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Journal of Functional Analysis 262 (2012) 3585–3601

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Existence of Klyachko models for $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$

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Received 6 October 2011; accepted 25 January 2012

Available online 8 February 2012

Communicated by P. Delorme

Abstract

We prove that any irreducible unitary representation of $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ admits an equivariant linear form with respect to one of the subgroups considered by Klyachko.

Keywords: Distinguished representations; Unitary dual; Highest derivatives; Mixed models

Contents

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2012.01.023

1. Introduction

Let F be either $\mathbb{R}$ or $\mathbb{C}$ and $G_n := GL(n, F)$. For any decomposition $n = r + 2k$ we consider a subgroup of G_n defined by

$$H_{r,2k} = \left\{ \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} \in G_n \colon u \in N_r,\ X \in M_{r\times 2k}(F) \text{ and } h \in Sp(2k) \right\}.$$

Here $N_r \subset G_r$ denotes the group of $r \times r$ upper unitriangular matrices and

$$Sp(2k) = \{ g \in G_{2k} \colon {}^t g J_k g = J_k \} \quad \text{where } J_k = \begin{pmatrix} & w_k \\ -w_k & \end{pmatrix} \tag{1}$$

and $w_k \in G_k$ is the permutation matrix with (i, j)-th entry equal to $\delta_{k+1-i,j}$. Let ψ be a non-trivial additive character of F. We associate to ψ the character ψ_r of N_r defined by

$$\psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r})$$

and the character $\phi_{r,2k}$ of $H_{r,2k}$ defined by

$$\phi_{r,2k} \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_r(u).$$

Let $\widehat{G_n}$ denote the unitary dual of G_n. For $\pi \in \widehat{G_n}$ we consider the space $\mathrm{Hom}_{H_{r,2k}}(\pi^\infty, \phi_{r,2k})$ of continuous $(H_{r,2k}, \phi_{r,2k})$-equivariant linear forms on the Frechét space π^∞ of smooth vectors in π. We refer to a non-zero element of $\mathrm{Hom}_{H_{r,2k}}(\pi^\infty, \phi_{r,2k})$ as a *Klyachko linear form* of type $(r, 2k)$. Let

$$\mathcal{M}_{r,2k} = \{ f : G_n \to \mathbb{C} \colon f \text{ is smooth and } f(hg) = \phi_{r,2k}(h) f(g),\ h \in H_{r,2k},\ g \in G_n \}.$$

If π is an irreducible Hilbert representation of G_n then a non-zero element $\ell \in \mathrm{Hom}_{H_{r,2k}}(\pi^\infty, \phi_{r,2k})$ defines a realization of π^∞ in the space of functions $\mathcal{M}_{r,2k}$ via $v \mapsto f_v : \pi^\infty \to \mathcal{M}_{r,2k}$ where $f_v(g) = \ell(\pi(g)v)$, $g \in G_n$. We therefore refer to $\mathcal{M}_{r,2k}$ as the *Klyachko model* of type $(r, 2k)$. With this relation in mind for the rest of this paper we focus on Klyachko linear forms rather than Klyachko models.

In order to formulate our main result we recall that the partition $\mathcal{V}(\pi)$, the $SL(2)$-type of π, is defined in [34, Section 2.2] for every $\pi \in \widehat{G_n}$ based on the classification of $\widehat{G_n}$. (See Section 2.4 below.)

Theorem A. *Let $\pi \in \widehat{G}_n$ and let r be the number of odd parts of the partition $\mathcal{V}(\pi)$. Then* $\mathrm{Hom}_{H_{r,n-r}}(\pi^\infty, \phi_{r,n-r}) \neq 0$.

An analogue of this finite family of spaces of linear forms associated with representations of $GL(n)$ over a finite field was first considered by Klyachko [17] followed by Inglis and Saxl [15] and Howlett and Zworestine [14]. In the finite field case the properties existence, disjointness and uniqueness of Klyachko linear forms hold for all irreducible representations.

Over a p-adic field, the problem was first considered by Heumos and Rallis [13] and further studied by Offen and Sayag [21–24] and Nien [19]. The outcome is disjointness and uniqueness of Klyachko linear forms for all irreducible admissible representations and existence for any representation in the unitary dual. The partition $\mathcal{V}(\pi)$ of an irreducible unitary representation π is also defined in [34] in the non-archimedean case and the analogue of Theorem A holds (see [24, (5.1)]).

The existence of a Klyachko linear form in the p-adic case is proved along the following lines. Let π be an irreducible unitary representation of $GL(n)$ over a p-adic field and let r be the number of odd parts of $\mathcal{V}(\pi)$. The case $r = 0$ is treated in two steps using the fact that generalized Speh representations are building blocks for the unitary dual. If π is a Speh representation a linear form invariant by the symplectic group is constructed on π by a global (automorphic) argument [21, Proposition 1]. For any π with $r = 0$, the invariant linear form is obtained by a construction for induced representations based on Bernstein's principle of meromorphic continuation [21, Proposition 2]. The general case, treated in [22], is obtained by a reduction to the case $r = 0$ using the theory of derivatives of Bernstein and Zelevinsky [7]. By the classification of $\widehat{G_n}$, Leibnitz rule [7, Lemma 4.5] and the $r = 0$ case the r-th derivative of π admits a linear form invariant by the symplectic group. The Klyachko linear form is obtained by composing it with the projection of π to its r-th derivative.

The scheme of the proof in the p-adic case, described above, serves us as a guideline to prove Theorem A. Nevertheless, certain difficulties are specific to the archimedean case. First steps towards a theory of derivatives for smooth Fréchet representations are taken in [2]. However, an appropriate Leibnitz rule is not yet formulated. We bypass the use of derivatives by applying the theory of adduced representations developed in [26]. Certain operations E and $\mathcal{I}$ between unitary representations of different groups are defined in [26]. We adapt these operations to products of twists of unitary representations by a (not necessarily unitary) character.

Given $\pi \in \widehat{G_n}$ let r be the number of odd parts in $\mathcal{V}(\pi)$. For $r = 0$ as in the p-adic case we apply global methods to treat Speh representations and the work of Carmona and Delorme [9] for induced representations. For $r > 0$ applying [11, Theorem B] we associate to π a representation σ of G_{n-r} which is a product of twists of unitary representations by characters. There is a linear map from the space of π to the space of $\mathcal{I}^{r-1}E(\sigma)$ which is, in particular, equivariant with respect to a Klyachko type subgroup $(H'_{n-r,r}, \phi'_{n-r,r})$ (see Section 5). By the $r = 0$ case σ^∞ admits a linear form invariant by the symplectic group $Sp(n-r)$. Composing it with a natural map from π^∞ to σ^∞ we obtain a Klyachko type linear form on π^∞. However, since σ may be reducible it is not clear whether this form is not identically zero. We overcome this obstacle by introducing a meromorphic family of equivariant linear forms. We apply an irreducibility result of Mœglin and Waldspurger [18, Proposition 1.9] to show that this meromorphic family is non-zero. By taking a leading term we obtain a Klyachko linear form on π. In order to justify that various maps are well defined and continuous on the level of smooth vectors we apply a result of Poulsen on smooth vectors in induced representations [25].

Theorem A is stated in terms of $(H_{r,n-r}, \phi_{r,n-r})$ (rather than $(H'_{n-r,r}, \phi'_{n-r,r})$) merely for compatibility with the original set up of Klyachko for the finite field case.

This work addresses existence of Klyachko linear forms in the archimedean case. Disjointness is obtained in [3]. Uniqueness, at this point, is only obtained for some special cases (e.g. in [29] the case $n = r$ and in [28,4] the case $r = 0$).

The paper is structured as follows. In Section 2 we give the necessary preliminaries regarding smooth vectors in induced representations, the unitary dual of $GL(n)$ and the irreducibility result mentioned above. We also recall a result of Carmona and Delorme [9] constructing invariant linear forms on families of induced representations. In Section 3 we recall the definition of the highest derivative with the needed adaptations. We also review a recent result of [11] which implies that the highest derivative of an odd representation is even. In Section 4 we prove our main result in the purely symplectic case (i.e. when $r = 0$). We address Speh representations by a global argument, similar to the p-adic case, and treat induced representations using the work of Carmona–Delorme. In Section 5 we provide the proof of the main theorem.

2. Preliminaries

2.1. Smooth vectors and induction

Let (π, V) be a continuous Hilbert representation of a Lie group G. A vector $v \in V$ is called smooth if the map $g \mapsto \pi(g)v : G \to V$ is infinitely differentiable. Both G and its Lie algebra $\mathfrak{g}$ act on the space of smooth vectors in V and we denote the corresponding representation by (π^∞, V^∞). It is naturally a Fréchet representation of G.

Theorem 2.1.1 *(Harish-Chandra). Let (π, V) be a unitary representation of a real reductive group G. Then π is irreducible if and only if π^∞ is irreducible* (*cf.* [36, Theorem 3.4.11]).

Remark 2.1.2. In fact [36] says that π is irreducible if and only if π_K, the underlying $(\mathfrak{g}, K)$-module with respect to a compact subgroup K of G, is irreducible. Since a G-invariant decomposition of π (resp. π^∞) clearly provides one of π^∞ (resp. π_K), the above theorem is indeed straightforward from [36].

Let G be a Lie group with a Lie algebra $\mathfrak{g}$. Denote by $\Delta_G : G \to \mathbb{R}_{>0}$ the modular function associated with G, i.e.

$$\Delta_G(g) = \left|\det\left(\mathrm{Ad}(g)|_{\mathfrak{g}}\right)\right|.$$

Let H be a closed subgroup of G, (σ, V) a Hilbert representation of H and $\delta : H \to \mathbb{R}_{>0}$ defined by $\delta(h) = \Delta_H(h)/\Delta_G(h)$.

Let W denote the Hilbert space of equivalence classes of measurable functions $f : G \to V$ such that

$$f(hg) = \delta^{\frac{1}{2}}(h)\sigma(h)f(g) \quad \text{and} \quad \|f\|_W^2 := \int_{H\backslash G} \left\|f(g)\right\|_V^2 \, dg < \infty.$$

Let (π, W) be the representation of G defined by $\pi(g)f(x) = f(xg)$, $x, g \in G$. Denote the representation (π, W) by $\mathrm{Ind}_H^G(\sigma)$, the normalized induction of σ from H to G. If (σ, V) is unitary then $\mathrm{Ind}_H^G(\sigma)$ is also unitary.

Recall the following result of Poulsen. It can be interpreted as a representation-theoretic version of Sobolev's embedding theorem.

Theorem 2.1.3. *(See [25, Theorem 5.1].) Let (σ, V) be a unitary representation of H and let $(\pi, W) = \mathrm{Ind}_H^G(\sigma)$. Then $\mathrm{Ind}_H^G(\sigma)^\infty$ consists of all infinitely differentiable functions $f \in W$ such that all their derivatives with respect to left-G-invariant differential operators on G are square integrable.*

We will apply Poulsen's Theorem for certain Hilbert representations induced from a twist of a unitary representation by a character. For the rest of this section let χ be a (not necessarily unitary) character of H that extends to a smooth function $\chi' : G \to \mathbb{C}^*$. Let (σ, V) be a unitary representation of H and $(\pi, W) = \mathrm{Ind}_H^G(\sigma)$.

There is an isomorphism of Hilbert representations $(\pi_\chi, W) \simeq \mathrm{Ind}_H^G(\sigma \otimes \chi)$ given by $f \mapsto \chi' f$, $f \in W$ where $\pi_\chi(g)f(x) = \chi'(x)^{-1}\chi'(xg)f(xg)$, $g, x \in G$. Since χ' is smooth it follows that $w \in W$ is smooth with respect to π_χ if and only if it is smooth with respect to π. The following corollaries are therefore immediate consequences of Poulsen's Theorem.

Corollary 2.1.4. *Every element of $\mathrm{Ind}_H^G(\sigma \otimes \chi)^\infty$ is an infinitely differentiable function on G with values in V.*

Corollary 2.1.5. *Suppose that $H\backslash G$ is compact and let $f \in \mathrm{Ind}_H^G(\sigma \otimes \chi)$. Then $f \in \mathrm{Ind}_H^G(\sigma \otimes \chi)^\infty$ if and only if $f : G \to V$ is an infinitely differentiable function.*

2.2. Induced representations of GL(n)

Let F be either $\mathbb{R}$ or $\mathbb{C}$ and let $G_n = GL(n, F)$. Let $K = K_n$ be the standard maximal compact subgroup of G_n, i.e. $O(n)$ if $F = \mathbb{R}$ and $U(n)$ if $F = \mathbb{C}$.

For a Hilbert representation (π, V) of G_n and $s \in \mathbb{C}$ we denote by $(|\ |^s\pi, V)$ the Hilbert representation on the same space V given by $g \mapsto |\det g|^s \pi(g)$.

Let $(n_1, \dots, n_k)$ be a decomposition of n and let $P = MU$ be the standard parabolic subgroup of G_n consisting of matrices in upper triangular block form, where

$$M = \{\mathrm{diag}(m_1, \dots, m_k)\colon\ m_i \in G_{n_i},\ i = 1, \dots, k\}$$

is the standard Levi subgroup of P and U is its unipotent radical. Let (σ_i, V_i) be a Hilbert representation of G_{n_i}, $i = 1, \dots, k$, and let $(\sigma, V) = (\sigma_1 \otimes \cdots \otimes \sigma_k, V_1 \otimes \cdots \otimes V_k)$ be the associated Hilbert representation of M. We also view (σ, V) as a representation of P where U acts trivially. We use the following standard notation for normalized parabolic induction to G_n

$$\sigma_1 \times \cdots \times \sigma_k = \mathrm{Ind}_P^{G_n}(\sigma).$$

For $\varphi \in \mathrm{Ind}_P^{G_n}(\sigma)$ and $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$ define the holomorphic section

$$\varphi_\lambda(g) = \left[\prod_{i=1}^{k} |\det m_i|^{\lambda_i}\right]\varphi(g) \tag{2}$$

where $g = umk \in G_n$, $u \in U$, $m = \operatorname{diag}(m_1, \dots, m_k) \in M$ and $k \in K_n$. We further associate to σ a family $I(\sigma, \lambda)$ of induced representations parameterized by $\lambda \in \mathbb{C}^k$ realized in the underlying vector space of $\operatorname{Ind}_P^{G_n}(\sigma)$. The representation $I(\sigma, \lambda)$ is defined by

$$\big(I(g, \sigma, \lambda)\varphi\big)_\lambda(x) = \varphi_\lambda(xg), \quad \varphi \in \operatorname{Ind}_P^{G_n}(\sigma),\ g, x \in G_n. \tag{3}$$

We have

$$I(\sigma, \lambda) \simeq |\ |^{\lambda_1}\sigma_1 \times \cdots \times |\ |^{\lambda_k}\sigma_k$$

and the underlying space for $I(\sigma, \lambda)^\infty$ is independent of λ (cf. Corollary 2.1.5).

2.3. *A result of Carmona–Delorme*

A result of Carmona–Delorme allows, in the setting of symmetric pairs, the construction of meromorphic families of invariant linear forms on induced representations. We recall the result in the special case of $(G_{2n}, Sp(2n))$ (where $Sp(2n)$ is defined by (1)).

Let $(n_1, \dots, n_k)$ be a decomposition of n and $P = MU$ the standard parabolic subgroup of G_{2n} of type $(2n_1, \dots, 2n_k)$ with unipotent radical U and standard Levi subgroup M. Let $\jmath = \operatorname{diag}(J_{n_1}, \dots, J_{n_k})$ where J_n is defined by (1) and

$$H = Sp(\jmath) = \left\{g \in G_{2n}\colon {}^t g \jmath g = \jmath\right\}.$$

Set $\tau(g) = \jmath\, {}^t g^{-1} \jmath^{-1}$ and let θ be the standard Cartan involution on G_{2n} (i.e., $\theta(g) = {}^t g^{-1}$ if $F = \mathbb{R}$ and $\theta(g) = {}^t \bar{g}^{-1}$ if $F = \mathbb{C}$). Note that $H = G^\tau$ and P is $\theta\tau$-stable. Let $\sigma_i \in \widehat{G_{2n_i}}$ and $0 \neq \ell_i \in \operatorname{Hom}_{Sp(2n_i)}(\sigma_i^\infty, \mathbb{C})$, $i = 1, \dots, k$. Set $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ and $\ell = \ell_1 \otimes \cdots \otimes \ell_k$. Thus $0 \neq \ell \in \operatorname{Hom}_{M \cap H}(\sigma^\infty, \mathbb{C})$. There is a permutation matrix $\eta \in G_{2n}$ so that ${}^t\eta \jmath \eta = J_n$ and therefore $\eta^{-1} Sp(\jmath)\eta = Sp(2n)$. The following is therefore an application of [9, Proposition 2 and Theorem 3].

Proposition 2.3.1. *With the above notation the integral*

$$\xi(\varphi; \ell, \lambda) = \int_{(M\cap H)\backslash H} \ell\big(\varphi_\lambda(h\eta)\big)\, dh, \quad \varphi \in (\sigma_1 \times \cdots \times \sigma_k)^\infty$$

converges absolutely for $\operatorname{Re}(\lambda_1) \gg \operatorname{Re}(\lambda_2) \gg \cdots \gg \operatorname{Re}(\lambda_k)$ *and extends to a meromorphic function of* $\lambda \in \mathbb{C}^k$. *Whenever holomorphic at* λ *it defines a non-zero element* $\xi(\ell, \lambda) \in \operatorname{Hom}_{Sp(2n)}(I(\sigma, \lambda)^\infty, \mathbb{C})$. (*See* (2) *for the definition of* φ_λ *and* (3) *for the definition of* $I(\sigma, \lambda)$.)

2.4. The unitary dual of $GL(n)$ and the $SL(2)$-type

The unitary dual $\widehat{G_n}$ of G_n was classified by Vogan in [35]. In [31], Tadić classified the unitary dual of $GL(n)$ over a p-adic field and expressed the classification in a uniform language for both the archimedean and non-archimedean cases. We recall the classification as it appears in [31, Theorem D]. (As noted in [31] Tadić' Theorem D is also valid in the archimedean case, see also [32].)

Let $\delta \in \widehat{G_r}$ be square-integrable (thus $r = 1$ if $F = \mathbb{C}$ and $r \in \{1, 2\}$ if $F = \mathbb{R}$). For an integer $t \geqslant 1$ denote by $U(\delta, t)$ the unique irreducible quotient of

$$|\ |^{\frac{t-1}{2}}\delta \times |\ |^{\frac{t-3}{2}}\delta \times \cdots \times |\ |^{\frac{1-t}{2}}\delta$$

and for $0 < \alpha < \frac{1}{2}$ let

$$\pi(\delta, t, \alpha) = |\ |^{\alpha} U(\delta, t) \times |\ |^{-\alpha} U(\delta, t).$$

For $r = 1$ the representation $U(\delta, t)$ is one-dimensional. For $r = 2$ it was constructed in [30] using the theory of automorphic forms. Later it was given an explicit Hilbert space model in [27].

Let B be the set of all representations of the form $U(\delta, t)$ or $\pi(\delta, t, \alpha)$ as above. Then for any $\pi_1, \ldots, \pi_k \in B$ the representation $\pi_1 \times \cdots \times \pi_k \in \widehat{G_n}$ for an appropriate n and any $\pi \in \widehat{G_n}$ is of this form for a uniquely determined multi-set $\{\pi_1, \ldots, \pi_k\}$ in B.

In particular, for any $\pi \in \widehat{G_n}$ there exist integers $k_1, \ldots, k_m$, $t_1, \ldots, t_m$, square integrable representations $\delta_i \in \widehat{G_{k_i}}$ and $-\frac{1}{2} < \alpha_i < \frac{1}{2}$ such that

$$\pi = |\ |^{\alpha_1} U(\delta_1, t_1) \times \cdots \times |\ |^{\alpha_m} U(\delta_m, t_m).$$

The following is therefore immediate from [18, Proposition 1.9].

Lemma 2.4.1. *Let $\pi_i \in \widehat{G_{n_i}}$, $i = 1, 2$. Then the set*

$$\left\{s \in \mathbb{C}\colon\ \pi_1 \times |\ |^s \pi_2 \textit{ is reducible}\right\}$$

is discrete in $\mathbb{C}$.

A partition of n is a multi-set of positive integers adding up to n. By abuse of notation we will sometimes denote a partition λ as a tuple $(n_1, \ldots, n_k)$ but we keep in mind that order is irrelevant. The integers $n_1, \ldots, n_k$ are referred to as the parts of λ. The transpose partition λ^t is the partition $(m_1, \ldots, m_l)$ where $m_i = \#\{j\colon 1 \leqslant j \leqslant k,\ i \leqslant n_j\}$ (l is the maximal integer so that $\{j\colon 1 \leqslant j \leqslant k,\ l \leqslant n_j\}$ is not empty). If λ and μ are partitions their union (as a multi-set) is denoted by (λ, μ). We call a partition even if all its parts are even and odd if all its parts are odd. For two natural numbers r and n let

$$\langle n\rangle_r = \overbrace{(n, \ldots, n)}^{r}$$

be the partition of nr with r equal parts.

The $SL(2)$-type associated to $\pi \in \widehat{G_n}$ is denoted by $\mathcal{V}(\pi)$ and characterized by the following properties. For any $\delta \in \widehat{G_r}$ square integrable, $0 < \alpha < \frac{1}{2}$, $\pi_1 \in \widehat{G_{n_1}}$ and $\pi_2 \in \widehat{G_{n_2}}$ we have

(1) $\mathcal{V}(U(\delta,n)) = \langle n\rangle_r$;
(2) $\mathcal{V}(\pi(\delta,n,\alpha)) = \langle n\rangle_{2r}$;
(3) $\mathcal{V}(\pi_1 \times \pi_2) = (\mathcal{V}(\pi_1), \mathcal{V}(\pi_2))$.

Definition 2.4.2. A representation $\pi \in \widehat{G_n}$ is called even if $\mathcal{V}(\pi)$ is even and odd if $\mathcal{V}(\pi)$ is odd. We denote by $r(\pi)$ the number of odd parts in $\mathcal{V}(\pi)$.

Note that a product of two even representations is even. The following statement is straightforward from the definitions and the classification of $\widehat{G_n}$.

Corollary 2.4.3. *Let $\pi \in \widehat{G_n}$. There is a decomposition $n = k + l$, $k, l \geqslant 0$, $\pi^e \in \widehat{G}_k$ an even representation and $\pi^o \in \widehat{G}_l$ an odd representation, uniquely determined up to isomorphism, such that $\pi = \pi^e \times \pi^o$.*

3. The highest derivative

The following convention will be used whenever convenient. For $n < m$ we view G_n as a subgroup of G_m through the imbedding $g \mapsto \operatorname{diag}(g, I_{m-n})$. This convention will freely be used throughout the paper for subgroups of G_n without further notice.

For subgroups A_i of G_{k_i}, $i = 1, 2$, by $(A_1 \times A_2) \ltimes M_{k_1\times k_2}(F)$ we mean the subgroup of $G_{k_1+k_2}$ consisting of matrices of the form

$$\operatorname{diag}(a_1, a_2) \ltimes X := \begin{pmatrix} a_1 & X \\ 0 & a_2 \end{pmatrix}, \quad a_i \in A_i,\ i = 1, 2,\ X \in M_{k_1\times k_2}(F).$$

In accordance with our convention, when $A_2 = \{e\}$ we also set $A_1 \ltimes M_{k_1\times k_2}(F) = (A_1 \times A_2) \ltimes M_{k_1\times k_2}(F)$.

For a representation (σ, V) of $A_1 \times A_2$ and a character χ of $M_{k_1\times k_2}(F)$ we denote by $(\sigma \ltimes \chi, V)$ the representation of $(A_1 \times A_2) \ltimes M_{k_1\times k_2}(F)$ defined by

$$(\sigma \ltimes \chi)\big(\operatorname{diag}(a_1, a_2) \ltimes X\big) = \chi(X)\sigma\big(\operatorname{diag}(a_1, a_2)\big), \quad a_i \in A_i,\ i = 1, 2,\ X \in M_{k_1\times k_2}(F).$$

We recall the archimedean analog, as formulated in [26], of the Bernstein–Zelevinsky notion of highest derivative [7].

Denote by P_n the "*mirabolic*" subgroup of G_n consisting of matrices with last row $e_n := (0, 0, \ldots, 0, 1)$, i.e. $P_n = G_{n-1} \ltimes F^{n-1}$. Note that

$$\Delta_{P_n}(g) = |\det g|, \quad g \in P_n.$$

The starting point of the archimedean theory of highest derivatives is the following

Theorem 3.0.1. *Let $\pi \in \widehat{G_n}$, then $\pi|_{P_n}$ is irreducible.*

Remark. The result was conjectured by Kirillov. In the p-adic case it was proved in [6], in the complex case in [26] and finally in the real case in [5].

For a Hilbert representation (σ, V) of G_n let $E(\sigma) = \sigma \ltimes \mathbf{1}_{F^n}$ be the associated representation of P_{n+1} on the same space V.

For a Hilbert representation (τ, V) of P_n let

$$\mathcal{I}(\tau) = \operatorname{Ind}_{P_n \ltimes F^n}^{P_{n+1}} (\tau \ltimes \hat{e}_n),$$

where $\hat{e}_n$ denotes the character of F^n defined by $\hat{e}_n(v) = \psi(e_n v)$. Note that $E|_{\widehat{G_n}} : \widehat{G_n} \to \widehat{P_{n+1}}$ and $\mathcal{I}|_{\widehat{P_n}} : \widehat{P_n} \to \widehat{P_{n+1}}$.

Based on Theorem 3.0.1 and Mackey theory it is shown in [26] that for $\pi \in \widehat{G_n}$ there exists a unique integer d, $1 \leqslant d \leqslant n$ and a unique $\sigma \in \widehat{G_{n-d}}$ such that

$$\pi|_{P_n} \simeq \mathcal{I}^{d-1} E(\sigma). \tag{4}$$

The representation σ is called the *highest derivative* (or *adduced*) of π and is denoted by $A(\pi)$. The integer d is called the *depth* of π and we denote it by $\operatorname{depth}(\pi)$.

Recursively we define $A^{j+1}(\pi) = A(A^j(\pi))$ as long as $A^j(\pi)$ is a representation of G_i for some integer $i \geqslant 1$. Let k be such that $A^k(\pi)$ is the trivial representation of G_0. The *depth sequence* of π is defined to be

$$\mathbf{d}(\pi) = (d_1, \ldots, d_k) \quad \text{where } d_{j+1} = \operatorname{depth}\big(A^j \pi\big),\ j = 0, \ldots, k-1. \tag{5}$$

The following theorem follows from [11, Theorem B].

Theorem 3.0.2. *Let $\pi \in \widehat{G_n}$ and $\mathbf{d}(\pi) = (d_1, \ldots, d_k)$ then $d_1 \geqslant \cdots \geqslant d_k$ and viewed as a partition $\mathbf{d}(\pi)$ satisfies*

$$\mathcal{V}(\pi) = \mathbf{d}(\pi)^t. \tag{6}$$

Corollary 3.0.3. *Let $\pi \in \widehat{G_n}$. Then*

(1) $\operatorname{depth}(\pi)$ *is the number of parts in $\mathcal{V}(\pi)$. In particular,* $\operatorname{depth}(\pi) \geqslant r(\pi)$ *and equality holds if and only if π is odd.*
(2) *If π is odd then $A(\pi)$ is even.*

Proof. We use the notation of the theorem. It is clear that d_1 is the number of parts in $\mathbf{d}(\pi)^t$. Since by definition $d_1 = \operatorname{depth}(\pi)$ the first part follows from (6). It follows from the definitions that $\mathbf{d}(A(\pi)) = (d_2, \ldots, d_k)$. Applying (6) again we obtain that $\mathcal{V}(A(\pi)) = \mathbf{d}(A(\pi))^t$ consists of parts of the form $m - 1$ where $1 < m$ is a part of $\mathbf{d}(\pi)^t = \mathcal{V}(\pi)$. The second part follows. $\square$

Let $n = m + r$. For Hilbert representations π of G_m and τ of P_r we set

$$\pi \times \tau = \operatorname{Ind}_{(G_m \times P_r) \ltimes M_{m \times r}(F)}^{P_n} \big((\pi \otimes \tau) \ltimes \mathbf{1}_{M_{m \times r}(F)}\big).$$

Lemma 3.0.4. *Let $s \in \mathbb{C}$ and consider the Hilbert representations π of G_m, σ of G_r and τ of P_r. We have*

(1) $E(|\ |^s\pi) = |\ |^s E(\pi)$;
(2) $\mathcal{I}(|\ |^s\tau) \simeq |\ |^s\mathcal{I}(\tau)$;
(3) $E(\pi\times\sigma) = \pi\times E(\sigma)$;
(4) $\mathcal{I}(\pi\times\tau) = \pi\times\mathcal{I}(\tau)$.

Proof. Part (1) is straightforward. Indeed, the underlying representation space of both $E(|\ |^s\pi)$ and $|\ |^s E(\pi)$ is that of π and the two actions by P_{m+1} are identical. For part (2) set $f_s(p) = |\det p|^s f(p)$, $p\in P_n$. The map $f\mapsto f_s$ is an isomorphism from $|\ |^s\mathcal{I}(\tau)$ to $\mathcal{I}(|\ |^s\tau)$. Parts (3) and (4) are proved in [26, Lemma 2.1(ii) and (iii)] when π, σ and τ are unitary. The proof of [26] is valid verbatim in the more general context of Hilbert representations. $\square$

Given a decomposition $n = m + r$ the Iwasawa decomposition on G_{n-1} implies that $P_n = [(G_m\times P_r)\ltimes M_{m\times r}(F)]K_{n-1}$. For $\lambda = (\lambda_1,\lambda_2)\in\mathbb{C}^2$ and $\varphi\in\pi\times\tau$ let

$$\varphi_\lambda(p) = |\det g_1|^{\lambda_1}|\det g_2|^{\lambda_2}\varphi(p),\quad p = \big[\mathrm{diag}(g_1,g_2)\ltimes X\big]k$$

where $g_1\in G_m$, $g_2\in P_r$, $X\in M_{m\times r}(F)$ and $k\in K_{n-1}$. It will also be convenient to denote by $I(\pi\otimes\tau,\lambda)$ the representation of P_n on the space of $\pi\times\tau$ defined by

$$\big(I(p,\pi\otimes\tau,\lambda)\varphi\big)_\lambda(x) = \varphi_\lambda(xp),\quad \varphi\in\pi\times\tau,\ p,x\in P_n.$$

Thus

$$I(\pi\otimes\tau,\lambda)\simeq |\ |^{\lambda_1}\pi\times|\ |^{\lambda_2}\tau$$

and the underlying space of $I(\pi\otimes\tau,\lambda)^\infty$ is independent of λ. The following is straightforward from Lemma 3.0.4.

Corollary 3.0.5. *Consider the Hilbert representations ϱ of G_r and π of G_m and let $\lambda\in\mathbb{C}^2$. Then for every $j\geqslant 0$ we have*

$$I\big(\pi\otimes\mathcal{I}^j E(\varrho),\lambda\big)\simeq\mathcal{I}^j E\big(I(\pi\otimes\varrho,\lambda)\big).$$

Let $S_{m,r}$ be the subgroup of G_n defined by $S_{m,r} = (G_m\times N_r)\ltimes M_{m\times r}(F)$. For a representation (σ,V) of G_m, by abuse of notation, we sometimes also denote by σ the representation of $S_{m,r}$ on V given by $(\sigma\otimes\psi_r)\ltimes\mathbf{1}_{M_{m\times r}(F)}$.

Proposition 3.0.6. *Let $d\leqslant n$, $Q = MU$ be a standard parabolic subgroup of G_{n-d} with its standard Levi decomposition ($M\simeq G_{m_1}\times\cdots\times G_{m_k}$), τ a non-zero unitary representation of M, $\lambda\in\mathbb{C}^k$ and $(\sigma,V) = \mathrm{Ind}_Q^{G_{n-d}}(\tau,\lambda)$. Let $\pi = \mathcal{I}^{d-1}E(\sigma)$ be the associated representation of P_n.*

(i) *We have $\pi\simeq\mathrm{Ind}_{S_{n-d,d}}^{P_n}((\sigma\otimes\psi_d)\ltimes\mathbf{1}_{M_{n-d\times d}(F)})$.*
(ii) *There is a continuous $S_{n-d,d}$-equivariant map $\mathrm{pr}_{d,\sigma}:\pi^\infty\to|\ |^{\frac{d-1}{2}}\sigma^\infty$, i.e.,*

$$\mathrm{pr}_{d,\sigma}\big(\pi(s)v\big) = \psi_d(u)|\det g|^{\frac{d-1}{2}}\sigma(g)\,\mathrm{pr}_{d,\sigma}(v),\quad v\in\pi^\infty \text{ and } s = \begin{pmatrix} g & X\\ 0 & u\end{pmatrix}\in S_{n-d,d}\tag{7}$$

where $g \in G_{n-d}$, $u \in N_d$ and $X \in M_{n-d\times d}(F)$. Furthermore, $\mathrm{pr}_{d,\sigma}$ *is not identically zero on any non-zero P_n-invariant subspace of π^∞.*

Proof. Part (i) follows by iteratively applying transitivity of induction. For part (ii) note that V^∞, the space of smooth vectors for σ, is also the space of smooth vectors of the representation $(\sigma \otimes \psi_d) \ltimes \mathbf{1}_{M_{n-d\times d}(F)}$ of $S_{n-d,d}$. Let

$$\tau_1 = (\tau \otimes \psi_d) \ltimes \mathbf{1}_{M_{n-d\times d}(F)}$$

be a unitary representation of the subgroup $Q_1 := (Q \times N_d) \ltimes M_{n-d\times d}(F)$ of $S_{n-d,d}$ and let

$$\eta_1 = (\chi_\lambda \times \mathbf{1}_{N_d}) \ltimes \mathbf{1}_{M_{n-d\times d}(F)}$$

be a character of Q_1 where χ_λ is the unramified character of Q associated to λ by

$$\chi_\lambda\big(\mathrm{diag}(g_1, \ldots, g_k)u\big) = \prod_{i=1}^{k} |\det g_i|^{\lambda_i}, \quad g_i \in G_{m_i},\ i = 1, \ldots, k,\ u \in U.$$

It follows from Corollary 2.1.4 that the elements of $\mathrm{Ind}_{Q_1}^{P_n}(\tau_1 \otimes \eta_1)^\infty$ are smooth functions on P_n with values in the space of τ. Let $\delta_1 = \Delta_{P_n}/\Delta_{S_{n-d,d}}$ then transitivity of induction gives the isomorphism

$$f \mapsto \varphi_f : \mathrm{Ind}_{Q_1}^{P_n}(\tau_1 \otimes \eta_1)^\infty \to \mathrm{Ind}_{S_{n-d,d}}^{P_n}\big((\sigma \otimes \psi_d) \ltimes \mathbf{1}_{M_{n-d\times d}(F)}\big)^\infty$$

where $\varphi_f(p)(s) = \delta_1^{\frac{1}{2}}(s) f(sp)$, $s \in S_{n-d,d}$, $p \in P_n$. Since f is a smooth function on P_n it now follows that φ_f is a smooth function on P_n with values in V. It further follows from Corollary 2.1.5 that $\varphi_f(p) \in V^\infty$ for $p \in P_n$.

To summarize so far, the elements of $\mathrm{Ind}_{S_{n-d,d}}^{P_n}((\sigma \otimes \psi_d) \ltimes \mathbf{1}_{M_{n-d\times d}(F)})^\infty$ are smooth functions on P_n with values in V^∞.

Thus, $\mathrm{pr}_{d,\sigma}(\varphi) := \varphi(e)$ is a well-defined linear transformation from $\mathrm{Ind}_{S_{n-d,d}}^{P_n}((\sigma \times \psi_d) \ltimes \mathbf{1}_{M_{n-d\times d}(F)})^\infty$ to V^∞. Evaluation at the identity is clearly not identically zero on any non-zero P_n-invariant space of smooth functions on P_n. The continuity of the evaluation morphism follows from [25, Lemma 5.2]. The equivariance property (7) is immediate from the definition of an induced representation. The proposition follows. □

4. Representations with symplectic models

The purpose of this section is to study linear forms invariant by the symplectic group on irreducible unitary representations of G_{2n}. The main result is

Theorem 4.0.1. *Let $\pi \in \widehat{G_{2n}}$ be an even representation then* $\mathrm{Hom}_{Sp(2n)}(\pi^\infty, \mathbb{C}) \neq 0$.

We begin with a result on Speh representations that we obtain by global means.

Proposition 4.0.2. *Let $n = 2mr$, $\delta \in \widehat{G_r}$ square integrable and $\pi = U(\delta, 2m) \in \widehat{G_n}$. Then* $\mathrm{Hom}_{Sp(2n)}(\pi^\infty, \mathbb{C}) \neq 0$.

Proof. If $r = 1$ then $\pi = \delta \circ \det$ is a character of G_n. The proposition is obvious in this case. Assume from now on that $r = 2$ (and in particular that $F = \mathbb{R}$). To complete the proposition we globalize π to a discrete automorphic representation for which the symplectic periods have already been studied.

Let Π be a cuspidal automorphic representation of $GL(2, \mathbb{A}_\mathbb{Q})$ with archimedean component $\Pi_\infty \simeq \delta$. The existence of Π is verified, for example, using the Jacquet–Langlands correspondence. Indeed, let D be the multiplicative group of the standard quaternion algebra defined over $\mathbb{Q}$. Let δ' be a representation of $D(\mathbb{R})$ associated with δ by the local Jacquet–Langlands correspondence [16, Section 5]. Since $\mathbb{R}^*\backslash D(\mathbb{R})$ is compact, it is easy to construct using the trace formula an automorphic representation Π' of $D(\mathbb{A}_\mathbb{Q})$ so that $\Pi'_\infty \simeq \delta'$ and Π'_p is unramified for all primes $p > 2$. It then follows from [16, Theorem 14.4] that π is associated by the global Jacquet–Langlands correspondence to a cuspidal automorphic representation Π of $GL(2, \mathbb{A}_\mathbb{Q})$. In particular $\Pi_\infty \simeq \delta$ as required.

Let ϱ be the unique irreducible quotient of $|\det|^{\frac{2m-1}{2}}\Pi \times |\det|^{\frac{2m-3}{2}}\Pi \times \cdots \times |\det|^{\frac{1-2m}{2}}\Pi$. It is a discrete automorphic representation of $GL(n, \mathbb{A}_\mathbb{Q})$ obtained by residues of Eisenstein series (see [18]). Furthermore, its local component at infinity is $\varrho_\infty = \pi$. Let ϱ_{aut} be the space of automorphic forms in (the unitary representation) ϱ. Based on [20, Theorem 3], the symplectic period defined on ϱ_{aut} by

$$\ell(\phi) = \int\limits_{Sp(n,\mathbb{Q})\backslash Sp(n,\mathbb{A}_\mathbb{Q})} \phi(h)\, dh$$

is not identically zero. Recall that $\varrho_{\mathrm{aut}} \simeq \bigotimes_{p \leqslant \infty} \tau_p$ where $\tau_\infty = (\rho_\infty)_K$ is the $(\mathfrak{g}, K)$-module of K-finite vectors in ϱ_∞ and τ_p is the smooth part of ϱ_p for $p < \infty$ [10]. There is therefore an automorphic form $\phi \in \varrho_{\mathrm{aut}}$ that as a vector is of the form $\phi_\infty \otimes \phi^\infty$ with $\phi_\infty \in \tau_\infty$ and $\phi^\infty \in \bigotimes_{p<\infty} \tau_p$ such that $\ell(\phi) \neq 0$. Define

$$\lambda(v) = \ell\big(v \otimes \phi^\infty\big), \quad v \in \tau_\infty.$$

Then λ is a non-zero $Sp(n) \cap K$ and $\mathfrak{sp}(n)$-invariant linear form on τ_∞ where $\mathfrak{sp}(n)$ is the Lie algebra of $Sp(n)$. By the automatic continuity for reductive symmetric spaces (cf. [33, Theorem 2.1] or [8, Theorem 1]) λ extends to an $Sp(n)$-invariant linear form on the smooth part of ρ_∞, i.e. it defines a non-zero element of $\mathrm{Hom}_{Sp(n)}(\pi^\infty, \mathbb{C})$. The proposition follows. □

Remark 4.0.3. In [12] an $Sp(n)$-invariant functional on the Speh representation $U(\delta, 2m)$ is constructed by purely local means using [27].

Proposition 4.0.4. *Let $(n_1, \ldots, n_k)$ be a decomposition of n. Let $(\sigma_i)_{i=1}^k$ with $\sigma_i \in \widehat{G_{2n_i}}$. Let $P = MU$ be a standard parabolic subgroup of G_{2n} of type $(2n_1, \ldots, 2n_k)$, $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ and $\lambda_0 \in \mathbb{C}^k$. If* $\mathrm{Hom}_{Sp(2n_i)}(\sigma_i^\infty, \mathbb{C}) \neq 0$ *for all i then* $\mathrm{Hom}_{Sp(2n)}(I(\sigma, \lambda_0)^\infty, \mathbb{C}) \neq 0$.

Proof. Let $0 \neq \ell_i \in \mathrm{Hom}_{Sp(2n_i)}(\sigma_i^\infty, \mathbb{C})$. By Proposition 2.3.1 and using its notation we obtain a non-zero meromorphic family of linear forms $\xi(\ell, \lambda) \in \mathrm{Hom}_{Sp(2n)}(I(\sigma, \lambda)^\infty, \mathbb{C})$. There exists a generic direction $\mu \in \mathbb{C}^k$ such that $\xi(\ell, \lambda_0 + z\mu)$ is meromorphic in a punctured neighborhood of $z = 0$ in $\mathbb{C}$. Let k_0 be the smallest integer k such that $z^k \xi(\ell, \lambda_0 + z\mu)$ is holomorphic at $z = 0$. We can now define

$$L = \lim_{z \to 0} z^{k_0} \xi(\ell, \lambda_0 + z\mu).$$

Thus $0 \neq L \in \mathrm{Hom}_{Sp(2n)}(I(\sigma, \lambda)^\infty, \mathbb{C})$. $\square$

Proof of Theorem 4.0.1. By the classification of the unitary dual and the recipe for the $SL(2)$-type we may write $\pi = I(\sigma, \alpha)$ where $\sigma = U(\delta_1, 2m_1) \otimes \cdots \otimes U(\delta_k, 2m_k)$ with δ_i square integrable and $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $-\frac{1}{2} < \alpha_i < \frac{1}{2}$, $i = 1, \ldots, k$. Let n_i be such that $\sigma_i := U(\delta_i, 2m_i) \in \widehat{G_{2n_i}}$. By Proposition 4.0.2 $\mathrm{Hom}_{Sp(2n_i)}(\sigma_i^\infty, \mathbb{C}) \neq 0$. The theorem therefore follows from Proposition 4.0.4. $\square$

5. Proof of Theorem A

As we recalled in Section 1, Theorem A is an analogue of a result over finite fields and over p-adic fields. For the sake of consistency we defined the Klyachko groups $H_{r,2k}$ as we did. In fact, the set up for the theory of highest derivatives is more compatible with a modified family of subgroups that we now introduce.

Fix a decomposition $n = 2k + r$ and let

$$H'_{2k,r} = \left\{ \begin{pmatrix} h & X \\ 0 & u \end{pmatrix} \in G_n \colon u \in N_r,\ X \in M_{2k \times r}(F) \text{ and } h \in Sp(2k) \right\}.$$

Let $\phi'_{2k,r}$ be the character of $H'_{2k,r}$ defined by

$$\phi'_{2k,r} \begin{pmatrix} h & X \\ 0 & u \end{pmatrix} = \psi_r(u).$$

Theorem 5.0.1. *For all $\pi \in \widehat{G_n}$ we have*

$$\mathrm{Hom}_{H'_{n-r(\pi),r(\pi)}}\left(\pi^\infty, \phi'_{n-r(\pi),r(\pi)}\right) \neq 0 \tag{8}$$

where $r(\pi)$ is given in Definition 2.4.2.

The scheme of the proof is as follows. The case where π is even is Theorem 4.0.1. Suppose that π is not even. By Corollary 2.4.3, π can be presented as $\pi = \pi^e \times \pi^o$ where $\pi^e \in \widehat{G_{2k_1}}$ is even and $\pi^o \in \widehat{G_t}$ is odd. Let $d = \mathrm{depth}(\pi^o)$. By Corollary 3.0.3 $d = r(\pi)$ and $A(\pi^o)$ is even and thus so is $\pi^e \times A(\pi^o)$. By Theorem 4.0.1, $\sigma = I(\pi^e \otimes A(\pi^o), (\frac{d}{2}, \frac{d-1}{2}))$ has a non-zero $Sp(n-d)$-invariant functional ℓ. Composing restriction of functions to P_n with the map defined by Proposition 3.0.6 there is a non-zero morphism (defined on smooth vectors) $T : \pi = I(\pi^e \otimes \pi^o) \to \sigma$.

Composing with the $Sp(n-d)$-invariant functional ℓ we construct an $(H'_{n-d,d}, \phi'_{n-d,d})$-equivariant linear form on π. However, it seems difficult to show that this linear form is non-zero.

Instead we repeat this construction for a family of representations $\pi_s = I(\pi^e \otimes \pi^o, (0, s))$. Since for generic s, $\sigma_s = I(\pi^e \otimes A(\pi^o), (\frac{d}{2}, s + \frac{d-1}{2}))$ is irreducible, the analogous map T_s has a dense image and thus its composition with ℓ is non-zero for generic s. Taking a leading term of the family of linear forms $\ell_s \circ T_s$ we obtain a non-zero equivariant linear form as required.

Proof of Theorem 5.0.1. Let $\pi \in \widehat{G_n}$. If $r(\pi) = 0$, i.e. π is even, then (8) follows from Theorem 4.0.1. Assume from now on that $r = r(\pi) > 0$ and let $k = (n - r)/2$. Note then that $H'_{2k,r}$ is a subgroup of P_n.

Write $\pi = \pi^e \times \pi^o$ where $\pi^e \in \widehat{G_{2k_1}}$ is even and $\pi^o \in \widehat{G_t}$ is odd as in Corollary 2.4.3. For $s \in \mathbb{C}$ let

$$\pi_s = I\big(\pi^e \otimes \pi^o, (0, s)\big)$$

be a representation of G_n and

$$\tau_s = I\left(\pi^e \otimes \big(\pi^o\big|_{P_t}\big), \left(\frac{1}{2}, s\right)\right)$$

a representation of P_n. By Corollary 2.1.5 restriction of functions to P_n is a well-defined (and clearly P_n-equivariant) map

$$\kappa_s : \pi_s^\infty \to \tau_s^\infty.$$

In the parameter s it is holomorphic and non-zero at each s.

Let $d = \operatorname{depth}(\pi^o)$. By Corollary 3.0.3 $d = r(\pi)$ and $A(\pi^o)$ is even. By (4) we have $\pi^o|_{P_t} = \mathcal{I}^{d-1}E(A(\pi^o))$. Let

$$\sigma_s = I\left(\pi^e \otimes A\big(\pi^o\big), \left(\frac{d}{2}, \frac{d-1}{2} + s\right)\right).$$

By Corollary 3.0.5 there is an isomorphism of Hilbert representations of P_n

$$\tau_s \simeq \mathcal{I}^{d-1}E(\sigma_s).$$

Denote by

$$\iota_s : \tau_s^\infty \to \mathcal{I}^{d-1}E(\sigma_s)^\infty$$

its restriction to the corresponding isomorphism between the spaces of smooth vectors. Thus

$$\iota_s \circ \kappa_s : \pi_s^\infty \to \mathcal{I}^{d-1}E(\sigma_s)^\infty$$

is a holomorphic family of non-zero P_n-equivariant maps. Let

$$\mathrm{pr}_{d,\sigma_s} : \mathcal{I}^{d-1}E(\sigma_s)^\infty \to \sigma_s^\infty$$

be the $S_{d,n-d}$-equivariant map provided by Proposition 3.0.6. It is defined by evaluation at the identity and therefore it is independent of s. By Proposition 3.0.6 its restriction to the image of $\iota_s \circ \kappa_s$ is non-zero. Thus

$$\mathrm{pr}_{d,\sigma_s} \circ \iota_s \circ \kappa_s : \pi_s^\infty \to \sigma_s^\infty$$

is non-zero. Since pr_{d,σ_s} is an evaluation map at e and κ_s is a restriction map to P_n, up to the identification given by the isomorphism ι_s, the map $\mathrm{pr}_{d,\sigma_s} \circ \iota_s \circ \kappa_s$ is also an evaluation at e. It therefore follows from [25, Lemma 5.2] that it is continuous. Note that $k = k_1 + \frac{t-r}{2}$. By (7) $\mathrm{pr}_{d,\sigma_s} \circ \iota_s \circ \kappa_s$ is, in particular, G_{2k}-equivariant.

It follows from Theorem 4.0.1 that

$$\mathrm{Hom}_{Sp(2k_1)}\bigl(\pi^e, \mathbb{C}\bigr) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{Sp(n-2k_1-r(\pi))}\bigl(A\bigl(\pi^o\bigr), \mathbb{C}\bigr) \neq 0$$

and therefore from Proposition 2.3.1 that there exists a non-zero holomorphic family of linear forms

$$\ell_s \in \mathrm{Hom}_{Sp(2k,F)}\bigl(\sigma_s^\infty, \mathbb{C}\bigr)$$

in a punctured disc centered at $s = 0$. By possibly taking a smaller disc it further follows from Lemma 2.4.1 that σ_s is irreducible in the punctured disc. By Theorem 2.1.1 in this punctured disc $\mathrm{pr}_{d,\sigma_s} \circ \iota_s \circ \kappa_s : \pi_s^\infty \to \sigma_s^\infty$ has a dense image and therefore the holomorphic family of linear forms $L_s := \ell_s \circ \mathrm{pr}_{d,\sigma_s} \circ \iota_s \circ \kappa_s$ on π_s^∞ is non-zero. By the equivariance property (7), $L_s \in \mathrm{Hom}_{H'_{2k,r}}(\pi_s^\infty, \phi'_{2k,r})$. There is therefore an integer a such that $0 \neq L := \lim_{s\to 0} s^a L_s$. Thus $0 \neq L \in \mathrm{Hom}_{H'_{2k,r}}(\pi^\infty, \phi'_{2k,r})$ and (8) follows. $\square$

Finally, we conclude Theorem A. Let τ be the involution on G_n defined by $g^\tau = w_n {}^t g^{-1} w_n$. Note that $H'_{2k,r} = H^\tau_{r,2k}$ and $\phi'_{2k,r}(h) = \phi_{r,2k}(h^\tau)$, $h \in H'_{2k,r}$. It follows that for any $\pi \in \widehat{G_n}$ we have

$$\mathrm{Hom}_{H_{r,2k}}\bigl(\pi^\infty, \phi_{r,2k}\bigr) \simeq \mathrm{Hom}_{H'_{2k,r}}\bigl(\bigl(\pi^\tau\bigr)^\infty, \phi'_{2k,r}\bigr).$$

By the Gelfand–Kazhdan Theorem $\pi^\tau \simeq \widetilde{\pi}$ where $\widetilde{\pi}$ denotes the dual of π (see e.g. [1, Theorem 2.4.2]) and therefore

$$\mathrm{Hom}_{H_{r,2k}}\bigl(\pi^\infty, \phi_{r,2k}\bigr) \simeq \mathrm{Hom}_{H'_{2k,r}}\bigl(\tilde{\pi}^\infty, \phi'_{2k,r}\bigr).$$

It further follows from the classification and the definition of the partition $\mathcal{V}(\pi)$ that $\mathcal{V}(\widetilde{\pi}) = \mathcal{V}(\pi)$ and hence $r(\widetilde{\pi}) = r(\pi)$. Theorem A therefore follows from Theorem 5.0.1.

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