Annals of Combinatorics 10 (2006) 255-269
0218-0006/06/020255-15
DOI 10.1007/s00026-006-0286-6

Annals of Combinatorics

The Asymptotic Behavior of Certain Birth Processes*

Siddhartha Sahi†
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
sahi@math.rutgers.edu

Received July 15, 2004

AMS Subject Classification: 60J80, 60J85, 05A19, 33D52

Abstract. We describe a connection between discrete birth process and a certain family of multivariate interpolation polynomials. This enables us to compute all asymptotic moments of the birth process, generalizing previously known results for the mean and variance.

Keywords: birth processes, cumulants, Capelli identity, interpolation polynomials, divisors, q-series

1. Introduction

In this paper we undertake the analysis of a certain class of birth processes. We establish a number of new results for these processes, including a precise description of their asymptotic behavior. As a consequence we also obtain simple proofs and generalizations of several results in the current literature on the subject.

A *discrete-time pure birth process* is a sequence of random variables $\{X_0, X_1, \ldots\}$, such that the possible values (states) for X_n are the integers $0, 1, 2, \ldots, n$. The process starts in state 0 and direct transitions are only possible from state m to state $m+1$. Such a process is characterized by its *transition probabilities*

$$\tau_{m,n} = \Pr\{X_{n+1} = m+1 \,|\, X_n = m\}. \tag{1.1}$$

Indeed, writing

$$P_{m,n} := \Pr\{X_n = m\},$$

we have $P_{0,0} = 1$, and the following recursion holds

$$P_{m,n} = (1-\tau_{m,n-1})P_{m,n-1} + \tau_{m-1,n-1}P_{m-1,n-1}. \tag{1.2}$$

In a number of important situations, the transition probabilities factorize in the form

$$\tau_{m,n} = \alpha_m \beta_n^{-1}, \tag{1.3}$$

* This research was supported by an NSF grant.
† The author is grateful to Larry Shepp and Dan Ocone for useful discussions.

where α_m and β_n depend only on m and n respectively.

Our first main result is the following "product formula" for the birth process.

Theorem 1.1. *Let X_n be as in* (1.1) – (1.3). *Then for each n we have*

$$\mathbf{E}\left(\prod_{j=0}^{X_n-1}\left[1+u\,\alpha_j^{-1}\right]\right)=\prod_{j=0}^{n-1}\left[1+u\beta_j^{-1}\right],$$

where $\mathbf{E}(f(X_n))$ *denotes the* expected value *of* $f(X_n)$.

We next consider the asymptotic behaviour of the process in a suitable regime for the transition probabilities, *viz.* we assume that the infinite products

$$\prod\nolimits_{j\geq 0}\alpha_j, \quad \prod\nolimits_{j\geq 0}\beta_j$$

are *absolutely convergent*.

By [13, Section 2.7], this means that the log series

$$\sum\log\alpha_j, \quad \sum\log\beta_j$$

are absolutely convergent. Moreover if we write

$$\gamma_j=1-\alpha_j\,, \quad \delta_j=\beta_j-1, \tag{1.4}$$

then absolute convergence is equivalent to

$$\sum|\gamma_j|<\infty, \quad \sum|\delta_j|<\infty \text{ and } \gamma_j\neq 1, \delta_j\neq -1. \tag{1.5}$$

We now introduce the random variable

$$Y_n=n-X_n\,. \tag{1.6}$$

Thus Y_n is the population "deficit" at time n.

Theorem 1.2. *Let Y_n be as in* (1.1) – (1.6). *Then Y_n converges in distribution to a random variable Y, valued in the set* $\{0,1,2,\ldots\}$ *and moreover*

$$\mathbf{E}\left(e^{\tau Y}\right)=\prod_{j=0}^{\infty}\frac{1-\gamma_j}{1-e^{\tau}\gamma_j}\prod_{j=0}^{\infty}\frac{1+e^{\tau}\delta_j}{1+\delta_j},$$

where the products converge uniformly in a strip

$$0\leq|\mathrm{Re}(\tau)|<\varepsilon.$$

In particular, one can compute explicitly the moments and central moments of Y, which are defined by

$$\mu_n'=\mathbf{E}\left[Y^n\right], \quad \mu_n=\mathbf{E}\left[(Y-\mathbf{E}[Y])^n\right].$$

To describe the result, we introduce the rational function

$$f_k(z) = \left(z\frac{d}{dz}\right)^k \left[\frac{z}{1-z}\right]. \tag{1.7}$$

We also recall the definition of the *Bell polynomials*

$$B_n(x_1, \ldots, x_n) = n! \sum_{\lambda} \prod_{i=1}^{n} \frac{[x_i/i!]^{m_i}}{m_i!}, \tag{1.8}$$

where the sum ranges over all partitions λ of n, written in the form

$$1m_1 + 2m_2 + \cdots + nm_n = n.$$

Theorem 1.3. *For Y as in Theorem 1.2, the moments of Y are given by*

$$\mu'_n = B_n(\kappa_1, \kappa_2, \ldots, \kappa_n), \tag{1.9}$$

$$\mu_n = B_n(0, \kappa_2, \ldots, \kappa_n), \tag{1.10}$$

where

$$\kappa_i = \sum_{j=0}^{\infty} \left[f_{i-1}(\gamma_j) - f_{i-1}(-\delta_j)\right]. \tag{1.11}$$

The numbers κ_i in formula (1.11) are in fact the *cumulants* of Y. We recall that the first two cumulants are simply the *mean* $\kappa_1 = \mu$ and the *variance* $\kappa_2 = \sigma^2$. Also, the functions f_k are closely related to the *Eulerian polynomials*. We postpone a general discussion of cumulants, Bell polynomials, and Eulerian polynomials to the next section.

An interesting situation arises when we put

$$\gamma_j = q^{j+1}, \quad \delta_j = 0. \tag{1.12}$$

In this case, the birth process has combinatorial interpretations in terms of *heaps* [12] and *random graphs* [1]. We recall the usual q-series notation

$$(x)_n = (1-x)(1-xq)\cdots\left(1-xq^{n-1}\right),$$

and also the "sum of divisors" functions

$$S_k(q) = \sum_{n\geq 1} \sigma_k(n)\, q^n \text{ where } \sigma_k(n) = \sum_{d|n} d^k.$$

Theorem 1.3 has the following very pretty specialization:

Proposition 1.4 (Uchimura). *Suppose Y is as in the previous theorem, and assume that γ_j, δ_j are as in (1.12). Then for all $k \geq 0$ we have*

$$\Pr\{Y = k\} = q^k \left(q^{k+1}\right)_\infty, \tag{1.13}$$

$$\kappa_{k+1}(Y) = S_k(q). \tag{1.14}$$

As a further consequence we obtain

Corollary 1.5. *We have the following q-series identities*

$$\sum_{k\geq 0} k^n q^k \left(q^{k+1}\right)_\infty = B_n\left(S_0,\ldots,S_{n-1}\right), \tag{1.15}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{\binom{k+1}{2}}}{(q)_k\left(1-q^k\right)^n} = \frac{1}{n!}\sum_{k=1}^{n} \left|s\left(n,k\right)\right| B_k\left(S_0,\ldots,S_{k-1}\right). \tag{1.16}$$

Here $s(n,k)$ are Stirling numbers of the first kind defined by

$$z(z-1)\cdots(z-n+1) = \sum_{k=1}^{n} s(n,k)\, z^k.$$

The identity (1.15) was first obtained by Uchimura in [12, Theorem 2.1]. It was independently rediscovered by Andrews-Crippa-Simon [1] along with (1.16) in a less explicit form. In fact, as pointed out in [1] and Dilcher [3], (1.16) can be easily derived from (1.15). Dilcher also discovered *finite* analogs of (1.16) which were subsequently generalized by Prodinger [8] and Fu-Lascoux [5].

Our work on this topic was motivated by the results of [2]. Our Theorem 1.3 is a far-reaching generalization of Theorems 3.2 – 3.4 of that paper.

The organization of the paper is as follows: In Section 1 we recall basic facts about moments and cumulants. In Section 2, we establish the product formula of Theorem 1.1. In Sections 3 and 4 we analyze the asymptotics to deduce the characteristic function of Theorem 1.2 and the moment formula of Theorem 1.3. Finally in Section 5, we consider transition probabilities satisfying formula (1.12), and deduce Proposition 1.4 and Corollary 1.5.

The probabilities $P_{m,n}$ are closely related to the more general interpolation polynomials $P_\lambda(x;\rho)$ defined in [10]. It would be interesting to obtain a statistical interpretation for the general $P_\lambda(x;\rho)$ which might shed light on some of their properties.

2. Preliminaries

2.1. Symmetric Functions

Let $e(x)$ and $h(x)$ be the elementary and complete symmetric functions in x, defined by the expansions

$$\prod_{j=1}^{n}\left(1+zx_j\right) = \sum_{i=0}^{n} z^i e_i(x),$$

and

$$\prod_{j=1}^{n}\left(1-zx_j\right)^{-1} = \sum_{i=0}^{\infty} z^i h_i(x).$$

The main result of [2, Theorem 3.1] can be reformulated as follows:

Proposition 2.1. *Let $P_{m,n} = \Pr(X_n = m)$ be as in* (1.1) – (1.6)*, then we have*

$$P_{m,n} = \left[\prod_{j=0}^{m-1} \alpha_j \prod_{j=0}^{n-1} \beta_j^{-1}\right] Q_{m,n}(\beta;\alpha),$$

where

$$Q_{m,n}(\beta;\alpha) = \sum_{i+j=n-m} (-1)^i h_i(\alpha_0,\ldots,\alpha_m)\, e_j(\beta_0,\ldots,\beta_{n-1}). \tag{2.1}$$

Since our notation is slightly different from [2], we provide a translation for the reader. Our transition probability $\tau_{m,n}$ in formula (1.1) corresponds $\lambda_{n,m}$ in [2, 1.3]. Our β_i and α_i correspond to β_i^{-1} and γ_j in [2, 3.2]. Our $P_{m,n}$ and $Q_{m,n}$ correspond to $P_{n,m}$ and $\widehat{P}_{n,m}$ in formulas [2, 1.4 and 3.4]. Our formula (2.1) above is simply the formula for $\widehat{P}_{n,m}$ in the proof of [2, Theorem 3.1]. Finally, we observe that the formula [2, 1.7] actually contains a misprint — the arguments of e_{n-i} should be $\frac{1}{\beta_0},\ldots,\frac{1}{\beta_{n-1}}$ instead of $\frac{1-\beta_0}{\beta_0},\ldots,\frac{1-\beta_{n-1}}{\beta_{n-1}}$.

We note that it is immediate from the definition that $Q_{m,n}(\beta;\alpha)$ is precisely the coefficient of t^{n-m} in the power series expansion in t of the rational function

$$\prod_{j=0}^{n-1}(1+t\beta_j) \Big/ \prod_{i=0}^{m}(1+t\alpha_i). \tag{2.2}$$

The same polynomial $Q_{m,n}(\beta;\alpha)$ arises in a completely different context in [6]. The Proposition 3.1 of [6] shows

Proposition 2.2. *The polynomial $Q_{m,n}(\beta;\alpha)$ is the unique polynomial in $\beta_0,\ldots,\beta_{n-1}$ with coefficients in the field $\mathbb{Q}(\alpha)$ satisfying the following properties*

(1) *It is symmetric in $\beta_0,\ldots,\beta_{n-1}$.*
(2) *Its highest degree term is $e_{n-m}(\beta_0,\ldots,\beta_{n-1})$.*
(3) *It vanishes if we set $\beta_0 = \alpha_0,\ldots,\beta_m = \alpha_m$.*

The vanishing property of $Q_{m,n}(\beta;\alpha)$ is easy to see from (2.2). The key point is that if we set $\beta_0 = \alpha_0,\ldots,\beta_m = \alpha_m$ in (2.2) we obtain a polynomial of degree t^{n-m-1}. Thus the coefficient of t^{n-m} is 0.

2.2. Cumulants

We recall that for a random variable X, the expression

$$\mathbf{E}[\exp(tX)] = \sum_{n=0}^{\infty} \mu'_n \frac{t^n}{n!}$$

is called the moment generating function. The cumulants κ_j are defined via the identity

$$\exp\left[\sum_{i=1}^{\infty} \kappa_i \frac{t^i}{i!}\right] = \sum_{n=0}^{\infty} \mu'_n \frac{t^n}{n!},$$

which implies polynomial expression for the cumulants κ_j in terms of the moments μ'_n, and vice versa. Indeed, rewriting the left side as

$$\prod_{i=1}^{\infty} \exp\left[\kappa_i t^i / i!\right] = \prod_{i=1}^{\infty} \left[\sum_{m_i=0}^{\infty} \frac{[\kappa_i / i!]^{m_i}}{m_i!} t^{i m_i} \right],$$

one arrives at formula (1.9) for μ'_n in terms of the Bell polynomial (1.8).

As explained previously, the first two cumulants are precisely the mean and standard deviation of X. The higher cumulants $(\kappa_j,\, j \geq 3)$ are measures of *non-normality* of X; they are all 0 if X is a Gaussian random variable. In statistics, κ_3/σ^3 is called the *skewness*, and κ_4/σ^4 is called the *kurtosis excess*.

The *central* moments of X are the moments of the random variable

$$Z = X - \mathbf{E}[X] = X - \kappa_1,$$

which has the generating function

$$\mathbf{E}[\exp(tZ)] = \mathbf{E}[\exp(tX)] / \exp(t\kappa_1) = \prod_{i=2}^{\infty} \exp\left[\kappa_i t^i / i!\right],$$

and therefore we get formula (1.10).

Next, one can write

$$f_k(z) = \left(z \frac{d}{dz} \right)^k \left[\frac{z}{1-z} \right] = \frac{z E_k(z)}{(1-z)^{k+1}},$$

where $E_0(z) = 1$, while for $k \geq 1$, $E_k(z)$ is a polynomial of degree $k-1$:

$$E_k(z) = \sum_{j=0}^{k-1} E(k, j) z^j.$$

These polynomials $E_k(z)$ are called the Eulerian polynomials and their coefficients $E(k, j)$ are called *Eulerian numbers*. They satisfy the Pascal triangle type relation:

$$E(k, j) = (k - j + 1) E(k-1, j-1) + (j+1) E(k-1, j).$$

The first few Eulerian polynomials are as follows:

$$E_1(z) = 1,$$

$$E_2(z) = 1 + z,$$

$$E_3(z) = 1 + 4z + z^2,$$

$$E_4(z) = 1 + 11z + 11z^2 + z^3,$$

$$E_5(z) = 1 + 26z + 66z^2 + 26z^3 + z^4.$$

3. The Product Formula

In this section we prove Theorem 1.1, as a straightforward consequence of the following result.

Proposition 3.1. *Let $Q_{m,n}$ be as in* (2.1)*, then we have the identity*

$$\sum_{m=0}^{n} Q_{m,n}(\beta;\alpha)\prod_{i=0}^{m-1}(u+\alpha_i)=\prod_{i=0}^{n-1}(u+\beta_i). \tag{3.1}$$

Proof. Let us write

$$\upsilon_m=\prod_{i=0}^{m-1}(u+\alpha_i).$$

Then we have

$$\upsilon_m=u^m+\text{ lower terms.}$$

Therefore we can invert this expansion and express u^i $(0\le i\le n)$ in terms of υ_i $(0\le i\le n)$.

In particular we have an expansion of the form

$$\sum_{m=0}^{n} R_{m,n}(\beta;\alpha)\prod_{i=0}^{m-1}(u+\alpha_i)=\prod_{i=0}^{n-1}(u+\beta_i), \tag{3.2}$$

with some unknown polynomials $R_{m,n}(\beta;\alpha)$. We will prove

$$R_{m,n}(\beta;\alpha)=Q_{m,n}(\beta;\alpha)$$

by verifying the properties of Proposition 2.2.

Clearly $R_{m,n}(\beta;\alpha)$ is symmetric in β. Also setting

$$\alpha_0=\cdots=\alpha_{n-1}=0$$

in (3.2) we deduce

$$\sum_{m=0}^{n} R_{m,n}(\beta;0)\,u^m=\prod_{i=0}^{n-1}(u+\beta_i).$$

The leading term in β of $R_{m,n}(\beta;\alpha)$ is

$$R_{m,n}(\beta;0)=e_{n-m}(\beta),$$

as desired.

To complete the proof, it suffices to show that for each k

$$R_{k,n}(\beta;\alpha)\,|_{\beta_0=\alpha_0,\ldots,\beta_k=\alpha_k}=0.$$

We will prove this by induction on k. First, in (3.2) we set

$$\beta_0=\alpha_0=-\,u,$$

then the right side is 0 because of the factor $u+\beta_0$, whereas on the left, all but one of the terms drops out because of the factor $u+\alpha_0$. Thus we get

$$R_{0,n}(\beta;\alpha)\,|_{\beta_0=\alpha_0}=0,$$

which is the desired result for $k=0$.

We now proceed by induction on k. Set

$$\beta_0=\alpha_0,\dots,\beta_{k-1}=\alpha_{k-1} \quad \text{and} \quad \beta_k=\alpha_k=-u$$

in (3.2). Once again the right side is 0 because of the factor $u+\beta_k$. On the left side, the terms for $m=0,\dots,k-1$ vanish by the inductive hypothesis, while the terms for $m=k+1,\dots,n-1$ vanish because of the presence of the factor $u+\alpha_k$. Thus we get

$$\left[R_{k,n}(\beta;\alpha)\,|_{\beta_0=\alpha_0,\dots,\beta_k=\alpha_k}\right]\prod_{i=0}^{k-1}(\alpha_i-\alpha_k)=0.$$

Since $\prod_{i=0}^{k-1}(\alpha_i-\alpha_k)$ is invertible in the field $\mathbb{Q}(\alpha)$, the result follows.

By Proposition 2.2 we deduce that for all m

$$R_{m,n}(\beta;\alpha)=Q_{m,n}(\beta;\alpha);$$

and so (3.2) becomes (3.1). ∎

Theorem 1.1 is now an immediate consequence.

Proof of Theorem 1.1. Combining Lemma 2.1 with the previous proposition, we obtain

$$\sum_{m=0}^{n}\frac{P_{m,n}}{\prod_{j=0}^{n-1}\beta_j^{-1}\prod_{i=0}^{m-1}\alpha_i}\prod_{i=0}^{m-1}(u+\alpha_i)=\prod_{i=0}^{n-1}(u+\beta_i).$$

Rewriting this, we get

$$\prod_{i=0}^{n-1}\left(1+u\beta_i^{-1}\right)=\sum_{m=0}^{n}P_{m,n}\prod_{i=0}^{m-1}\left(1+u\alpha_i^{-1}\right)=\mathbf{E}\left[\prod_{i=0}^{X_n-1}\left(1+u\alpha_i^{-1}\right)\right]. \quad ∎$$

In view of the standard identity

$$\prod_{i=1}^{k}(1+ux_i)=\sum_{m=0}^{k}e_m(x_1,\dots,x_k)\,u^m,$$

the previous result can also be reformulated as

Corollary 3.2. $\mathbf{E}\left[e_m\left(\alpha_0^{-1},\dots,\alpha_{X_n-1}^{-1}\right)\right]=e_m\left(\beta_0^{-1},\dots,\beta_{n-1}^{-1}\right).$

4. The Characteristic Function

In this section, we will prove Theorem 1.2. For this we define

$$\phi_{k,n}(z) = \prod_{j=0}^{n-1} \frac{1+z\delta_j}{1+\delta_j} \prod_{j=0}^{n-k-1} \frac{1-\gamma_j}{1-z\gamma_j}. \tag{4.1}$$

Lemma 4.1. *Let Y_n be as in* (1.1) – (1.6)*, then*

$$\Pr\{Y_n = k\} = \frac{1}{k!}\phi_{k,n}^{(k)}(0).$$

Here $f^{(k)}(z)$ denotes the k-th derivative of f.

Proof. By definition,

$$\Pr\{Y_n = k\} = \Pr\{X_n = n-k\} = P_{n-k,n}.$$

Therefore by Proposition 2.1 we get

$$\Pr\{Y_n = k\} = \left[\prod_{j=0}^{n-k-1} \alpha_j \prod_{j=0}^{n-1} \beta_j^{-1}\right] Q_{n-k,n}(\beta;\alpha).$$

By (1.4), we have

$$\alpha_j = 1-\gamma_j, \quad \beta_j = 1+\delta_j.$$

Now the recursion (1.2) for $P_{m,n}$ yields the following recursion for $Q_{m,n}$

$$Q_{m,n} = (\beta_{n-1} - \alpha_m) Q_{m,n-1} + Q_{m-1,n-1}.$$

In particular we see that the recursion is *unchanged* if we add the same constant to all α_i and β_j. Thus we get

$$Q_{m,n}(\beta;\alpha) := Q_{m,n}(1+\delta; 1-\gamma) = Q_{m,n}(\delta; -\gamma).$$

So we have

$$\Pr\{Y_n = k\} = \left[\frac{\prod_{j=0}^{n-k-1}(1-\gamma_j)}{\prod_{j=0}^{n-1}(1+\delta_j)}\right] Q_{n-k,n}(\delta;-\gamma).$$

Now by formula (2.2), $Q_{n-k,n}$ is the coefficient of z^k in the expansion of

$$\prod_{j=0}^{n-1}(1+z\delta_j) \prod_{j=0}^{n-k-1}(1-z\gamma_j)^{-1}.$$

Therefore $\Pr\{Y_n = k\}$ is the coefficient of z^k in $\phi_{k,n}$, and the result follows by Taylor's formula. ∎

We next study the behaviour of the corresponding infinite products.

Lemma 4.2. *Let* γ_j *be as in* (1.1) – (1.5) *then*

$$0 \leq \gamma_j < 1 \text{ for all } j\,.$$

Furthermore, we have

$$\sup \gamma_j = \max \gamma_j < 1.$$

Proof. Since the products $\prod_{j=0}^{\infty} \alpha_j$, $\prod_{j=0}^{\infty} \beta_j$ are absolutely convergent, we have

$$\lim_j \alpha_j = \lim_j \beta_j = 1.$$

In particular, for all m we have

$$\alpha_m = \lim_n \alpha_m \beta_n^{-1} = \lim_n \tau_{m,n} \in [0,\, 1].$$

Again, by the absolute convergence of $\prod_{j=0}^{\infty} \alpha_j$ we have

$$\alpha_j \neq 0 \text{ for all } j.$$

Hence we have

$$\alpha_j \in (0,\, 1],$$

and therefore

$$\gamma_j = 1 - \alpha_j \in [0,\, 1),$$

which proves the first part of the lemma.

For the second part, we merely observe that since $\gamma_j \to 0$, the sequence $\{\gamma_j\}$ attains its supremum. ∎

Lemma 4.3. *Let* γ_j, δ_j *be as in* (1.1) – (1.5) *then there is* $R > 1$*, such that the infinite product*

$$\phi(z) = \prod_{j=0}^{\infty} \frac{1-\gamma_j}{1-z\gamma_j} \prod_{j=0}^{\infty} \frac{1+z\delta_j}{1+\delta_j}$$

converges uniformly and absolutely on the disc $\{|z| \leq R\}$*, and defines a holomorphic function there.*

Proof. It suffices to consider the products $\prod (1+z\delta_j)$, $\prod \frac{1}{1-z\gamma_j}$ separately.

By (1.5), the series

$$\sum |z\delta_j| = |z| \sum |\delta_j|$$

converges uniformly on compact sets. Therefore by [9, Theorem 15.4], the product

$$\prod (1+z\delta_j)$$

converges uniformly on compact sets and defines an entire function.

By the previous lemma, we can choose $R > 1$ such that

$$\sup \gamma_j < R^{-1} < 1.$$

Then on the disc $\{|z| \leq R\}$, we have

$$\sum \left| \frac{z\gamma_j}{1 - z\gamma_j} \right| \leq \frac{|z|}{m} \sum |\gamma_j| ,$$

where

$$m = \inf |1 - z\gamma_j| > 1 - R \sup \gamma_j > 0.$$

Therefore again by [9, Theorem 15.4], the product

$$\prod \frac{1}{1 - z\gamma_j} = \prod \left(1 + \frac{z\gamma_j}{1 - z\gamma_j} \right),$$

converges uniformly to a holomorphic function on the disc $\{|z| \leq R\}$. ∎

Proposition 4.4. *Let Y_n be as in* (1.1) – (1.6)*, and define*

$$y_k = \lim_{n \to \infty} \Pr\{Y_n = k\}.$$

Then the limit exists, and moreover

$$\sum_{k=0}^{\infty} y_k z^k = \prod_{j=0}^{\infty} \frac{1 - \gamma_j}{1 - z\gamma_j} \prod_{j=0}^{\infty} \frac{1 + z\delta_j}{1 + \delta_j}. \tag{4.2}$$

In particular, the series is absolutely convergent on the disc $\{|z| \leq R\}$, where $R > 1$ is as in the previous lemma.

Proof. Let us fix k and consider the sequence $\phi_{k,n}$ from (4.1) as $n \to \infty$. By Lemma 4.3, we have

$$\lim_{n \to \infty} \phi_{k,n} \to \phi$$

uniformly on $\{|z| \leq R\}$. Note that the limit is *independent* of k.

Therefore, by an elementary result in complex analysis [9, Corollary 10.27], we get

$$\lim_n \phi_{k,n}^{(k)} \to \phi^{(k)}.$$

Therefore by Lemma 4.1 we get

$$\lim_{n \to \infty} \Pr\{Y_n = k\} = \lim_{n \to \infty} \frac{1}{k!} \phi_{k,n}^{(k)}(0) = \frac{1}{k!} \phi^{(k)}(0).$$

This identifies y_k with the Taylor coefficients of ϕ and the result follows. ∎

We can now prove Theorem 1.2.

Proof of Theorem 1.2. By the previous proposition, Y_n converges in distribution to the random variable Y, such that

$$\Pr\{Y = k\} = y_k.$$

Let R be as in the previous proposition and set

$$\varepsilon = \log R > 0.$$

Then for

$$|\mathrm{Re}\tau| \leq \varepsilon,$$

we have

$$|e^{\tau}| \leq R.$$

Therefore we may substitute

$$z = e^{\tau} \tag{4.3}$$

in formula (4.2), to get

$$\mathbf{E}\left[e^{\tau Y}\right] = \sum_{k=0}^{\infty} y_k e^{\tau k} = \prod_{j=0}^{\infty} \frac{1-\gamma_j}{1-e^{\tau}\gamma_j} \prod_{j=0}^{\infty} \frac{1+e^{\tau}\delta_j}{1+\delta_j},$$

and the result follows. ∎

5. The Cumulant Formula

Proof of Theorem 1.3. The cumulants of Y are given by

$$\kappa_{k+1} = \left(\frac{d}{d\tau}\right)^k \left[\log \mathbf{E}\left(e^{\tau Y}\right)\right]\big|_{\tau=0}.$$

Or, equivalently, in terms of the variable z from (4.3)

$$\kappa_{k+1} = \left(z\frac{d}{dz}\right)^k \left[\log \mathbf{E}\left(z^{Y}\right)\right]\big|_{z=1} = \left(z\frac{d}{dz}\right)^k \log \phi(z)\big|_{z=1},$$

where, as in (4.2), we have

$$\phi(z) = \prod_{j=0}^{\infty} \frac{1-\gamma_j}{1-z\gamma_j} \prod_{j=0}^{\infty} \frac{1+z\delta_j}{1+\delta_j},$$

with uniform convergence for $|z| \leq R$, with $R > 1$.

Moreover we have

$$\phi(1) = 1.$$

Therefore $\phi(z)$ is analytic and *non-zero* in a small disk D centered at 1.

This means we can take the principal branch of the logarithm of both sides to get

$$\log \mathbf{E}\left(z^{Y}\right) = \sum_{j=0}^{\infty} \left(\log\left[1+z\delta_j\right] - \log\left[1-z\gamma_j\right]\right) + \text{const.}$$

For z in D, the series on the right converges uniformly and therefore we can differentiate under the summation sign.

Finally, substituting $z = zy$ we conclude that

$$\left[z\frac{d}{dz}\right]^{k+1} \log(1-zy)\big|_{z=1} = \left[z\frac{d}{dz}\right]^{k+1} \log(1-z)\big|_{z=y} = -f_k(y),$$

Hence we get

$$\kappa_{k+1} = \left[z\frac{d}{dz}\right]^{k+1} \log\phi(z)\big|_{z=1} = \sum_{j=0}^{\infty}\left[f_k(\gamma_j) - f_k(-\delta_j)\right].$$

This proves formula (1.11), and the rest of the theorem follows as discussed in Section 2. ∎

6. Divisors and q-Series

Let Y_n be as in (1.1) – (1.4), (1.6) and set

$$\gamma_j = q^{j+1}, \quad \delta_j = 0$$

as in (1.12) . If we have $0 \leq q < 1$, then the conditions (1.5) are satisfied and all of the results of the previous sections hold.

We can now prove Proposition 1.4.

Proof of Proposition 1.4. As before we write

$$y_k = \Pr(Y = k).$$

Then by formulas (4.2) and (1.12) we get

$$\sum_{k=0}^{\infty} y_k z^k = \prod_{j=0}^{\infty} \frac{1-q^{j+1}}{1-zq^{j+1}} = \frac{(q)_\infty}{(zq)_\infty}.$$

By an elementary identity due to Euler [7, Ex. I.2.4], we have

$$\frac{1}{(x)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q)_k}.$$

Therefore comparing coefficients we get

$$y_k = \frac{(q)_\infty}{(q)_k} q^k = q^k \left(q^{k+1}\right)_\infty.$$

Next, we have the series expansion

$$f_k(z) = \left[z\frac{d}{dz}\right]^k \left(\frac{z}{1-z}\right) = \sum_{d=1}^{\infty} d^k z^d.$$

Therefore by Theorem 1.3 we get

$$\kappa_{k+1} = \sum_{l=0}^{\infty} f_k\left(q^{l+1}\right) = \sum_{d=1}^{\infty}\sum_{l=0}^{\infty} d^k q^{d(l+1)}.$$

Replacing $d(l+1)$ by n, we obtain

$$\kappa_{k+1} = \sum_{n=1}^{\infty}\sum_{d|n} d^k q^n = S_k(q),$$

which proves formula (1.14). ∎

Corollary 1.5 is an immediate consequence.

Proof of Corollary 1.5. The q-series of formula (1.15) is

$$\sum_{k\geq 0} k^n q^k \left(q^{k+1}\right)_\infty .$$

In view of the previous lemma this can be written as

$$\sum_{k=0}^{\infty} k^n \Pr(Y=k) = \mathbf{E}(Y^n) = \mu'_n.$$

Therefore formula (1.15) follows from formula (1.9).

We next consider formula (1.16). As explained in [1, p. 49] and Dilcher [3, Lemma 1], using the binomial theorem, together with the following standard q-series identity [7, Ex. I.2.4]

$$(z)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q)_n} q^{\binom{n}{2}} z^n,$$

we can rewrite the left side of (1.16) as

$$\sum_{k=0}^{\infty} \binom{k+n-1}{n} q^k \left(q^{k+1}\right)_\infty = \mathbf{E}\left[\frac{Y(Y+1)\cdots(Y+n-1)}{n!}\right].$$

Since we have

$$Y(Y+1)\cdots(Y+n-1) = \sum_{k=1}^{n} |s(n,k)| Y^k,$$

formula (1.16) follows from (1.15). ∎

References

1. G. Andrews, D. Crippa, and K. Simon, q-Series arising from the study of random graphs, SIAM J. Discrete Math. **10** (1997) 41–56.
2. T. Bickel, N. Galli, and K. Simon, Birth processes and symmetric polynomials, Ann. Combin. **5** (2001) 123–139.
3. K. Dilcher, Some q-series identities related to divisor functions, Discrete Math. **145** (1995) 83–93.
4. W. Feller, An Introduction to Probability Theory and Its Applications, 2nd Ed., John Wiley and Sons, Inc., New York-London-Sydney, 1971.
5. A.M. Fu and A. Lascoux, q-Identities from Lagrange and Newton interpolation, Adv. Appl. Math. **31** (2003) 527–531.
6. F. Knop and S. Sahi, Difference equations and symmetric polynomials defined by their zeros, Internat. Math. Res. Notices **10** (1996) 473–486.
7. I. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, 1995.
8. H. Prodinger, Some applications of the q-Rice formula, Random Structures Algorithms **19** (2001) 552–557.

9. W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.
10. S. Sahi, The spectrum of certain invariant differential operators associated to Hermitian symmetric spaces, Progr. Math. **123** (1994) 569–576.
11. S. Sahi, Interpolation, integrality and a generalization of Macdonald's polynomials, Internat. Math. Res. Notices **10** (1996) 457–471.
12. K. Uchimura, Divisor generating functions and insertion into a heap, Discrete Appl. Math. **18** (1987) 73–81.
13. E. Whittaker and G. Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, 1927.