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Binomial coefficients and Littlewood-Richardson coefficients for interpolation polynomials and Macdonald polynomials

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Dedicated to Gregg Zuckerman on his 60th birthday.

Abstract. We establish a precise relationship between binomial coefficents and Littlewood-Richardson coefficients for interpolation polynomials and Macdonald polynomials, and obtain explicit formulas for both kinds of coefficients.

Introduction

Let $\mathbb{F} = \mathbb{Q}(q,t)$ denote the field of rational functions in q,t. In ([**18**], [**6**], [**19**], [**4**]) the author and F. Knop introduced two inhomogeneous polynomial bases

$$\{G_\eta : \eta \in \mathcal{C}_n\} \subset \mathbb{F}[x_1, \ldots, x_n], \ \{R_\lambda : \lambda \in \mathcal{P}_n\} \subset \mathbb{F}[x_1, \ldots, x_n]^{S_n} \tag{0.1}$$

whose index sets are, respectively, compositions and partitions of length n:

$$\mathcal{C}_n := \{\eta = (\eta_1, \ldots, \eta_n) : \eta_i \in \mathbb{Z}_{\geq 0}\}, \ \mathcal{P}_n := \{\lambda \in \mathcal{C}_n : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}.$$

R_λ and G_η are called interpolation polynomials and, as shown in ([**19**], [**4**]), their top degree terms are, respectively, the symmetric and nonsymmetric Macdonald polynomials of type A ([**12**], [**17, 3, 11**]).

In this paper we prove several new results about R_λ and G_η. We first introduce common notation to avoid having to state the results twice. Thus we write

$$\{h_v(x) : v \in L\} \subset \mathcal{R}$$

to denote *either* of the two situations in (0.1).

The index set L admits a partial order $\supseteq$, which, together with the "rank" function $|v| = v_1 + \cdots + v_n$, makes L into a graded poset ([**2**]). Furthermore there is a certain map $u \mapsto \overline{u} : L \to \mathbb{F}^n$ such that h_v is characterized as the unique polynomial in $\mathcal{R}$ of degree $|v|$ satisfying

$$h_v(\overline{v}) = 1; \text{ and } h_v(\overline{u}) = 0 \text{ unless } u \supseteq v. \tag{0.2}$$

We refer the reader to sections 0.3 and 1.1 for precise definitions of $\supseteq$ and $u \mapsto \overline{u}$ in the symmetric and non-symmetric cases, respectively.

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0.1. Binomial coefficients. Our first result is a formula for the special values $h_v(\bar{u})$, which are called *binomial coefficients* in [**15, 21**]; we define

$$b_{uv} := h_v(\bar{u}) \tag{0.3}$$

We denote by $:\supset$ the covering relation of $\supseteq$; thus we have

$$u :\supset v \text{ iff } u \supseteq v \text{ and } |u| = |v| + 1$$

The b_{uv} are *explicitly* known if $u :\supset v$ (see [**1, 15**] and formulas (0.15),(0.18) below); to emphasize this fact we write

$$a_{uv} = \begin{cases} b_{uv} & \text{if } u :\supset v \\ 0 & \text{else} \end{cases} .$$

Consider the $L \times L$ matrices $A = (a_{uv})$, $B = (b_{uv})$, and the diagonal matrix $Z = (|\overline{u}|\,\delta_{uv})$, where we define for any n-tuple, e.g. for $y \in \mathbb{F}^n$

$$|y| := y_1 + \cdots + y_n$$

By (0.2), (0.3) B is unitriangular and hence invertible. We denote its inverse by

$$B^{-1} = (b'_{uv})$$

Theorem 0.1. .

(1) *The following recursions characterize* b_{uv} *and* b'_{uv}*:*

$$\text{(i) } b_{uu} = 1, \text{ (ii) } (|\overline{u}| - |\overline{v}|)b_{uv} = \textstyle\sum_{w:\supset v} b_{uw}(|\overline{w}| - |\overline{v}|)a_{wv}. \tag{0.4}$$

$$\text{(i) } b'_{uu} = 1, \text{ (ii) } (|\overline{u}| - |\overline{v}|)b'_{uv} = \textstyle\sum_{w\subset: u} a_{uw}(|\overline{w}| - |\overline{u}|)b'_{wv}. \tag{0.5}$$

(2) *The matrices* A, B, Z *satisfy the commutation relations*

$$\text{(i) } [Z, B] = B[Z, A], \text{ (ii) } \left[Z, B^{-1}\right] = -[Z, A]B^{-1} \tag{0.6}$$

(3) *Let* $\mathfrak{C}_{uv} := \{\mathbf{w} = (w_0, w_1, \cdots, w_k) \mid w_0 = u, w_k = v, w_i :\supset w_{i+1}\}$*; then*

$$b_{uv} = \textstyle\sum_{\mathbf{w}\in\mathfrak{C}_{uv}} wt(\mathbf{w}) \quad \text{with } wt(\mathbf{w}) = \prod_{i=0}^{k-1}\left[\frac{|\overline{w_i}| - |\overline{w_{i+1}}|}{|\overline{w_0}| - |\overline{w_{i+1}}|} a_{w_i, w_{i+1}}\right]. \tag{0.7}$$

$$b'_{uv} = \textstyle\sum_{\mathbf{w}\in\mathfrak{C}_{uv}} wt'(\mathbf{w}) \quad \text{with } wt'(\mathbf{w}) = \prod_{i=0}^{k-1}\left[\frac{|\overline{w_{i+1}}| - |\overline{w_i}|}{|\overline{w_i}| - |\overline{w_k}|} a_{w_i, w_{i+1}}\right]. \tag{0.8}$$

0.2. Littlewood Richardson coefficients. Our second result concerns the *Littlewood Richardson* coefficients $c_{uv} := c_{uv}(p)$, which are defined for each $p \in \mathcal{R}$ by the product expansion

$$p(x)\,h_v(x) = \textstyle\sum_u c_{uv} h_u(x). \tag{0.9}$$

Theorem 0.2. *The following recursion characterizes* $c_{uv} := c_{uv}(p)$*:*

(0.10)

$$\text{(i) } c_{uu} = p(\overline{u}) \text{ (ii) } [|\overline{u}| - |\overline{v}|]c_{uv} = \sum_{w:\supset v} c_{uw}[|\overline{w}| - |\overline{v}|]a_{wv} - \sum_{w\subset: u}[|\overline{u}| - |\overline{w}|]a_{uw}c_{wv}$$

The matrices $C = C(p) = (c_{uv})$ *and* $D = D(p) = (p(\overline{u})\,\delta_{uv})$ *satisfy:*

$$\text{(i) } C = B^{-1}DB, \text{ (ii) } [Z, C] = [C, [Z, A]]. \tag{0.11}$$

Of special interest are the Littlewood Richardson coefficients for h_w, which are defined as follows:

$$c^u_{vw} := c_{uv}(h_w). \tag{0.12}$$

These can be expressed entirely in terms of binomial coefficients. Define

$$\mathfrak{C}^u_{vw}(z) = \mathfrak{C}_{uz} \times \mathfrak{C}_{zw} \times \mathfrak{C}_{zv}, \ \mathfrak{C}^u_{vw} = \cup_z \mathfrak{C}^u_{vw}(z)$$

and for $\omega = \left(\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3\right) \in \mathfrak{C}^u_{vw}$ define

$$wt\left(\omega\right) = wt'\left(\mathbf{w}^1\right) wt\left(\mathbf{w}^2\right) wt\left(\mathbf{w}^3\right) \ .$$

THEOREM 0.3. *The coefficient c^u_{vw} is given explicitly as follows:*

$$c^u_{vw} = \sum\nolimits_z b'_{uz} b_{zw} b_{zv} = \sum\nolimits_{\omega \in \mathfrak{C}^u_{vw}} wt\left(\omega\right) \tag{0.13}$$

0.3. The symmetric case. We now make the above results explicit in the symmetric case, and give an application of Theorem 0.3 to symmetric Macdonald polynomials.

DEFINITION 0.4. For $\lambda \in \mathcal{P}_n$ we define

$$\bar{\lambda} = \left(\bar{\lambda}_1, \cdots, \bar{\lambda}_n\right) \text{ where } \bar{\lambda}_i = q^{\lambda_i} t^{1-i}$$

For $\lambda, \mu \in \mathcal{P}_n$ we write $\lambda \supseteq \mu$ if $\lambda_i \leq \mu_i$ for all i, so that the diagram of λ contains that of μ. We write $\lambda :\supset \mu$ if $\lambda \supseteq \mu$ and $|\lambda| = |\mu| + 1$.

By [**19**], [**4**] for each $\lambda \in \mathcal{P}_n$ there exists a unique polynomial $R_\lambda\left(x\right)$ in $\mathbb{F}\left[x_1, \ldots, x_n\right]^{S_n}$ such that

$$\deg\left(R_\lambda\right) = |\lambda|\,,\ R_\lambda\left(\bar{\mu}\right) = \delta_{\lambda\mu} \text{ for } |\mu| \leq |\lambda|$$

DEFINITION 0.5. For $\lambda, \mu \in \mathcal{P}_n$ we define the symmetric binomial coefficient to be $b_{\lambda\mu} = R_\lambda\left(\bar{\mu}\right)$. If $\lambda :\supset \mu$ we write $a_{\lambda\mu} = b_{\lambda\mu}$.

Our result give an explicit formula for $b_{\lambda\mu}$. To state this formula we recall some standard notation related to partitions from [**12**].

The Young diagram of a partition λ is a left-justified array of boxes with λ_i boxes in row i. Transposing the diagram of λ gives the diagram of a new partition, usually denoted λ', such that λ'_j is the length of the jth column of the diagram of λ. If $s = (i, j)$ is the box in row i and column j; we define the *arm* and *leg* of s to be

$$a\left(s\right) = \lambda_i - j,\ l\left(s\right) = \lambda'_j - i.$$

and we define the (q, t)-*hooklengths* of λ as in [**12**, VI.8.1,1']:

$$c_\lambda(s) = 1 - q^{a(s)} t^{l(s)+1},\ c_\lambda = \prod\nolimits_{s\in\lambda} c_\lambda(s) \tag{0.14}$$
$$c'_\lambda(s) = 1 - q^{a(s)+1} t^{l(s)},\ c'_\lambda = \prod\nolimits_{s\in\lambda} c'_\lambda(s)$$

If $\lambda \supseteq \mu$ we write λ/μ for the "skew" diagram consisting of the boxes in λ which are not in μ. If $\lambda :\supset \mu$ then λ/μ consists of a single box.

A standard skew tableau of shape λ/μ is a labelling of the boxes of λ/μ by the numbers $1, 2, \cdots, k$ where $k = |\lambda| - |\mu|$, such that the labels increase from left to right along each row and from top to bottom along each column. We write $ST_{\lambda/\mu}$ for the set of such tableaux, which can also be regarded as sequences of partitions

$$\lambda = \lambda^0 :\supset \lambda^1 :\supset \cdots :\supset \lambda^k = \mu$$

where λ^i is obtained from λ by deleting the boxes with labels $1, \cdots, i$.

THEOREM 0.6. *If $\lambda :\supset \mu$ with $\lambda/\mu = (i, j)$, let R_i and C_j denote the (other) boxes in row i and column j, respectively, then we have*

$$a_{\lambda\mu} = t^{1-i} \prod_{s\in C_j} \frac{c_\lambda(s)}{c_\mu(s)} \prod_{s\in R_i} \frac{c'_\lambda(s)}{c'_\mu(s)}. \tag{0.15}$$

Definition 0.7. If $T \in ST_{\lambda/\mu}$ with $T = \left(\lambda = \lambda^0 :\supset \lambda^1 :\supset \cdots :\supset \lambda^k = \mu\right)$ we define

$$wt\left(T\right) = \prod_{i=0}^{k-1} \left[\frac{\left|\overline{\lambda^i}\right| - \left|\overline{\lambda^{i+1}}\right|}{\left|\overline{\lambda}\right| - \left|\overline{\lambda^{i+1}}\right|} a_{\lambda_i,\lambda_{i+1}}\right],\ wt'\left(T\right) = \prod_{i=0}^{k-1} \left[\frac{\left|\overline{\lambda^i}\right| - \left|\overline{\lambda^{i+1}}\right|}{\left|\overline{\mu}\right| - \left|\overline{\lambda^i}\right|} a_{\lambda_i,\lambda_{i+1}}\right]$$

Theorem 0.8. *If $\lambda \not\supseteq \mu$ then $b_{\lambda\mu} = 0$. If $\lambda \supseteq \mu$ then we have*

$$b_{\lambda\mu} = \sum\nolimits_{T \in ST_{\lambda/\mu}} wt\left(T\right)$$

Moreover if we define

$$b'_{\lambda\mu} := \sum\nolimits_{T \in ST_{\lambda/\mu}} wt'\left(T\right)$$

then we have

$$\sum\nolimits_{\mu} b_{\lambda\mu} b'_{\mu\nu} = \delta_{\lambda\nu}$$

Definition 0.9. For $p\left(x\right) \in \mathbb{F}\left[x_1, \ldots, x_n\right]^{S_n}$ we define its Littlewood-Richardson coefficients $c_{\lambda\mu} = c_{\lambda\mu}\left(p\right)$ via the product expansion

$$p\left(x\right) R_\mu\left(x\right) = \sum\nolimits_\lambda c_{\lambda\mu} R_\lambda\left(x\right).$$

We also define

$$c_{\mu\nu}^\lambda = c_{\lambda\mu}\left(R_\nu\right) = c_{\lambda\nu}\left(R_\mu\right)$$

Theorem 0.10. *The coefficients $c_{\lambda\mu} = c_{\lambda\mu}\left(p\right)$ are characterized as follows*

$$\text{(i) } c_{\lambda\lambda} = p\left(\overline{\lambda}\right) \quad \text{(ii) } [|\overline{\lambda}| - |\overline{\mu}|]c_{\lambda\mu} = \sum_{\nu :\supset \mu} c_{\lambda\nu}[|\overline{\nu}| - |\overline{\mu}|]a_{\nu\mu} - \sum_{\nu \subset: \lambda} [|\overline{\lambda}| - |\overline{\nu}|]a_{\lambda\nu}c_{\nu\mu}$$

Moreover we have

$$c_{\mu\nu}^\lambda = \sum\nolimits_\kappa b'_{\lambda\kappa} b_{\kappa\mu} b_{\kappa\nu}$$

0.4. Macdonald polynomials. We now give an application of Theorem 0.10 to Macdonald polynomials.

Let $J_\lambda\left(x; q, t\right)$ be the "integral form" of the symmetric Macdonald polynomial as in [**12**, VI.8.3]. The J_λ are orthogonal with respect to the (q, t)-inner product $\langle ., . \rangle$ defined in [**12**, VI.1.5]. By [**12**, VI.8.7] we have

$$\langle J_\lambda, J_\mu \rangle = j_\lambda \delta_{\lambda\mu} \text{ where } j_\lambda = c_\lambda c'_\lambda \tag{0.16}$$

Using Theorem 0.3 we can obtain an explicit formula for the scalar product $\langle J_\lambda, J_\mu J_\nu \rangle$.

Definition 0.11. For λ, μ, ν in $\mathcal{P}_n$ we define

$$n\left(\lambda\right) = \sum\nolimits_i \left(i-1\right)\lambda_i = \sum\nolimits_{(i,j)\in\lambda} \left(i-1\right) = \sum\nolimits_j \lambda'_j \left(\lambda'_j - 1\right)/2$$
$$n\left(\lambda, \mu, \nu\right) = n\left(\lambda\right) - n\left(\mu\right) - n\left(\nu\right)$$

Theorem 0.12. *We have*

$$\langle J_\lambda, J_\mu J_\nu \rangle = \begin{cases} c_{\mu\nu}^\lambda j_\mu j_\nu q^{-n\left(\lambda', \mu', \nu'\right)} t^{2n(\lambda,\mu,\nu)} & \text{if } |\lambda| = |\mu| + |\nu| \\ 0 & \text{else} \end{cases} \tag{0.17}$$

0.5. Remarks.

(1) The definitions and notations for the symmetric interpolation polynomials are slightly different in [**19**], [**4**], and [**15**]. The precise connection between these definitions is explained on P. 471 of [**21**].

(2) The nonsymmetric analog of Theorem 0.6 is contained in [**1**, Cor 4.2], and we give a concise reformulation. Suppose $\eta :\supset \gamma \in \mathcal{C}_n$, let $1 \leq i_1 < \ldots < i_k \leq n$ be the corresponding indices as in (1.1), and for $1 \leq j \leq n$ define constants $a_j, a'_j \in \mathbb{F}$ as follows:

$$a_j = \begin{cases} \overline{\gamma}_{i_l} & j \in [i_{l-1}, i_l) \\ q\overline{\gamma}_{i_1} & j \geq i_k \end{cases} , \quad a'_j = \begin{cases} \overline{\gamma}_{i_l} & j \in (i_{l-1}, i_l] \\ q\overline{\gamma}_{i_1} & j > i_k \end{cases} .$$

Then we have

$$a_{\eta\gamma} = \frac{a_n - t^{1-n}}{1-t} \cdot \prod_{j=1}^{n} \frac{a'_j - t\overline{\gamma}_j}{a_j - \overline{\gamma}_j} \tag{0.18}$$

The analogs of Theorems 0.8 and 0.10 are straightforward.

(3) The nonsymmetric analog of Theorem 0.12 involves three steps. We sketch the argument below and we leave the details to the interested reader.
- The first step is to define the analog of the (q,t)-scalar product for nonsymmetric Macdonald polynomials. This involves a reinterpretation of the results of [**14**] along the lines of [**20**]. Note however that the natural scalar product is *Hermitian* (with $q^* = q^{-1}, t^* = t^{-1}$).
- The second step is to define the integral form of the nonsymmetric Macdonald polynomials and compute its norm explicitly.
- Finally one needs to compute the precise normalization constant relating the integral nonsymmetric Macdonald polynomial and the top term of the nonsymmetric interpolation polynomial.

(4) The results of this paper in the limiting case of Jack polynomials were obtained in [**22**].

1. Proofs of Theorems 0.1, 0.2, 0.3

1.1. Preliminaries. In this section we recall the definition of the partial order $\supseteq$ and the map $u \mapsto \overline{u}$ on the index set L. For $L = \mathcal{P}_n$ these are defined as in Definition 0.4.

For $L = \mathcal{C}_n$, the definition of $\supseteq$ is due to [**4**]. For γ, η in $\mathcal{C}_n$, we write $\eta :\supset \gamma$ if there are indices $1 \leq i_1 < \ldots < i_k \leq n$ such that

$$\eta_i = \begin{cases} \gamma_{i_1} + 1 & \text{if } i = i_k \\ \gamma_{i_{j+1}} & \text{if } i = i_j,\ j < k \\ \gamma_i & \text{otherwise} \end{cases} \tag{1.1}$$

Definition 1.1. [**4**] We define the partial order $\supseteq$ on $\mathcal{C}_n$ to be the transitive closure of $:\supset$; conversely $:\supset$ is the covering relation of $\supseteq$.

For $L = \mathcal{C}_n$ the definition of $\overline{u}$ is due to ([**6**], [**19**], [**4**]), and involves the permutation action of the symmetric group S_n on n-tuples (in $\mathcal{C}_n, \mathbb{F}^n, \mathbb{Z}^n$, etc.). The S_n-orbit of $\eta \in \mathcal{C}_n$ contains a unique partition that we denote η_+. The set $\{\sigma \in S_n : \sigma(\eta_+) = \eta\}$ contains a unique element of minimal length that we denote by σ_η. (Here, as usual, the length of a permutation σ is the number of σ-inversions, i.e. pairs of indices $1 \leq i < j \leq n$ such that $\sigma(i) > \sigma(j)$.)

DEFINITION 1.2. For η in $\mathcal{C}_n$ we define $\overline{\eta} \in \mathbb{F}^n$ to be

$$\overline{\eta} := \sigma_\eta \left(\overline{\eta_+}\right) \tag{1.2}$$

REMARK 1.3. The restrictions of $(\supseteq, u \mapsto \overline{u})$ from $\mathcal{C}_n$ to $\mathcal{P}_n$ *agree* with the corresponding structures on $\mathcal{P}_n$.

Let L denote $\mathcal{C}_n$ or $\mathcal{P}_n$, and let $\mathcal{R}$ denote $\mathbb{F}[x_1, \ldots, x_n]$ or $\mathbb{F}[x_1, \ldots, x_n]^{S_n}$ accordingly. We recall that $|u| := u_1 + \cdots + u_n$, and for $d \in \mathbb{Z}_{\geq 0}$ we define

$$\mathcal{R}_d = \{p \in \mathcal{R} \mid \deg(p) \leq d\}, \ L_d = \{u \in L \mid |u| \leq d\}, \ \bar{L}_d = \{\overline{u} \mid u \in L_d\}$$

The following result is key to the definition of interpolation polynomials h_v.

PROPOSITION 1.4. [**19, 4**] *A polynomial in $\mathcal{R}_d$ is determined by its values on $\bar{L}_d$.*

We briefly sketch the argument. In the symmetric case the main idea goes back to [**18**] and arose in connection with author's joint work with B. Kostant [**8, 9**] on the Capelli identity. Evaluation gives a linear map $Ev : \mathcal{R}_d \to \mathbb{F}^{\bar{L}_d}$ and the proposition asserts that this is an isomorphism. We first note that both spaces have dimension $\#(L_d)$; this is obvious for $\mathbb{F}^{\bar{L}_d}$, while for $\mathcal{R}_d$ it follows by expressing a (symmetric) polynomial in terms of (symmetric) monomials. Therefore it suffices to prove that Ev is surjective, which can be carried out by induction on d.

Interpolation polynomials are images of delta functions under Ev^{-1}.

DEFINITION 1.5. $h_v(x)$ is the unique polynomial in $\mathcal{R}_{|v|}$ satisfying

$$h_v(\overline{u}) = \delta_{uv} \text{ for all } u \in L_{|v|}$$

The following "extra" vanishing result relates $h_v(x)$ and $\supseteq$.

PROPOSITION 1.6. [**6, 4**] *We have $h_v(\overline{u}) = 0$ unless $u \supseteq v$.*

1.2. Proofs. The proof of Theorem 0.1 depends on the following simple identity for $h_v(x)$.

PROPOSITION 1.7. *Let $|x|$ denote $x_1 + \cdots + x_n$, then we have*

$$(|x| - |\overline{v}|)\, h_v(x) = \textstyle\sum_{w:\supset v} (|\overline{w}| - |\overline{v}|) a_{wv} h_w(x) \tag{1.3}$$

PROOF. Both sides of (1.3) are polynomials of degree $d = |v| + 1$. By Proposition 1.4 it suffices to show that they agree on $\bar{L}_d$. Now let $x = \overline{u}$, then by formula (0.2) both sides vanish if $|u| < d$ and both become $(|\overline{u}| - |\overline{v}|)a_{uv}$ if $|u| = d$. □

PROOF OF THEOREM 0.1. We first prove (0.4). By formulas (0.2), (0.3) we get $b_{uu} = h_u(\overline{u}) = 1$, which is (0.4i). Next (0.4ii) follows from Proposition 1.7 by setting $x = \overline{u}$ in (1.3) and using formulas (0.2), (0.3). Finally (0.4) characterizes b_{uv} by induction on $|u| - |v|$.

Next note that (0.6i) is equivalent (0.4ii), and (0.6ii) is equivalent (0.5ii). Also (0.6ii) is equivalent to (0.6i) since

$$\left[Z, B^{-1}\right] = -B^{-1}(ZB - BZ)B = -B^{-1}[Z, B]B$$

This proves (0.6) and (0.5ii). Now (0.5i) is obvious, and (0.5) characterizes b'_{uv} by induction on $|u| - |v|$.

We next prove (0.7). Let $\bar{b}_{uv}$ temporarily denote the sum in (0.7), It suffices to verify that $\bar{b}_{uv}$ satisfies the recursion (0.4). Now (0.4i) holds since $\bar{b}_{uu}$ involves the single chain $\mathbf{w} = (u, u)$ whose weight is the empty product 1. For (0.4ii) we observe that

$$wt(\mathbf{w}) = wt(\bar{\mathbf{w}}) \frac{|\overline{w_{k-1}}| - |\overline{v}|}{|\overline{u}| - |\overline{v}|} a_{w_{k-1},v} \text{ where } \bar{\mathbf{w}} = (w_0, w_1, \cdots, w_{k-1})$$

Therefore collecting the terms in (0.7) with $w_{k-1} = w$, we get

$$\bar{b}_{uv} = \sum_{w:\supset v} \left[\sum_{\bar{\mathbf{w}} \in \mathfrak{C}_{uw}} wt(\bar{\mathbf{w}}) \right] \frac{|\overline{w}| - |\overline{v}|}{|\overline{u}| - |\overline{v}|} a_{wv} = \sum_{w:\supset v} \bar{b}_{uw} \frac{|\overline{w}| - |\overline{v}|}{|\overline{u}| - |\overline{v}|} a_{wv},$$

which is (0.4ii). Therefore $\bar{b}_{uv} = b_{uv}$ for all u, v. The proof of (0.8) is similar. □

Proof of Theorem 0.2. We first prove (0.11). Substituting $x = \overline{w}$ in (0.9) we get

$$p(\overline{w}) h_v(\overline{w}) = \sum_u c_{uv} h_u(\overline{w}).$$

By (0.2,0.3) this becomes

$$d_{ww} b_{wv} = \sum_u b_{wu} c_{uv},$$

Hence we obtain the matrix identity $DB = BC$, which is equivalent to (0.11i).

To prove (0.11ii) we calculate as follows:

$$[Z, C] = \left[Z, B^{-1}DB\right] = \left[Z, B^{-1}\right] DB + B^{-1}[Z, D] B + B^{-1}D[Z, B]$$

The middle term vanishes since Z and D are both diagonal matrices. The first and last terms can be computed by formula (0.6) and we get

$$[Z, C] = -[Z, A] B^{-1}DB + B^{-1}DB[Z, A] = -[Z, A] C + C[Z, A] = [C, [Z, A]]$$

We now prove (0.10). Since B is unitriangular, (0.11i) implies that C and D have the same diagonal entries, which is (0.10i). Next (0.10ii) is equivalent to (0.11ii). Finally (0.11) characterizes c_{uv} by induction on $|u| - |v|$. □

Proof of Theorem 0.3. For $p = h_w$, the diagonal matrix $D = D(h_w)$ has diagonal entries $d_{zz} = h_w(\bar{z}) = b_{zw}$. By formula (0.11ii) we have

$$c^u_{vw} = \sum_z b'_{uz} d_{zz} b_{zv} = \sum_z b'_{uz} b_{zw} b_{zv}$$

which is the first equality in (0.13). The second equality follows from (0.7), (0.8). □

2. Proofs of Theorems 0.6, 0.8, 0.10, 0.12

2.1. Preliminaries. In this section we recall some basic results on the symmetric interpolation polynomials $R_\lambda(x)$, which are needed for the proofs of Theorems 0.6, 0.8, 0.10, 0.12 below.

We write $\mathcal{R}^{(n)}_d := \left\{ p \in \mathbb{F}[x_1, \ldots, x_n]^{S_n} \mid \deg(p) \le d \right\}$ and define the symmetrized monomials

$$m_\lambda = \sum_{\sigma \in D(\lambda)} x_1^{\sigma_1} \cdots x_n^{\sigma_n} \text{ for } \lambda \in \mathcal{P}_n$$

where $D(\lambda)$ denotes the set of all distinct rearrangements of λ. Also define maps $\omega : \mathcal{R}_d^{(n-1)} \to \mathcal{R}_d^{(n)}$, $\tau : \mathcal{R}_d^{(n)} \to \mathcal{R}_d^{(n)}$, $\tau' : \mathcal{R}_d^{(n-1)} \to \mathcal{R}_d^{(n-1)}$, $\upsilon : \mathcal{R}_d^{(n)} \to \mathcal{R}_{d+n}^{(n)}$

$$\omega\left(m_{\lambda_1,\ldots,\lambda_{n-1}}\right) = m_{\lambda_1,\ldots,\lambda_{n-1},0}, \text{ extended by linearity} \tag{2.1}$$
$$(\tau f)(x_1,\ldots,x_n) = f\left(x_1 - t^{1-n},\ldots,x_n - t^{1-n}\right)$$
$$(\tau' f)(x_1,\ldots,x_{n-1}) = f\left(x_1 + t^{1-n},\ldots,x_{n-1} + t^{1-n}\right)$$
$$(\upsilon f)(x) = f\left(q^{-1}x\right)\prod\nolimits_{j=1}^{n}\frac{x_i - t^{1-n}}{\bar{\lambda}_i - t^{1-n}}$$

Proposition 2.1. *If* $\lambda \in \mathcal{P}_n$ *with* $|\lambda| = d$ *there is a unique* $R_\lambda(x) \in \mathcal{R}_d^{(n)}$ *with*

$$R_\lambda(\overline{\mu}) = \delta_{\lambda\mu} \text{ for all partitions } \mu \text{ with } |\mu| \leq d \tag{2.2}$$

Moreover if $\lambda_n > 0$ *, then*

$$R_\lambda = \upsilon(R_{\lambda-\varepsilon}) \text{ for } \varepsilon = (1,\cdots,1) \tag{2.3}$$

If $\lambda_n = 0$ *then there is a unique* $S(x) \in \mathcal{R}_{d-n}^{(n)}$ *such that*

$$R_\lambda = (\tau\omega\tau')(R_{\lambda^-}) + \upsilon(S) \text{ for } \lambda^- = (\lambda_1,\ldots,\lambda_{n-1}) \tag{2.4}$$

This is proved in ([**19**], [**4**]). The function $S(x)$ is chosen by induction so that the right side of (2.4) vanishes for $\bar{\mu}$ if $|\mu| \leq d$ and $\mu_n > 0$. One verifies that (2.3) and (2.4) serve to define $R_\lambda(x)$ by induction on $n + |\lambda|$.

As shown in ([**19**], [**4**]), the polynomials $R_\lambda(x)$ are eigenfunctions for certain difference operators. We recall the result below:

Proposition 2.2. *Let* D_1 *be the operator defined by*

$$D_1 = \sum\nolimits_k A_k(x)(1 - T_k).$$

where

$$A_k(x) = \left(1 - t^{1-n}x_k^{-1}\right)\prod\nolimits_{l\neq k}\frac{x_k - tx_l}{x_k - x_l}$$

and T_k *is the k-th* q^{-1}*-shift operator*

$$T_k f(x_1,\ldots,x_n) = f\left(x_1,\ldots,q^{-1}x_k,\ldots,x_n\right)$$

then we have

$$D_1 R_\mu(x) = \left[\sum\nolimits_k\left(t^{k-1} - \bar{\mu}_k^{-1}\right)\right]R_\mu(x) \tag{2.5}$$

2.2. Proofs.

Proof of Theorem 0.6. We evaluate (2.5) at $x = \bar{\lambda}$ to get

$$\left[\sum\nolimits_k\left(t^{k-1} - \bar{\mu}_k^{-1}\right)\right]R_\mu\left(\bar{\lambda}\right) = \left[\sum\nolimits_k A_k\left(\bar{\lambda}\right)\right]R_\mu\left(\bar{\lambda}\right) - \sum\nolimits_k A_k\left(\bar{\lambda}\right)R_\mu\left(\overline{\lambda - \varepsilon_k}\right)$$

which we rewrite as follows:

$$\left[\sum\nolimits_k A_k\left(\bar{\lambda}\right) - \sum\nolimits_k\left(t^{k-1} - \bar{\mu}_k^{-1}\right)\right]R_\mu\left(\bar{\lambda}\right) = \sum\nolimits_k A_k\left(\bar{\lambda}\right)R_\mu\left(\overline{\lambda - \varepsilon_k}\right) \tag{2.6}$$

As in Lemma 3.5 of [**19**], we have

$$A_k\left(\bar{\lambda}\right)R_\mu\left(\overline{\lambda - \varepsilon_k}\right) = 0 \text{ if } k \neq i$$

and by Lemma 3.3 of [**19**], we have

$$\sum\nolimits_k A_k\left(\bar{\lambda}\right) = \sum\nolimits_k\left(t^{k-1} - \bar{\lambda}_k^{-1}\right)$$

Since $\bar{\mu}_k = \bar{\lambda}_k$ for $k \neq i$, (2.6) can be rewritten as

$$\left(\bar{\mu}_i^{-1} - \bar{\lambda}_i^{-1}\right) R_\mu\left(\bar{\lambda}\right) = A_i\left(\bar{\lambda}\right) = \left(1 - t^{1-n}\bar{\lambda}_i^{-1}\right) \prod\nolimits_{l\neq i} \frac{\bar{\lambda}_i - t\bar{\lambda}_l}{\bar{\lambda}_i - \bar{\lambda}_l}$$

Substituting $\bar{\mu}_i = q^{-1}\bar{\lambda}_i$ and rewriting, we get

$$R_\mu\left(\bar{\lambda}\right) = \frac{\bar{\lambda}_i - t^{1-n}}{q-1} \prod_{l\neq i} \frac{\bar{\lambda}_i - t\bar{\lambda}_l}{\bar{\lambda}_i - \bar{\lambda}_l}$$

To complete the proof of (0.15) it suffices to verify the two identities

$$\prod_{s\in C_j} \frac{c_\lambda(s)}{c_\mu(s)} = \prod_{l<i} \frac{\bar{\lambda}_i - t\bar{\lambda}_l}{\bar{\lambda}_i - \bar{\lambda}_l} \tag{2.7}$$

$$t^{1-i} \prod_{s\in R_i} \frac{c'_\lambda(s)}{c'_\mu(s)} = \frac{\bar{\lambda}_i - t^{1-n}}{q-1} \prod_{l>i} \frac{\bar{\lambda}_i - t\bar{\lambda}_l}{\bar{\lambda}_i - \bar{\lambda}_l}. \tag{2.8}$$

Now C_j consists of boxes $\{(l,j) \mid l < i\}$. For $s = (l,j) \in C_j$ we have

$$a_\lambda(s) = a_\mu(s) = \lambda_l - j \ = \lambda_l - \lambda_i; \qquad l_\lambda(s) = l_\mu(s) + 1 = i - l.$$

$$\frac{c_\lambda(s)}{c_\mu(s)} = \frac{1 - q^{a_\lambda(s)}t^{l_\lambda(s)+1}}{1 - q^{a_\mu(s)}t^{l_\mu(s)+1}} = \frac{1 - q^{\lambda_l - \lambda_i}t^{i-l+1}}{1 - q^{\lambda_l-\lambda_i}t^{i-l}} = \frac{q^{\lambda_i}t^{1-i} - q^{\lambda_l}t^{2-l}}{q^{\lambda_i}t^{1-i} - q^{\lambda_l}t^{1-l}} = \frac{\bar{\lambda}_i - t\bar{\lambda}_l}{\bar{\lambda}_i - \bar{\lambda}_l}$$

which implies (2.7).

We now prove (2.8). Denote the left and right sides of (2.8) by $X(\lambda, i)$ and $Y(\lambda, i)$ respectively. First suppose $j = 1$. Then R_i is the empty set and $X(\lambda, i) = t^{1-i}$. Also we have $\lambda_i = 1$ and $\lambda_l = 0$ for $l > i$ therefore we get

$$Y(\lambda, i) = \frac{qt^{1-i} - t^{1-n}}{q-1} \prod_{l=i+1}^{n} \frac{qt^{1-i} - t^{2-l}}{qt^{1-i} - t^{1-l}} = t^{1-i} = X(\lambda, i)$$

Now suppose $j > 1$. Let k be the largest index such that $\lambda_k > 0$ and define

$$\lambda^* = (\lambda_1 - 1, \cdots, \lambda_k - 1, 0, \cdots, 0)$$

Note that necessarily $k \geq i$. Now we have

$$\frac{X(\lambda, i)}{X(\lambda^*, i)} = \frac{c'_\lambda(1,i)}{c'_\mu(1,i)} = \frac{1 - q^j t^{k-i}}{1 - q^{j-1}t^{k-i}}$$

Also for $l \leq k$ the ratios $\frac{\bar{\lambda}_i - t\bar{\lambda}_l}{\bar{\lambda}_i - \bar{\lambda}_l}$ are unchanged when we replace λ by λ^*. Thus

$$\begin{aligned} \frac{Y(\lambda, i)}{Y(\lambda^*, i)} &= \frac{q^{\lambda_i}t^{1-i} - t^{1-n}}{q^{\lambda_i - 1}t^{1-i} - t^{1-n}} \prod_{l=k+1}^{n} \frac{q^{\lambda_i}t^{1-i} - t^{2-l}}{q^{\lambda_i}t^{1-i} - t^{1-l}} \prod_{l=k+1}^{n} \frac{q^{\lambda_i - 1}t^{1-i} - t^{1-l}}{q^{\lambda_i-1}t^{1-i} - t^{2-l}} \\ &= \frac{q^{\lambda_i}t^{1-i} - t^{2-(k+1)}}{q^{\lambda_i - 1}t^{1-i} - t^{2-(k+1)}} = \frac{1 - q^j t^{k-i}}{1 - q^{j-1}t^{k-i}} = \frac{X(\lambda, i)}{X(\lambda^*, i)} \end{aligned}$$

and the identity $X(\lambda, i) = Y(\lambda, i)$ follows by induction on $|\lambda|$. □

Theorems 0.8, 0.10 now follow from Theorems 0.1, 0.2, respectively. For the proof of Theorems 0.12 we need a preliminary result.

Lemma 2.3. *Let k_λ be the coefficient of $m_\lambda(x)$ in $R_\lambda(x)$; then*

$$k_\lambda = (-1)^{|\lambda|}\, t^{2n(\lambda)} q^{-n\left(\lambda'\right)} / c'_\lambda. \tag{2.9}$$

Proof. We proceed by induction on $n+|\lambda|$. The result is obvious for $n+|\lambda| = 0$, and so we may suppose $n+|\lambda| = 0$.

If $\lambda_n = 0$ then $R_\lambda = (\tau\omega\tau')(R_{\lambda^-}) + \upsilon(S)$ by formula (2.4). Now τ, τ' do not change the leading terms of a polynomial, ω maps m_{λ^-} to m_λ, and the coefficient of m_λ in $\upsilon(S)$ is 0. Therefore we deduce that $k_\lambda = k_{\lambda^-}$, and since the right side of (2.9) is unchanged under passage from λ to λ^- the equality (2.9) holds by induction.

If $\lambda_n > 0$ then let $\mu = \lambda - \varepsilon$. By formula (2.3) we deduce

$$k_\mu/k_\lambda = q^{|\mu|}\textstyle\prod_{j=1}^n \left(\bar{\lambda}_i - t^{1-n}\right) = (-1)^n q^{|\mu|} t^{n(1-n)} \textstyle\prod_{j=1}^n \left(1 - q^{\lambda_i} t^{n-i}\right)$$

Therefore by induction we get

$$k_\lambda = \frac{(-1)^{|\mu|} t^{2n(\mu)} q^{-n(\mu')}/c'_\mu}{(-1)^n q^{|\mu|} t^{n(1-n)} \prod_{j=1}^n (1-q^{\lambda_i} t^{n-i})} = (-1)^{|\lambda|} \frac{t^{2n(\mu)+n(n-1)} q^{-n(\mu')-|\mu|}}{c'_\mu \prod_{j=1}^n (1-q^{\lambda_i} t^{n-i})}$$

To complete the proof its suffices to verify the following identities for $\lambda = \mu + \varepsilon$

$$2n(\lambda) = 2n(\mu) + n(n-1)\,,\ n(\lambda') = n(\mu') + |\mu|\,,\ c'_\lambda = c'_\mu \textstyle\prod_{j=1}^n \left(1 - q^{\lambda_i} t^{n-i}\right)$$

whose (easy) verifications we leave to the reader. □

Proof of Theorem 0.12. For two polynomials $p(x)$, $q(x)$ in $\mathbb{F}[x_1, \ldots, x_n]$ we write $p \sim_d q$ if $p - q$ has total degree $< d$.

As shown in [**12**], the coefficient of m_λ in $J_\lambda(x)$ is c_λ. Therefore if we define

$$r_\lambda = c_\lambda / k_\lambda = (-1)^{|\lambda|} t^{-2n(\lambda)} q^{n(\lambda')} j_\lambda \tag{2.10}$$

then by Lemma 2.3 we get

$$J_\lambda(x) \sim_{|\lambda|} r_\lambda R_\lambda(x)$$

Therefore if $d = |\mu| + |\nu|$ then by Definition 0.9 we get

$$J_\mu J_\nu \sim_d r_\mu r_\nu R_\mu R_\nu = \textstyle\sum_{|\lambda| \le d} r_\mu r_\nu c^\lambda_{\mu\nu} R_\lambda \sim_d \sum_{|\lambda| = d} r_\mu r_\nu r_\lambda^{-1} c^\lambda_{\mu\nu} J_\lambda.$$

Since since the first and last polynomials are homogenous of degree d, they are equal. Therefore by (0.16) we get

$$\langle J_\lambda, J_\mu J_\nu \rangle = \begin{cases} r_\mu r_\nu r_\lambda^{-1} c^\lambda_{\mu\nu} j_\lambda & \text{if } |\lambda| = |\mu| + |\nu| \\ 0 & \text{else} \end{cases}.$$

To complete the proof it suffices to verify that

$$r_\mu r_\nu r_\lambda^{-1} j_\lambda = j_\mu j_\nu q^{-n(\lambda', \mu', \nu')} t^{2n(\lambda,\mu,\nu)} \text{ if } |\lambda| = |\mu| + |\nu|$$

which follows immediately from (2.10). □

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