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The FKG Inequality for Partially Ordered Algebras

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Abstract The FKG inequality asserts that for a distributive lattice with log-supermodular probability measure, any two increasing functions are positively correlated. In this paper we extend this result to functions with values in partially ordered algebras, such as algebras of matrices and polynomials.

Keywords FKG inequality · Distributive lattice · Ahlswede-Daykin inequality · Correlation inequality · Partially ordered algebras

1 Introduction

Let 2^S be the lattice of all subsets of a finite set S, partially ordered by set inclusion. A function $f : 2^S \to \mathbb{R}$ is said to be increasing if $f(\alpha) - f(\beta)$ is positive for all $\beta \subseteq \alpha$. (Here and elsewhere by a positive number we mean one which is ≥ 0.)

Given a probability measure μ on 2^S we define the expectation and covariance of functions by

$$E_\mu(f) := \sum_{\alpha \in 2^S} \mu(\alpha) f(\alpha),$$

$$C_\mu(f, g) := E_\mu(fg) - E_\mu(f) E_\mu(g).$$

The FKG inequality [8] asserts if f, g are increasing, and μ satisfies

$$\mu(\alpha \cup \beta)\mu(\alpha \cap \beta) \geq \mu(\alpha)\mu(\beta) \quad \text{for all } \alpha, \beta \subseteq S. \tag{1}$$

then one has $C_\mu(f, g) \geq 0$.

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A special case of this inequality was previously discovered by Harris [11] and used by him to establish lower bounds for the critical probability for percolation. Fortuyn et al. [8] were motivated by questions of statistical mechanics, especially the study of correlation in the Ising-Potts and related models. The inequality was also discovered independently by Sarkar [21] in connection with reliability theory in statistics. Other proofs were subsequently found by Holley [12] and Preston [16].

Over the years, this inequality has proved to be extremely useful in many fields, and there is a long and growing list of applications to statistics, combinatorics, graph theory, physics etc. Indeed it is virtually impossible for us to list all the applications of this inequality. However as a starting point, we refer the interested reader to [2–6, 9, 10, 13–15, 22, 23] and to the references therein.

Our purpose in this paper is to prove a generalization of the FKG inequality, where one considers functions with values in a partially ordered algebra, such as a matrix algebra or a polynomial algebra.

Here and elsewhere by an "algebra" we will mean an algebra over $\mathbb{R}$ which is not necessarily associative, commutative or unital. Also by a "convex cone" we will mean a non-empty subset of a vector space over $\mathbb{R}$, which is closed under addition and under multiplication by positive scalars.

Definition 1 A partially ordered algebra is a pair (A, P) where A is an algebra and P is a convex cone satisfying

$$x, y \in P \quad \Rightarrow \quad xy \in P.$$

We will say that the elements of P are *positive*. Now the notion of an increasing function from 2^S to A makes sense (with the same definition), as does the expected value of such a function. However since A is not necessarily commutative, some care is needed with the concept of covariance, which we define as

$$C_\mu(f, g) := E_\mu(f \cdot g) - E_\mu(f) \cdot E_\mu(g),$$

where $a \cdot b = (ab + ba)/2$ denotes the anticommutator in A.

Our first main result is the following generalization of the FKG inequality:

Theorem 2 *Suppose (A, P) is a partially ordered algebra and μ satisfies* (1). *Then any two increasing functions $f, g : 2^S \to A$ have positive covariance.*

The proof of the theorem involves several new ideas. Apart from the possible non-commutativity and non-associativity of A, the main difficulty is that the square of an arbitrary element in A need not be positive. This is a real problem because the proofs of many inequalities in real analysis rely crucially on the positivity of x^2. Indeed it is rather surprising that the FKG inequality holds without this assumption.

Our techniques also enable us to deduce the following closely related result:

Theorem 3 *Suppose (A, P) and μ are as above, A is associative and commutative, $f : 2^S \to P$ is increasing, and $g : 2^S \to P$ is decreasing, then $E_\mu(f^2)E_\mu(g^2) - E_\mu(fg)^2$ is positive.*

This theorem is a version of the Cauchy-Schwartz inequality. Again, it is somewhat remarkable that it holds in this generality!

We conclude this section with a few remarks.

First of all, the results of this paper apply to the following situations (among others):

1. A is the algebra of $n \times n$ real matrices and P consists of matrices which preserve a convex cone in $\mathbb{R}^n$.
2. A is the algebra of polynomials (or formal power series) and P consists of those polynomials (or power series) which have positive coefficients.

Second, it is well known that the classical FKG inequality follows from the more general "four function theorem" of Ahlswede and Daykin [1] (see [18] for a further generalization). Therefore it is natural to ask whether this theorem holds in the present setting. Unfortunately this turns out to be false, even under the additional assumptions of associativity and commutativity, as we show in this paper by an explicit counterexample in the polynomial algebra $\mathbb{R}[x]$. This seems to be connected with the failure of the cone P to be closed under *division*.

Third, we note that in the special case of product measures, i.e. those which satisfy

$$\mu(\alpha \cup \beta)\mu(\alpha \cap \beta) = \mu(\alpha)\mu(\beta) \quad \text{for all } \alpha, \beta \subseteq S.$$

The results of this paper were obtained earlier in [20].

Finally, we remark that there exists another, as yet conjectural, generalization of the FKG inequality of the form

$$E_n(f_1, \ldots, f_n) \geq 0,$$

where E_n is a certain correlation functional of the n increasing positive functions $f_1, \ldots, f_n : 2^S \to \mathbb{R}_{\geq 0}$, with respect to an FKG measure. The case $n = 2$ is the classical FKG inequality, while for $n = 3, 4, 5$ these functionals were first introduced in [17]. The general definition may be found in [19] together with a proof under some additional assumptions. It seems quite likely that the ideas of this paper have some bearing on this issue and we shall consider this possibility in a subsequent paper.

2 Proof of the FKG Inequality

Let (A, P), μ and $f, g : 2^S \to A$ be as in the statement of the FKG inequality (Theorem 2). In the special case where $f = g$, the covariance is simply the *variance* $V_\mu(f) = C_\mu(f, f) = E_\mu(f^2) - E_\mu(f)^2$. Therefore Theorem 2 clearly implies

Theorem 4 *For* (A, P), μ, f *as above*, *the variance* $V_\mu(f)$ *is positive*.

For $A = \mathbb{R}$, Theorem 4 is trivial, since the variance can be re-expressed as a weighted sum of squares. However in the present setting, it turns out that Theorem 4 *implies* Theorem 2 via a simple polarization argument, as shown below:

Proof (of Theorem 2) Note that the covariance is unchanged if we add a constant to f or g. Therefore replacing f and g by $f - E_\mu(f)$ and $g - E_\mu(g)$ we may assume that $E_\mu(f) = E_\mu(g) = 0$. It suffices then to prove that $E_\mu(f \cdot g)$ is positive.

Now given (A, P), let $A' = A[x, y]$ be the polynomial algebra in two variables, and let P' consist of polynomials with coefficients in P. Also define $h : 2^S \to A'$ by

$$h(\alpha) = xf(\alpha) + yg(\alpha).$$

Then (A', P') is a partially ordered algebra and h is an increasing function (with $E_\mu(h) = 0$). Therefore by Theorem 4, the variance belongs to P'. But we have

$$V_\mu(h) = E_\mu([xf + yg]^2) = x^2 E_\mu(f^2) + 2xyE_\mu(f \cdot g) + y^2 E_\mu(g^2).$$

Hence $C_\mu(f, g) = E_\mu(f \cdot g)$ belongs to P. □

The rest of this section will be devoted to the proof of Theorem 4. First of all, we define the support of the measure μ to be the set

$$\text{supp}(\mu) = \{\alpha : \mu(\alpha) \neq 0\}.$$

Since our argument involves division by $\mu(\alpha)$, a little extra care is required for $\alpha \notin \text{supp}(\mu)$. The main point is that in the formula for variance, the values $f(\alpha)$ for $\alpha \notin \text{supp}(\mu)$ play no role whatsoever. Therefore we shall prove the positive variance theorem under the weaker (but equivalent) assumption that f is increasing on the support of μ (and not necessarily on all of 2^S).

We fix x in S, and write S_x for the set $S \setminus \{x\}$; and for $\alpha \subseteq S_x$, we write αx for the set $\alpha \cup \{x\}$. Now we define a measure μ_x and a function f_x on 2^{S_x} as follows:

$$\mu_x(\alpha) := \mu(\alpha) + \mu(\alpha x),$$
$$f_x(\alpha) := \begin{cases} \frac{\mu(\alpha)f(\alpha)+\mu(\alpha x)f(\alpha x)}{\mu_x(\alpha)} & \text{if } \mu_x(\alpha) \neq 0, \\ 0 & \text{if } \mu_x(\alpha) = 0. \end{cases}$$

It follows from the definitions that

$$E_{\mu_x}(f_x) = E_\mu(f).$$

For the proof of the positive variance theorem we need two preliminary lemmas.

Lemma 5 *If μ is a probability measure satisfying (1), then so is μ_x.*

Proof Although this is well known (see e.g. [7]) we reproduce the calculation for the sake of completeness. Fix α, β in S_x and write $\gamma = \alpha \cup \beta$ and $\delta = \alpha \cap \beta$, then we need to show that

$$\mu_x(\gamma)\mu_x(\delta) - \mu_x(\alpha)\mu_x(\beta) \geq 0.$$

To simplify notation we write, respectively,

$$a_0, b_0, c_0, d_0 = \mu(\alpha), \mu(\beta), \mu(\gamma), \mu(\delta),$$
$$a_1, b_1, c_1, d_1 = \mu(\alpha x), \mu(\beta x), \mu(\gamma x), \mu(\delta x).$$

Applying (1) to various combinations of the sets α, αx, β, βx we obtain

$$c_0 d_0 \geq a_0 b_0, \qquad c_1 d_1 \geq a_1 b_1 \quad \text{and} \quad c_1 d_0 \geq a_1 b_0, a_0 b_1; \tag{2}$$

while on the other hand we have

$$\begin{aligned}
&\mu_x(\gamma)\mu_x(\delta) - \mu_x(\alpha)\mu_x(\beta) \\
&\quad = [c_0 + c_1][d_0 + d_1] - [a_0 + a_1][b_0 + b_1] \\
&\quad = [c_0 d_0 - a_0 b_0] + [c_1 d_1 - a_1 b_1] + [c_1 d_0 + c_0 d_1 - a_1 b_0 - a_0 b_1].
\end{aligned}$$

Now (2) implies that the first two expressions on the right are positive, and it remains only to prove that the third expression is positive. If $c_1 d_0 = 0$ then by (2) we have $a_1 b_0 = a_0 b_1 = 0$ and the third expression reduces to $c_0 d_1$. If $c_1 d_0 \neq 0$, then applying (2) twice we obtain

$$\begin{aligned}
c_1 d_0 - a_1 b_0 - a_0 b_1 + c_0 d_1 &\geq c_1 d_0 - a_1 b_0 - a_0 b_1 + \frac{a_0 a_1 b_0 b_1}{c_1 d_0} \\
&= \left(1 - \frac{a_1 b_0}{c_1 d_0}\right)(c_1 d_0 - a_0 b_1) \geq 0.
\end{aligned}$$

□

Lemma 6 *If μ is an FKG measure and f is increasing on supp(μ), then f_x is increasing on supp(μ_x)*

Proof For this we fix subsets $\alpha \supset \beta$ in supp(μ_x) and write

$$a_0, b_0, a_1, b_1 = \mu(\alpha), \mu(\beta), \mu(\alpha x), \mu(\beta x)$$

as before. By assumption, these are all non-zero, and we have

$$\begin{aligned}
&\mu_x(\alpha)\mu_x(\beta)[f_x(\alpha) - f_x(\beta)] \\
&\quad = (b_0 + b_1)[a_0 f(\alpha) + a_1 f(\alpha x)] - (a_0 + a_1)[b_0 f(\beta) + b_1 f(\beta x)] \\
&\quad = a_1 b_1 [f(\alpha x) - f(\beta x)] + a_0 b_0 [f(\alpha) - f(\beta)] \\
&\qquad + a_1 b_0 [f(\alpha x) - f(\beta)] + a_0 b_1 [f(\alpha) - f(\beta x)].
\end{aligned}$$

Since $\mu_x(\alpha)\mu_x(\beta) \neq 0$ it suffices to show that the above expression is positive. The first two terms are each positive since f is increasing, while the last two terms can be rewritten as follows:

$$\begin{aligned}
&\frac{1}{2}[a_1 b_0 + a_0 b_1][f(\alpha x) - f(\beta x) + f(\alpha) - f(\beta)] \\
&\quad + \frac{1}{2}[a_1 b_0 - a_0 b_1][f(\alpha x) - f(\alpha) + f(\beta x) - f(\beta)].
\end{aligned}$$

By the FKG condition on μ, we have $a_1 b_0 \geq a_0 b_1$. Since f is increasing, it follows that the above expression is positive. □

Proof (of Theorem 4) We proceed by induction on $|S|$, the result being obvious for $S=\emptyset$.

Now suppose $S\neq\emptyset$, and fix $x\in S$. In view of the two previous lemmas, the induction hypothesis implies that $V_{\mu_x}(f_x)$ is positive. Therefore it suffices to prove that $V_\mu(f)-V_{\mu_x}(f_x)$ is positive.

Since $E_{\mu_x}(f_x)$ is equal to $E_\mu(f)$, we get

$$\begin{aligned}V_\mu(f)-V_{\mu_x}(f_x)&=E_\mu(f^2)-E_{\mu_x}(f_x^2)\\&=\sum_{\alpha\subseteq S_x}[\mu(\alpha)f(\alpha)^2+\mu(\alpha x)f(\alpha x)^2-\mu_x(\alpha)f_x(\alpha)^2].\end{aligned}$$

We prove that each summand is positive. This is obvious if $\mu_x(\alpha)=0$, for then $\mu(\alpha x)=\mu(\alpha)=0$ as well. Otherwise, writing $a_0,a_1=\mu(\alpha),\mu(\alpha x)$, the expression becomes

$$\begin{aligned}&[a_0f(\alpha)^2+a_1f(\alpha x)^2]-(a_0+a_1)\left[\frac{a_0f(\alpha)+a_1f(\alpha x)}{a_0+a_1}\right]^2\\&=\frac{1}{a_0+a_1}[(a_0+a_1)(a_0f(\alpha)^2+a_1f(\alpha x)^2)-(a_0f(\alpha)+a_1f(\alpha x))^2]\\&=\frac{a_0a_1}{a_0+a_1}[f(\alpha)^2+f(\alpha x)^2-2f(\alpha)\cdot f(\alpha x)]\\&=\frac{a_0a_1}{a_0+a_1}[f(\alpha x)-f(\alpha)]^2.\end{aligned}$$

which is positive since f is increasing. □

We close this section with the remark that the proof of Theorem 4 does not require that P be closed under multiplication, but only that the square of a positive element is positive.

3 The Proof of Theorem 3

In this section we will prove Theorem 3. We will deduce it from the following result, which is of independent interest.

Theorem 7 *Suppose* (A,P) *is a partially ordered algebra,* μ *satisfies* (1), *and* $f:2^S\to A$ *is an increasing function. Then the following expression is positive*

$$\Phi(\mu,S,f):=\sum_{\{\omega,\omega^c\}}\mu(\omega)\mu(\omega^c)[f(\omega)-f(\omega^c)]^2$$

(*where the sum ranges over all unordered pairs of complementary subsets of* S.)

Proof If $\mu(\omega)\mu(\omega^c) = 0$ for all ω, then $\Phi(\mu, S, f) = 0$. Therefore we may assume that $M = \sum_\omega \mu(\omega)\mu(\omega^c)$ is positive. Now define

$$\nu(\omega) = \frac{\mu(\omega)\mu(\omega^c)}{M} \quad \text{and} \quad g(\omega) = f(\omega) - f(\omega^c).$$

An easy calculation shows that ν is a probability measure satisfying (1), and that g is an increasing function with $E_\nu(g) = 0$. Hence by Theorem 4 $V_\nu(g) = E_\nu(g^2)$ is positive. But this may be rewritten as follows:

$$\begin{aligned} E_\nu(g^2) &= \sum_\omega \frac{\mu(\omega)\mu(\omega^c)}{M}[f(\omega) - f(\omega^c)]^2 \\ &= \frac{2}{M}\sum_{\{\omega,\omega^c\}} \mu(\omega)\mu(\omega^c)[f(\omega) - f(\omega^c)]^2 = \frac{2}{M}\Phi(\mu, S, f). \end{aligned}$$

Therefore it follows that $\Phi(\mu, S, f)$ is positive. □

Again, the proof of this theorem merely requires that P be closed under squaring, rather than multiplication. We now make two observations needed in the proof below. The first remark is that the previous result holds even if μ is not a probability measure, but merely a positive measure satisfying (1)—this is completely obvious. The second remark is that if f_1, f_2 are increasing and P-valued functions, then the product $f_1 f_2$ is also increasing and P-valued. To see this, we choose $\alpha \supseteq \beta$, and calculate as follows:

$$f_1(\alpha)f_2(\alpha) - f_1(\beta)f_2(\beta) = f_1(\alpha)[f_2(\alpha) - f_2(\beta)] + [f_1(\alpha) - f_1(\beta)]f_2(\beta).$$

Since f_1, f_2 are increasing and P-valued, it follows that the expression above positive.

We are now ready to prove Theorem 3.

Proof (of Theorem 3) Since A is associative and commutative, we obtain

$$\begin{aligned} &E_\mu(f^2)E_\mu(g^2) - E_\mu(fg)^2 \\ &\quad = \left(\sum \mu(\alpha)f(\alpha)^2\right)\left(\sum \mu(\beta)g(\beta)^2\right) - \left(\sum \mu(\alpha)f(\alpha)g(\alpha)\right)^2 \\ &\quad = \sum_{\alpha,\beta} \mu(\alpha)\mu(\beta)f(\alpha)^2 g(\beta)^2 - \sum_{\alpha,\beta} \mu(\alpha)\mu(\beta)f(\alpha)g(\alpha)f(\beta)g(\beta) \\ &\quad = \sum_{\{\alpha,\beta\}} \mu(\alpha)\mu(\beta)[f(\alpha)g(\beta) - g(\alpha)f(\beta)]^2. \end{aligned}$$

Grouping together all the terms with a fixed union and intersection, we can rewrite this as follows:

$$\sum_{\omega_0\subseteq\omega_1} \sum_{\substack{\{\alpha,\beta\} \\ \alpha\cap\beta=\omega_0,\alpha\cup\beta=\omega_1.}} \mu(\alpha)\mu(\beta)[f(\alpha)g(\beta) - g(\alpha)f(\beta)]^2,$$

and it suffices to show that the inner sum is positive for each $\omega_0 \subseteq \omega_1$.

Each term of the inner sum corresponds to a pair of complementary subsets of the set $T = \omega_1 \setminus \omega_0$. More precisely we can write

$$\alpha = \gamma \cup \omega_0, \beta = \gamma^c \cup \omega_0,$$

where γ is a subset of T and γ^c denotes its complement $T \setminus \gamma$. Define a measure ν and a function h on 2^T by

$$\nu(\gamma) = \mu(\gamma \cup \omega_0),$$
$$h(\gamma) = f(\gamma \cup \omega_0) g(\gamma^c \cup \omega_0).$$

Then the inner sum can be rewritten as

$$\sum_{\{\gamma,\gamma^c\}} \nu(\gamma)\nu(\gamma^c)[h(\gamma) - h(\gamma^c)]^2 = \Phi(\nu, T, h),$$

where Φ is as in the previous theorem.

Since g is a decreasing P-valued function, $g(\gamma^c \cup \omega_0)$ is an increasing P-valued function of γ. Hence by the remark preceding the proof, h is also an increasing function. Moreover the measure ν, although not necessarily a probability measure, clearly satisfies (1). Therefore by the previous theorem, the expression $\Phi(\nu, T, h)$ is positive. □

4 The Ahlswede-Daykin Inequality

The Ahlswede-Daykin "four function theorem" [1] asserts that if a, b, c, d: $2^S \to \mathbb{R}_{\geq 0}$ are four functions such that for all $a, \beta \subseteq S$,

$$a(\alpha \cup \beta) b(\alpha \cap \beta) - c(\alpha) d(\beta) \geq 0 \tag{3}$$

then the following inequality holds:

$$\left[\sum_\alpha a(\alpha)\right]\left[\sum_\alpha b(\alpha)\right] - \left[\sum_\alpha c(\alpha)\right]\left[\sum_\alpha d(\alpha)\right] \geq 0. \tag{4}$$

This easily implies the FKG inequality by taking

$$a = fg\mu, \qquad b = \mu, \qquad c = f\mu, \qquad d = g\mu.$$

Therefore it is natural to ask whether this results holds in the present setting of partially ordered algebras. Unfortunately this turns out to be false, as the following example shows.

Let A be the polynomial algebra $\mathbb{R}[x]$ and let P be the set of polynomials with positive coefficients. Let S be a set with a single element, so that 2^S has two elements $\{S, \emptyset\}$. We now define four functions a, b, c, d from $2^S \to P$. In order to simplify the notation we write a_0, a_1 etc. instead of $a(\emptyset), a(S)$ etc. Our functions are defined as follows:

$$\begin{array}{lll} a_0 = x^2, & b_0 = 1, & c_0 = d_0 = x, \\ a_1 = x^3 + 2x^2 + x + 1, & b_1 = x + 1, & c_1 = d_1 = x^2 + x + 1. \end{array}$$

We verify that these functions satisfy the Ahlswede-Daykin hypotheses (3). There are 3 expressions to calculate, viz.

$$\begin{aligned} a_0 b_0 - c_0^2 &= x^2 - x^2 = 0, \\ a_1 b_0 - c_1 c_0 &= (x^3 + 2x^2 + x + 1) - x(x^2 + x + 1) = x^2 + 1, \\ a_1 b_1 - c_1^2 &= (x^3 + 2x^2 + x + 1)(x + 1) - (x^2 + x + 1)^2 \\ &= (x^4 + 3x^3 + 3x^2 + 2x + 1) - (x^4 + 2x^3 + 3x^2 + 2x + 1) \\ &= x^3. \end{aligned}$$

These all belong to P as desired.

However the Ahlswede-Daykin conclusion (4) does not hold, since we have

$$\begin{aligned} &(a_0 + a_1)(b_0 + b_1) - (c_0 + c_1)^2 \\ &\quad = (x^3 + 3x^2 + x + 1)(x + 2) - (x^2 + 2x + 1)^2 \\ &\quad = (x^4 + 5x^3 + 7x^2 + 3x + 2) - (x^4 + 4x^3 + 6x^2 + 4x + 1) \\ &\quad = x^3 + x^2 - x + 1, \end{aligned}$$

which does not belong to P.

Why does this inequality fail to hold? A key step in the proof of the Ahlswede-Daykin inequality is the following elementary fact for real numbers: if a, b, c, d are positive real numbers such that $a \geq c, d$ and $ab \geq cd$ then

$$a + b - c - d \geq 0.$$

Indeed this is obvious if $a = 0$ (which forces $c = d = 0$); while if $a \neq 0$, then we calculate

$$\begin{aligned} a(a + b - c - d) &= a^2 + ab - ac - ad \\ &\geq a^2 + cd - ac - ad \\ &= (a - c)(a - d) \geq 0 \end{aligned}$$

and the desired result follows upon dividing by a.

It is this last division step that breaks down in the general setting. In a partially ordered algebra the quotient of two positive elements need not be positive. For instance, in the polynomial algebra we have

$$(x^3 + 1)/(x + 1) = x^2 - x + 1.$$

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