{T}_\alpha \mathcal{T}_{s_\theta}^{-1} \mathcal{T}_{03}^{-1} \mathcal{T}_\alpha \mathcal{T}_{03} \mathcal{T}_{s_\theta} \mathcal{T}_{01} \mathcal{T}_\alpha^{-1}, \tag{3.10}$$

which shows that indeed \mathcal{T}_{02} is expressible in terms of the other generators. □

LEMMA 3.6. *In defining the group \tilde{A} , if $l_0 = 1$, the relations in Definition 1 b) are superfluous.*

PROOF. Because $\mathcal{D}^{-k} \mathcal{T}_\alpha \mathcal{D}^k = \mathcal{T}_{s_\theta}^k \mathcal{T}_\alpha \mathcal{T}_{s_\theta}^{-k}$ the result follows from Proposition 1.6. Also, by the same principle, it is enough to prove that the simple lace Coxeter relations between $\mathcal{T}_{03}^{-1} \mathcal{T}_{02} \mathcal{T}_{03}$ and \mathcal{T}_α hold. By the Lemma 2.1 this is equivalent with the double lace Coxeter relations between $\mathcal{T}_{02} \mathcal{T}_{03}$ and \mathcal{T}_α .

Let us consider the relation

$$\mathcal{T}_{02} \mathcal{T}_{03} \mathcal{T}_\alpha \mathcal{T}_{02} \mathcal{T}_{03} \mathcal{T}_\alpha = \mathcal{T}_\alpha \mathcal{T}_{02} \mathcal{T}_{03} \mathcal{T}_\alpha \mathcal{T}_{02} \mathcal{T}_{03}.$$

It essentially says that $\mathcal{T}_{02} \mathcal{T}_{03}$ commutes with $\mathcal{T}_\alpha \mathcal{T}_{02} \mathcal{T}_{03} \mathcal{T}_\alpha$. But since

$$\mathcal{T}_{02} \mathcal{T}_{03} = C \mathcal{T}_{01}^{-1} \mathcal{T}_{s_\theta}^{-1}$$

and C is central this is equivalent to the fact that $\mathcal{T}_{01}^{-1} \mathcal{T}_{s_\theta}^{-1}$ and $\mathcal{T}_\alpha \mathcal{T}_{01}^{-1} \mathcal{T}_{s_\theta}^{-1} \mathcal{T}_\alpha$ commute. This is always true since inside the affine Artin group generated by \mathcal{T}_{01} and $\mathcal{T}_i, i \neq 0$ since it is equivalent, using for example the relations from Proposition 1.4 and Remark 1.5, with the fact that $\mathcal{X}_{\alpha_0^{-1}\theta}^{01}$ and $\mathcal{X}_{-\alpha+\alpha_0^{-1}\theta}^{01}$ commute. The proof is completed. □

LEMMA 3.7. *In the above definition, if $l_0 = 2$, only one of the relations (0.1) should be imposed.*

PROOF. We will prove that if we impose only one relation in (0.1), say

$$\mathcal{T}_{01}\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha = \mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{01} \tag{3.11}$$

the other two easily following from this one and the fact that \mathcal{C} is central. We will illustrate this briefly.

The above relation says that \mathcal{T}_{01} and $\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_\alpha$ commute. Since \mathcal{T}_{01} also commutes with \mathcal{C} , $\mathcal{T}_\alpha^{-1}\mathcal{T}_{01}^{-1}\mathcal{T}_\alpha^{-1}$ (this is just the braid relation) and $\mathcal{T}_\alpha\mathcal{T}_{s_\theta}^{-1}\mathcal{T}_\alpha$ (this is a relation which could be checked easily), it follows that \mathcal{T}_{01} commutes with their product, which is

$$\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}^{-1}\mathcal{T}_\alpha\mathcal{T}_\alpha^{-1}\mathcal{T}_{01}^{-1}\mathcal{T}_\alpha^{-1}\mathcal{T}_\alpha\mathcal{T}_{s_\theta}^{-1}\mathcal{T}_\alpha\mathcal{C} = \mathcal{T}_\alpha^{-1}\mathcal{T}_{03}\mathcal{T}_\alpha.$$

We have just proved that

$$\mathcal{T}_{01}\mathcal{T}_\alpha^{-1}\mathcal{T}_{03}\mathcal{T}_\alpha = \mathcal{T}_\alpha^{-1}\mathcal{T}_{03}\mathcal{T}_\alpha\mathcal{T}_{01}.$$

The argument is the same if we choose to keep any other relation. □

3.4. The main result. Our main goal is to prove that the double affine Artin group $\mathcal{A}_{\tilde{W}}$ is isomorphic to the group $\tilde{\mathcal{A}}$. Let us define the candidates for isomorphisms between the two groups. Let the map

$$\phi : \tilde{\mathcal{A}} \rightarrow \mathcal{A}_{\tilde{W}}$$

be defined as the extension to a group morphism of the map

$$\phi(\mathcal{T}_i) = T_i \text{ (for } i \neq 0), \quad \phi(\mathcal{T}_{01}) = T_0, \quad \phi(\mathcal{T}_{03}) = X_{\alpha_0^{-1}\theta}T_{s_\theta}^{-1}, \quad \phi(\mathcal{T}_{02}) = T_0^{-1}X_{\alpha_0}.$$

The next Proposition shows that this could indeed be done.

PROPOSITION 3.8. *The map $\phi : \tilde{\mathcal{A}} \rightarrow \mathcal{A}_{\tilde{W}}$ is well defined.*

PROOF. From Remark 1.9 and Proposition 3.2 it follows that the images of the generators of the group $\tilde{\mathcal{A}}$ satisfy the required braid relations inside the double affine Artin group $\mathcal{A}_{\tilde{W}}$. Also,

$$\begin{aligned} \phi(\mathcal{C}) &= \phi(\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_{03}\mathcal{T}_{s_\theta}) \\ &= T_0T_0^{-1}X_{\alpha_0}X_{\alpha_0^{-1}\theta}T_{s_\theta}^{-1}T_{s_\theta} \\ &= X_{\alpha_0^{-1}\delta} \end{aligned}$$

which is central in the double affine Artin group. The only thing which needs explanation is the fact that the image of relation (0.1) holds if $l_0 = 2$. In fact, as it follows from Lemma 3.7, we need to do this only for one relation, say

$$\mathcal{T}_{01}\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha = \mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{01}. \tag{3.12}$$

Indeed,

$$\begin{aligned} \phi(\mathcal{T}_{01}\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha) &= T_0T_\alpha^{-1}T_0^{-1}X_{\alpha_0}T_\alpha \\ &= T_0T_\alpha^{-1}T_0^{-1}T_\alpha^{-1}X_{\alpha_0+\alpha} && \text{by (3.6)} \\ &= T_\alpha^{-1}T_0^{-1}T_\alpha^{-1}T_0X_{\alpha_0+\alpha} && \text{by the braid relation} \\ &= T_\alpha^{-1}T_0^{-1}T_\alpha^{-1}X_{\alpha_0+\alpha}T_0 && \text{by (3.7)} \\ &= T_\alpha^{-1}T_0^{-1}X_{\alpha_0}T_\alpha T_0 && \text{by (3.6)} \\ &= \phi(\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{01}). \end{aligned}$$

The proof is completed. □

Let the map

$$\psi : \mathcal{A}_{\tilde{W}} \rightarrow \tilde{\mathcal{A}}$$

be defined as the extension to a group morphism of the map

$$\psi(T_i) = T_i \text{ (for } i \neq 0), \quad \psi(T_0) = \mathcal{T}_{01}, \quad \psi(X_{a_0^{-1}\theta}) = \mathcal{T}_{03}\mathcal{T}_{s_\theta}, \quad \psi(X_{a_0^{-1}\delta}) = \mathcal{C}.$$

PROPOSITION 3.9. *The map $\psi : \mathcal{A}_{\tilde{W}} \rightarrow \tilde{\mathcal{A}}$ is well defined.*

PROOF. As noted in Remark 1.9, the elements for which we defined ψ are enough to generate the double affine Artin group. This has the advantage of reducing the number of relations for us to check to the images by ψ of the following

- (1) braid relations for T_i ($i \neq 0$) and T_0 ;
- (2) braid relations for T_i ($i \neq 0$) and $X_{a_0^{-1}\theta}T_{s_\theta}^{-1}$;
- (3) $X_{a_0^{-1}\delta}$ is central;
- (4) relations from Proposition 3.1.

The only nontrivial check will be to prove that the images of the relations stated in Proposition 3.1 hold inside the group $\tilde{\mathcal{A}}$. Let us consider first the case when $l_0 = 1$. We have to prove that

$$\mathcal{T}_{01}\psi(X_\alpha)\mathcal{T}_{01} = \psi(X_{\alpha_0+\alpha}).$$

Since $T_\alpha X_{-\theta}T_\alpha = X_{\alpha-\theta}$ we know that

$$X_\alpha = T_\alpha X_{-\theta}T_\alpha X_\theta.$$

Therefore we want that

$$\mathcal{T}_{01}\mathcal{T}_\alpha\mathcal{T}_{s_\theta}^{-1}\mathcal{T}_{03}^{-1}\mathcal{T}_\alpha\mathcal{T}_{03}\mathcal{T}_{s_\theta}\mathcal{T}_{01} = \mathcal{C}\mathcal{T}_\alpha\mathcal{T}_{s_\theta}^{-1}\mathcal{T}_{03}^{-1}\mathcal{T}_\alpha$$

or equivalently

$$\mathcal{T}_{s_\theta}\mathcal{T}_\alpha^{-1}\mathcal{T}_{01}\mathcal{T}_\alpha\mathcal{T}_{s_\theta}^{-1}\mathcal{T}_{03}^{-1}\mathcal{T}_\alpha\mathcal{T}_{03}\mathcal{T}_{s_\theta}\mathcal{T}_{01}\mathcal{T}_\alpha^{-1}\mathcal{T}_{03} = \mathcal{C}.$$

This immediately follows from equation (3.10).

In the case $l_0 = 2$ we have to prove that

$$\mathcal{T}_{01}\psi(X_{\alpha-a_0^{-1}\theta}) = \psi(X_{\alpha-a_0^{-1}\theta})\mathcal{T}_{01}.$$

As before,

$$X_{\alpha-a_0^{-1}\theta} = T_\alpha X_{-a_0^{-1}\theta}T_\alpha,$$

hence our statement is proved as follows

$$\begin{aligned} \mathcal{T}_{01}\psi(X_{\alpha-a_0^{-1}\theta}) &= \mathcal{T}_{01}\mathcal{T}_\alpha\mathcal{T}_{s_\theta}^{-1}\mathcal{T}_{03}^{-1}\mathcal{T}_\alpha \\ &= \mathcal{C}^{-1}\mathcal{T}_{01}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha && \text{by (3.9)} \\ &= \mathcal{C}^{-1}\mathcal{T}_{01}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_\alpha\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha \\ &= \mathcal{C}^{-1}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha && \text{by the braid relations} \\ &= \mathcal{C}^{-1}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_\alpha\mathcal{T}_\alpha^{-1}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{01} && \text{by (0.1)} \\ &= \mathcal{C}^{-1}\mathcal{T}_\alpha\mathcal{T}_{01}\mathcal{T}_{02}\mathcal{T}_\alpha\mathcal{T}_{01} \\ &= \mathcal{T}_\alpha\mathcal{T}_{s_\theta}^{-1}\mathcal{T}_{03}^{-1}\mathcal{T}_\alpha\mathcal{T}_{01} && \text{by (3.9)} \\ &= \psi(X_{\alpha-a_0^{-1}\theta})\mathcal{T}_{01}. \end{aligned}$$

The proof of the Proposition is now complete. □

Our main result is the following.

THEOREM 3.10. *The groups $\mathcal{A}_{\tilde{W}}$ and $\tilde{\mathcal{A}}$ are isomorphic.*

PROOF. The morphisms constructed in Proposition 3.8 and Proposition 3.9 are inverses for each other, as we could easily check this on generators. \square

In other words, the definition of the group $\tilde{\mathcal{A}}$ could serve as a definition for the double affine Artin group. We will state this explicitly, by keeping only the non-redundant relations.

THEOREM 3.11. *Let A be an irreducible affine Cartan matrix subject to our restriction and $S(A)$ its Dynkin diagram. The double affine Artin group $\mathcal{A}_{\tilde{W}}$ is given by generators and relations as follows:*

Generators: one generator T_i for each node, with the exception of the affine node for which we have three generators T_{01}, T_{02} and T_{03} .

Relations: a) Braid relations for each pair of generators associated to any pair of distinct nodes (note that there are three generators associated to the affine node).

b) If there are double laces connecting the affine node with the node α (i.e. $l_0 = 2$) the following relation also holds

$$T_{01}T_{\alpha}^{-1}T_{03}T_{\alpha} = T_{\alpha}^{-1}T_{03}T_{\alpha}T_{01} \tag{3.13}$$

c) The element

$$X_{a_0^{-1}\delta} := T_{01}T_{02}T_{03}T_{s_{\theta}} \quad \text{is central} \tag{3.14}$$

The elliptic Artin group has the same description except for the last relation which is replaced by

$$X_{a_0^{-1}\delta} = 1 \tag{3.15}$$

By Definition 1.13 we obtain a new description of the double affine Hecke algebra. Note that the elements appearing in (1.10) and (1.11) are T_{03} and T_{02} , respectively.

The same result at the level of Weyl groups is also of interest.

THEOREM 3.12. *Let A be an irreducible affine Cartan matrix subject to our restriction and $S(A)$ its Dynkin diagram. The double affine Weyl group \tilde{W} is given by generators and relations as follows:*

Generators: one generator s_i for each node, with the exception of the affine node for which we have three generators s_{01}, s_{02} and s_{03} .

Relations: a) Braid relations for each pair of generators associated to any pair of distinct nodes (note that there are three generators associated to the affine node).

b) If there are double laces connecting the affine node with the node α (i.e. $l_0 = 2$) the following relation also holds

$$s_{01}s_{\alpha}s_{03}s_{\alpha} = s_{\alpha}s_{03}s_{\alpha}s_{01}. \tag{3.16}$$

c) All generators have order two.

d) If, with the usual notation, the following relation holds

$$\tau_{a_0^{-1}\delta} := s_{01}s_{02}s_{03}s_{\theta} \quad \text{is central} \tag{3.17}$$

The elliptic Weyl group has the same description except for the last relation which is replaced by

$$\tau_{a_0^{-1}\delta} = 1 \tag{3.18}$$

4. Automorphisms of double affine Artin groups

4.1. A reflection representation for the double affine Weyl group.

The fact that the generators of the double affine Weyl group in Theorem 3.12 have order two suggests that it may have a faithful representation in which the generators will act as reflections. We construct here such a representation.

Assume we have fixed an irreducible affine Cartan matrix A of rank n satisfying our assumptions. Let us define the following $n + 4$ -dimensional real vector space

$$V := \mathfrak{h}^* \oplus \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2 \oplus \mathbb{R}\Lambda_1 \oplus \mathbb{R}\Lambda_2$$

together with the nondegenerate bilinear form (\cdot, \cdot) which extends the natural scalar product on \mathfrak{h}

$$\begin{aligned} (\delta_i, \Lambda_j) &= \delta_{ij}, \quad i, j = 1, 2 \\ (\delta_i, \alpha_k) &= 0, \quad i = 1, 2, \quad k = 1 \cdots n \\ (\Lambda_i, \alpha_k) &= 0, \quad i = 1, 2, \quad k = 1 \cdots n. \end{aligned}$$

By r we will denote the maximum number of laces in the Dynkin diagram of A and by $\mathring{R}_s, \mathring{R}_l$ the short, respectively long, roots in \mathring{R} . The vector space V contains the subset \tilde{R} defined as

$$\tilde{R} = (\mathring{R}_s + \mathbb{Z}\delta_1 + \mathbb{Z}\delta_2) \cup (\mathring{R}_l + r\mathbb{Z}\delta_1 + r\mathbb{Z}\delta_2), \quad \text{if } A \neq A_{2n}^{(2)},$$

$$\tilde{R} = (\mathring{R}_s + \mathbb{Z}\delta_1 + \mathbb{Z}\delta_2) \cup (\mathring{R}_l + r\mathbb{Z}\delta_1 + r\mathbb{Z}\delta_2) \cup \left(\frac{1}{2}\mathring{R}_l + \mathbb{Z}\delta_1 + \mathbb{Z}\delta_2 + \frac{1}{2}(\delta_1 + \delta_2)\right), \quad \text{for } A_{2n}^{(2)}.$$

For each $\tilde{\alpha} \in \tilde{R}$ and $v \in V$ define

$$s_{\tilde{\alpha}}(v) = v - \frac{2(v, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})} \tilde{\alpha}.$$

The $s_{\tilde{\alpha}}$ are reflections of V . We consider an action of the double affine Weyl group \tilde{W} on the vector space V by letting the simple reflections s_i acting as s_{α_i} , and s_{01}, s_{03} and s_{02} acting as $s_{a_0^{-1}(\delta_1 - \theta)}, s_{a_0^{-1}(\delta_2 - \theta)}$ and $s_{a_0^{-1}(\delta_1 + \delta_2 - \theta)}$, respectively. Also, the element $\tau_{a_0^{-1}\delta}$ acts as

$$\tau_{a_0^{-1}\delta}(x) := x + (x, \delta_2)a_0^{-1}\delta_1 - (x, \delta_1)a_0^{-1}\delta_2.$$

PROPOSITION 4.1. *The above action of the generators of the double affine Weyl group extend to a faithful representation*

$$\rho : \tilde{W} \rightarrow \text{GL}(V).$$

PROOF. All the relations in the Theorem 3.12 are easily verified. Hence, we defined indeed a representation of the double affine Weyl group. To check that it is faithful we use Proposition 1.1 which has as a consequence the fact that any element of \tilde{W} can be put in the form $\psi\lambda_\mu\tau_\beta$ and the following explicit actions of λ_μ and τ_β :

$$\begin{aligned} \lambda_\mu(x) &= x - (x, \mu)\delta_1 + (x, \delta_1)\left(\mu - \frac{1}{2}|\mu|^2\delta_1\right) \\ \tau_\beta(x) &= x - (x, \beta)\delta_2 + (x, \delta_2)\left(\beta - \frac{1}{2}|\beta|^2\delta_2\right). \end{aligned}$$

The verification is straightforward. □

From now on we will not distinguish between the double affine Weyl group and its image inside $GL(V)$ via the above representation.

Note the vector space $V_{(0,0)} := \mathfrak{h}^* \oplus \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2$ is invariant under the action of \tilde{W} and that the element $\tau_{a_0^{-1}\delta}$ acts trivially on it. We immediately obtain that the restriction to $V_{(0,0)}$ gives a faithful representation of the elliptic Weyl group.

LEMMA 4.2. *All the reflections $s_{\tilde{\alpha}}$ for $\alpha \in \tilde{R}$ belong to the double affine Weyl group.*

PROOF. This is a simple check, using the fact that

$$ws_{\tilde{\alpha}}w^{-1} = s_{w(\tilde{\alpha})}$$

for any $w \in \tilde{W}$ and $\tilde{\alpha} \in \tilde{R}$, and the fact that $s_{a_0^{-1}(\delta_1-\theta)}$ and $s_{a_0^{-1}(\delta_2-\theta)}$ belong to the double affine Weyl group. \square

We consider next the following subgroup of the full automorphisms group of \tilde{W}

$$\mathfrak{G}(\tilde{W}) := \{f \in Aut(\tilde{W}) \mid f(s_i) = s_i, i \neq 0, \text{ and } f(\tau_{a_0^{-1}\delta}) = \tau_{a_0^{-1}\delta}\}.$$

Since the generators $s_{a_0^{-1}(\delta_1-\theta)}$ and $s_{a_0^{-1}(\delta_2-\theta)}$ could be replaced by $s_{a_0^{-1}(\bar{\delta}_1-\theta)}$ and $s_{a_0^{-1}(\bar{\delta}_2-\theta)}$ as long as $\bar{\delta}_1$ and $\bar{\delta}_2$ generate the lattice $\mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2$, for any matrix u in $SL(2, \mathbb{Z})$ we can define such an automorphism which sends s_{01} and s_{03} to $s_{a_0^{-1}(u \cdot \delta_1 - \theta)}$ and $s_{a_0^{-1}(u \cdot \delta_2 - \theta)}$, respectively, Here $SL(2, \mathbb{Z})$ acts as usually on the lattice generated by δ_1 and δ_2 . We will denote this automorphism by \underline{u} . Let us observe that

$$\underline{u}(s_{m\delta_1+n\delta_2-\theta}) = s_{u \cdot (m\delta_1+n\delta_2)-\theta}.$$

The following result is an immediate consequence of the above remarks.

PROPOSITION 4.3. *The map described above is an injective morphism*

$$v : SL(2, \mathbb{Z}) \rightarrow \mathfrak{G}(\tilde{W}).$$

Let us note that the above morphism could be extended to $GL(2, \mathbb{Z})$. The element

$$e := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

gives rise to an involution \underline{e} of \tilde{W} which normalizes the group of automorphisms given by the modular group. For example

$$eu_{12}e = u_{21}^{-1}.$$

The fact that such automorphisms exist follows automatically from Theorem 2.6 since the double affine Weyl group is a quotient of the triple group by relations which are fixed by the action of B_3 and by the fact that \mathfrak{c}^2 acts trivially on this quotient. We preferred the explicit geometrical argument since it shows that actually $SL(2, \mathbb{Z})$ acts faithfully on the double affine Weyl group.

4.2. Automorphisms of double affine Artin groups. Returning to the double affine Artin group, denote by T_{0i} the images of the elements \mathcal{T}_{0i} in $\mathcal{A}_{\tilde{W}}$. We will be interested in automorphisms of the double affine Artin group which descend to automorphism of the double affine Weyl group. Let us call the group of such automorphisms $Aut(\mathcal{A}_{\tilde{W}}; \tilde{W})$. As for double affine Weyl groups, let us consider the following group

$$\mathfrak{G}(\mathcal{A}_{\tilde{W}}) := \{f \in Aut(\mathcal{A}_{\tilde{W}}; \tilde{W}) \mid f(T_i) = T_i, i \neq 0, \text{ and } f(X_{a_0^{-1}\delta}) = X_{a_0^{-1}\delta}\}.$$

As we remarked in the case of the double affine Weyl group, the action of the braid group on three letters described in Theorem 2.6 descends to the double affine Artin group, since this is a factor of the triple group by a relation which is invariant under B_3 . It is obvious that these descents belong to $\mathfrak{G}(\mathcal{A}_{\tilde{W}})$. Even more, the anti-involution ϵ , which fixes \mathcal{T}_{02} and interchanges \mathcal{T}_{01} and \mathcal{T}_{03} , also descends to the double affine Artin group. To keep our notation simple and because no confusion will arise we will use the same symbols to denote automorphisms of $\mathcal{A}_{\tilde{W}}$ which descend from those of \mathcal{A} .

It easy to see that ϵ interchanges X_μ with $Y_{-\mu}$ for any element of the root lattice. On any group we have a canonical anti-involution which sends any element to its inverse. If we consider its composition with ϵ we obtain an involution $\underline{\epsilon}$ which could be described as follows: it sends each of the generators $T_i, i \neq 0$ to its inverse and interchanges X_μ with Y_μ for any element of the root lattice. The descent of $\underline{\epsilon}$ to the double affine Weyl group is \underline{e} .

This involution – or rather its descent to the double affine Hecke algebra – plays a central role in the theory of Macdonald polynomials, where is responsible for the so-called difference Fourier transform. To mention only one of its many implications we note that the difference Fourier transform was the crucial ingredient in Cherednik’s proof [3] of the Macdonald evaluation–duality conjecture. Let us state this result explicitly.

THEOREM 4.4. *The map which sends each of the generators $T_i, i \neq 0$ to its inverse and interchanges X_μ with Y_μ for any $\mu \in \dot{Q}$ can be uniquely extended to an automorphism*

$$\underline{\epsilon} : \mathcal{A}_{\tilde{W}} \rightarrow \mathcal{A}_{\tilde{W}}$$

of the double affine Artin groups. The square of $\underline{\epsilon}$ is the identity isomorphism.

This a trivial consequence of our description of the double affine Artin group, however it is quite hard to prove it starting from the original description of Cherednik. See [12, Chapter 3], [5, Theorem 2.2] and [13, Theorem 4.2] for a different proof. We restate Theorem 2.6 in the double affine Artin group context.

THEOREM 4.5. *The map sending \mathfrak{a} and \mathfrak{b} to $\underline{\mathfrak{a}}$ and $\underline{\mathfrak{b}}$, respectively, defines a group morphism*

$$\Upsilon : B_3 \rightarrow \mathfrak{G}(\mathcal{A}_{\tilde{W}}).$$

By w_o we denote the longest element of the the Weyl group \dot{W} . The following result is well known.

LEMMA 4.6. *Inside the affine Artin group \mathcal{A}_W, T_0 commutes with $T_{s_o}^{-1}T_{w_o}$.*

Next we will describe the action of the center of the braid group B_3 on the double affine Artin group.

COROLLARY 4.7. *The generator c of the center of the braid group B_3 acts (via Υ) on the double affine Artin group as the conjugation by T_{w_\circ} .*

PROOF. By the above Theorem c acts as conjugation by T_{s_θ} on the elements T_{0i} . By the above Lemma this is the same as conjugation by T_{w_\circ} . \square

By Deligne [4], the center of the Artin group $\mathcal{A}_{\check{W}}$ is generated by T_{w_\circ} when $w_\circ = -1$ or by $T_{w_\circ}^2$ if $w_\circ \neq -1$. The latter occurs only for types $A_n, n \geq 2, D_{2n+1}$ and E_6 .

COROLLARY 4.8. *The above action of B_3 will give a morphism from $\text{PSL}(2, \mathbb{Z})$ (if $w_\circ = -1$) or $\text{SL}(2, \mathbb{Z})$ (if $w_\circ \neq -1$) to the outer automorphism group of the double affine Artin group.*

The Corollary 4.8 recovers results of Cherednik [3, Theorem 4.3]. The Theorem 4.5 also seems to follow from his work although no reference was made to the braid group on three letters. The implications of the above Corollary are extremely important. To give one example, it leads to projective representations of the modular group expressed in terms of special values of Macdonald polynomials. In turn these give identities involving special values of Macdonald polynomials at roots of unity.

Our approach allows us to say more about this action.

THEOREM 4.9. *The morphism $\Upsilon : B_3 \rightarrow \mathfrak{G}(\mathcal{A}_{\check{W}})$ is injective.*

PROOF. The following diagram is commutative

$$\begin{array}{ccc}
 B_3 & \xrightarrow{\Upsilon} & \mathfrak{G}(\mathcal{A}_{\check{W}}) \\
 \pi \downarrow & & \downarrow \\
 \text{SL}(2, \mathbb{Z}) & \xrightarrow{v} & \mathfrak{G}(\check{W})
 \end{array}$$

Because v is injective, any element in the kernel of Υ must be contained in the kernel of π , which is spanned by c^2 . From the above Corollary it is clear that the image by Υ of such an element cannot act trivially on $\mathfrak{G}(\mathcal{A}_{\check{W}})$. The Theorem is proved. \square

This Theorem also implies that the original action of B_3 on the triple group is also faithful.

THEOREM 4.10. *The morphism $\Upsilon : B_3 \rightarrow \text{Aut}(\mathcal{A})$ is injective.*

We close by noting that all the automorphisms of the double affine Artin groups given by B_3 descend faithfully to the corresponding double affine Hecke algebras, elliptic Artin groups and elliptic Hecke algebras. For Hecke algebras there is no action of B_3 on the parameters except for the non-reduced case where it acts by permuting the t_{0i} .

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109

E-mail address: bogdion@umich.edu

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08903

E-mail address: sahi@math.rutgers.edu