Linear Algebra and its Applications 432 (2010) 2524–2528

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

A note on the resolvent of a nonnegative matrix and its applications

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ARTICLE INFO

Article history:
Received 24 September 2009
Accepted 6 November 2009
Available online 20 February 2010

Submitted by R. Bhatia

AMS classification:
15B48
15B36
91A40
91B66

Keywords:
Nonnegative matrix
Inverse *M*-matrix
Page-rank
Leontief model

ABSTRACT

We prove a conjecture of Dubey et al. on the change in the resolvent of a nonnegative matrix if its entries are decreased, and discuss applications to mathematical economics.

0. Introduction

Nonnegative matrices arise naturally in many applications and mathematical results about them can lead to fundamental insights. We mention two examples from economics: (i) application of Perron–Frobenius theory [13] to the Leontief model of an economy [8,7,2]; (ii) application of geometric convexity [11] to strategic market games [1,5,12].

In this paper we prove a conjecture of Dubey et al. [4] on the resolvent of a nonnegative matrix, which was motivated by their analysis of competition in social networks such as the internet [3]. In addition to the game-theoretic context of [3], we provide a second application of our main result to

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.11.004

the open Leontief model. It is our hope and expectation that the reader will find our result useful in other contexts as well.

1. The main result

We need some notation in order to state our main result. The *resolvent* of an $n \times n$ matrix X is the matrix $R(z,X) := (X - zI)^{-1}$ defined for all scalars z outside the set S_X of eigenvalues of X. It was introduced by Fredholm in his seminal paper on operator theory [6] and played a key role in subsequent work of Hilbert and von Neumann. We consider here a slight variant of the resolvent, $Y(t,X) := (I - tX)^{-1}$; this admits a power series expansion

$$Y(t,X) = I + tX + t^2X^2 + \cdots, \quad |t| < 1/r_X, \tag{1}$$

where $r_X := \max\{|\lambda| : \lambda \in S_X\}$ is the spectral radius of X. Note that by (1) if X is nonnegative and $0 \leqslant t < 1/r_X$ then $Y(t,X)$ is also nonnegative.

Theorem 1. *Suppose $X' = (x'_{ij})$ is a nonnegative matrix obtained from $X = (x_{ij})$ by decreasing a single entry x_{hk}. Then for all indices i,j and for all t in $[0, 1/\max(r_X, r_{X'}))$, the entries of $Y := Y(t,X)$ and $Y' := Y(t,X')$ satisfy*:

$$y_{ij}y'_{ik} \leqslant y'_{ij}y_{ik} \quad \text{and} \quad y_{ij}y'_{hj} \leqslant y'_{ij}y_{hj}. \tag{2}$$

Proof. It suffices to prove the first inequality in (2), since the second then follows by transposing X. Also note that, replacing X, X' by tX, tX' if necessary, we may assume without loss of generality that $t = 1$, so that (1) becomes

$$Y = I + X + X^2 + \cdots \tag{3}$$

Now the matrix entries of the various powers of X are given as follows:

$$\left(X^2\right)_{ij} = \sum_p x_{ip}x_{pj}, \left(X^3\right)_{ij} = \sum_{p,q} x_{ip}x_{pq}x_{qj}, \ldots$$

Therefore by (3) we deduce that the entries of Y are given by

$$y_{ij} = \sum_{\alpha \in A} x_\alpha,$$

where A is the set of all finite sequences of indices that start at i and end at j, and x_α denotes the corresponding product of matrix entries as follows:

$$x_{(i,i_1,\ldots,i_{m-1},j)} \cong x_{i,i_1}x_{i_1,i_2} \ldots x_{i_{m-1},j}.$$

[For the single term sequence (i), $x_{(i)}$ is the empty product 1.]

With analogous notation we have

$$y'_{ij} = \sum_{\alpha \in A} x'_\alpha, \; y_{ik} = \sum_{\beta \in B} x_\beta, \; y'_{ik} = \sum_{\beta \in B} x'_\beta,$$

where B is the set of all finite sequences that start at i and end at k. Thus the assertion (2) of the theorem can be reformulated as follows

$$\sum_{(\alpha,\beta) \in A \times B} x_\alpha x'_\beta \leqslant \sum_{(\alpha,\beta) \in A \times B} x'_\alpha x_\beta. \tag{4}$$

While for all (α, β) we do have $x'_\beta \leqslant x_\beta$ and $x'_\alpha \leqslant x_\alpha$, it does not follow, nor indeed is it true, that $x_\alpha x'_\beta \leqslant x'_\alpha x_\beta$. Therefore an argument is required to establish (4). We adopt the following strategy: since the sums in (4) are absolutely convergent, they are invariant under rearrangement, and it suffices to exhibit a bijection $(\alpha, \beta) \mapsto (\bar{\alpha}, \bar{\beta})$, from the set $A \times B$ to itself, such that

$$x_\alpha x'_\beta \leqslant x'_{\bar{\alpha}} x_{\bar{\beta}} \tag{5}$$

Given $(\alpha, \beta) \in A \times B$ we consider two cases. If the sequence α does not contain the index k then we put $(\bar{\alpha}, \bar{\beta}) = (\alpha, \beta)$. However if α does contain k, then we define $\bar{\beta}$ by stripping off from α all the indices after the *last* occurrence of k, and we define $\bar{\alpha}$ by appending these stripped-off indices to β.

This map is its own inverse, hence a bijection, and it remains only to verify (5). To this end we note that since X' and X agree in all columns other than column k, we have $x'_{pq} = x_{pq}$ if $q \neq k$. Hence if γ is any sequence that does not contain k, except perhaps as its first term, then we have

$$x'_\gamma = x_\gamma.$$

In the first case $(\bar{\alpha}, \bar{\beta}) = (\alpha, \beta)$, we have $x_\alpha = x'_\alpha$ because α does not contain k, and (5) follows since $x'_\beta \leqslant x_{\bar{\beta}}$. In the second case, the two sides of Eq. (5) are actually *equal* because they differ only in the treatment of the stripped-off indices, which do not include k.

2. Applications

2.1. The Google Page-rank model

Dubey et al. [3] consider a class of non-cooperative games involving firms that compete for customers in a social network. For simplicity we shall restrict ourselves to a discussion of their "quasilinear" model, which is a simplified version of the Google Page-rank model of internet usage, but which already contains many essential features of the general class.

This model involves a discrete Markovian birth–death process that is specified by a nonnegative vector $v = (v_i)$ and a nonnegative matrix $X = (x_{ij})$. Here v_i represents the number of births (initial visits) per unit time in site i, and x_{ij} is the transition probability from site j to site i. The matrix X is column-substochastic $\left(\sum_i x_{ij} < 1\right)$ since there is a positive probability of death (logging off). The steady state vector p (Page-rank) satisfies $p = v + Xp$, whence we get

$$p = (I - X)^{-1} v.$$

By the Perron–Frobenius theorem (see [13]), the spectral radius of a substochastic matrix is less than 1. Therefore $Y = (I - X)^{-1}$ is a nonnegative matrix given by (3) and Theorem 1 is applicable.

Suppose $\overline{X}$ is obtained from X by *increasing* some entries in column k of X while maintaining substochasticity; and let $\overline{Y} = (I - \overline{X})^{-1}$. Assume that $X, \overline{X}$ are *irreducible* [13, Definition 1.6], then $Y, \overline{Y}$ are strictly positive and our main result has the following consequence, which was conjectured by Dubey et al.

Corollary 2. *The following inequality holds for all* (i, j):

$$\frac{\overline{y}_{ik}}{\overline{y}_{ij}} \geqslant \frac{y_{ik}}{y_{ij}}. \tag{6}$$

Proof. First suppose that only a single entry of X, say x_{hk}, has been increased. Then (6) is equivalent to the first inequality in (2), albeit with X replaced by $\overline{X}$ and X' replaced by X. For the general case of (6), we simply increase the entries of column k one at a time, and iterate (2).

2.2. The open Leontief model

The open Leontief model of an economy [2,7,8,13] deals with the case of n industries each producing exactly one good. The production of one unit of good j requires inputs $x_{ij} \geqslant 0$ of the other goods i. Goods are measured in "dollars-worth" units, and one usually assumes that every industry runs at a profit, *i.e.* it costs less than a dollar to produce one dollar's worth of any good. This means that the technology matrix $X = (x_{ij})$ is column-substochastic.

In order to produce a vector $p = (p_i)$ of goods, the production process consumes Xp, leaving only the excess vector $c = p - Xp$ available for outside use. One thinks of c as a 'demand' vector and p as a 'supply' vector, and solving for p in terms of c one gets

$$p = (I - X)^{-1} c.$$

Since X is column-stochastic the spectral radius of X is less than 1, and $Y = (I - X)^{-1}$ is a nonnegative matrix given by (3). The ijth entry of Y is the partial derivative $y_{ij} = \partial p_i / \partial c_j$ and represents the increase in supply of good i in response to a 1 unit increase in the demand of good j. We shall refer to Y as the *impact* matrix.

For simplicity we shall also assume that there is sufficient interconnectivity among the goods under consideration so that X is *irreducible* [13, Definition 1.6]. This implies that Y is strictly positive; hence an increase in the demand of any one good leads to an increase in the supply of every good [2,7].

For the Leontief model our main result can be interpreted as describing the effect of a *change in technology*. Suppose there is an improvement in the production technology of good k that reduces the required input x_{hk} of good h, then the new technology matrix X' is as in the statement of Theorem 1.

It follows from (3) that the new impact matrix $Y' = (I - X')^{-1}$ is entrywise smaller than Y. The *impact reduction percentage* is given by

$$r_{ij} := 100 * \left(y_{ij} - y'_{ij}\right) / y_{ij} \tag{7}$$

and Theorem 1 implies the following property of the matrix $R = (r_{ij})$.

Corollary 3. *The largest entry in any row of R occurs in column k. The largest entry in any column of R occurs in row h.*

Proof. For the first statement, we need to show that for all $i \neq k$ and all j, we have $r_{kj} \geqslant r_{ij}$. By formula (7) this is equivalent to $y'_{kj}/y_{kj} \leqslant y'_{ij}/y_{ij}$, which in turn follows from the first inequality in (2). The second statement of the corollary follows analogously from the second inequality in (2).

Appendix

We sketch here an alternative proof of Theorem 1, which was provided by the referee. This proof, while shorter, is somewhat less self-contained, in that it relies on earlier results in the literature.

The first result is the Sherman–Morrison formula [10] for the inverse of a matrix after a rank 1 update. In the context of Theorem 1, assuming $t = 1$ without loss of generality, and writing $\alpha = x_{hk} - x'_{hk}$, we get

$$y'_{ij} = y_{ij} - \alpha \frac{y_{ih} y_{kj}}{1 + \alpha y_{kh}}.$$

Using this formula, the first inequality in (2) reduces to showing

$$y_{ij} y_{kk} - y_{ik} y_{kj} \geqslant 0.$$

The quantity on the left is the determinant of an *almost principal* minor of the inverse M-matrix Y, and is thus non-negative by a result of Markham [9]. The main ingredient in Markham's result (and indeed his proof) is the fact that a principal minor of an inverse M-matrix is itself an inverse M-matrix, and thus has non-positive off-diagonal cofactors. The relevant fact about principal minors in turn follows easily by examining the Schur complement in the formula for the inverse of a partitioned matrix.

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