Fourier coefficients of automorphic forms, character variety orbits, and small representations

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Text. We consider the Fourier expansions of automorphic forms on general Lie groups, with a particular emphasis on exceptional groups. After describing some principles underlying known results on $GL(n)$, $Sp(4)$, and G_2, we perform an analysis of the expansions on split real forms of E_6 and E_7 where simplifications take place for automorphic realizations of real representations which have small Gelfand–Kirillov dimension. Though the character varieties are more complicated for exceptional groups, we explain how the nonvanishing Fourier coefficients for small representations behave analogously to Fourier coefficients on $GL(n)$. We use this mechanism, for example, to show that the minimal representation of either E_6 or E_7 never occurs in the cuspidal automorphic spectrum. We also give a complete description of the *internal Chevalley modules* of all complex Chevalley groups – that is, the orbit decomposition of the Levi factor of a maximal parabolic on its unipotent radical. This generalizes classical results on trivectors and in particular includes a full description of the complex character variety orbits for all maximal parabolics. The results of this paper have been applied in the string theory literature to the study of BPS instanton contributions to graviton scattering (Green et al., 2011, [12]).

Video. For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=ELkyOT8c28I.

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1. Introduction

The most common way to dissect a modular form is to take its Fourier expansion. Any smooth function f on the complex upper half plane $\mathbb{H} = \{x + iy \mid y > 0\}$ that is periodic in x with period r has an absolutely convergent Fourier series expansion

$$f(x+iy) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi inx/r}, \tag{1.1}$$

with coefficients that depend on y. The holomorphy of f further demands that each coefficient $a_n(y)$ satisfy a first order differential equation, whose general solution is a scalar multiple of $e^{-2\pi iny/r}$. The property that modular forms are bounded as $y \to \infty$ requires this solution to vanish if $n < 0$, so f furthermore has the form

$$f(x+iy) = \sum_{n \geqslant 0} a_n e^{2\pi in(x+iy)/r} \tag{1.2}$$

for some coefficients $a_n \in \mathbb{C}$. This is the well-known q-expansion of a classical holomorphic modular form. Any expression of the form (1.2) is of course periodic; the modularity of f is deeper and comes from identities satisfied by the a_n. A similar argument applies to the non-holomorphic Maass forms, which are instead eigenfunctions of the non-euclidean laplacian $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. In this case a differential equation and boundedness condition are again used to pin down the coefficient $a_n(y)$ as a Bessel function times a scalar coefficient a_n.

The Fourier expansions of classical, holomorphic modular forms reveal a tremendous amount of arithmetic information, such as Hecke eigenvalues and point counts of elliptic curves over varying finite fields. They also play a crucial analytic role, as they completely determine the form and provide the Dirichlet series coefficients for its L-functions. Thanks to the work of Hecke [14], Maass [24], Jacquet and Langlands [16], and Atkin and Lehner [2], there is now a very complete theory of Fourier expansions for all $GL(2)$ automorphic forms, that is, the classical holomorphic modular forms, the non-holomorphic Maass forms, and Eisenstein series.

The expansion (1.1)–(1.2) can be interpreted group theoretically as follows. Suppose F is now a function on a Lie group G that is left invariant under a discrete subgroup Γ. This includes the case of classical modular forms for $G = SL(2, \mathbb{R})$ or $GL(2, \mathbb{R})$, by setting $F(g) = f(g \cdot i)$, where $g \cdot i = (\operatorname{sgn} \det g)\frac{ai+b}{ci+d}$ is the point in the upper half plane mapped from i by the fractional linear transformation corresponding to $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Suppose that U is an abelian subgroup for which $\Gamma \cap U \setminus U$ is compact and consequently has finite volume (normalized to be 1) under its Haar measure du. We may then expand

$$F(g) = \sum_{\chi \in \mathfrak{C}(\Gamma \cap U)} F_\chi(g), \tag{1.3}$$

where $\mathfrak{C}(\Gamma \cap U)$ is the group[2] of characters on U which are trivial on $\Gamma \cap U$, and

$$F_\chi(g) = \int_{\Gamma \cap U \setminus U} F(ug)\chi(u)^{-1}\, du. \tag{1.4}$$

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[2] Even though the notation does not explicitly reflect the ambient group U, we will primarily work with arithmetic subgroups Γ and unipotent groups U, making it possible to recover U by taking the Zariski closure of $\Gamma \cap U$.

Note that each term F_χ satisfies the transformation law

$$F_\chi(ug) = \chi(u)F_\chi(g), \quad u \in U, \ g \in G, \tag{1.5}$$

and is so determined by its restriction to the quotient $U \setminus G$ – much like the functions $a_n(y)$ in (1.1) depend only on a single real variable. Thus (1.4) is the Fourier expansion of $F(ug)$ evaluated at $u = e$, and each F_χ is a Fourier coefficient.

In our example of the hyperbolic upper half plane $\mathbb{H}$, U corresponds to the subgroup of 2×2 unit upper triangular matrices, which is isomorphic to the real line $\mathbb{R}$; $\varGamma \cap U$ corresponds to $r\mathbb{Z}$ under this isomorphism. The ability to write F_χ there in terms of an unknown scalar multiple of an explicit special function depended on a differential equation; in more modern terms, it has to do with dimensionality of certain functionals on a representation space. In any event, it is a special circumstance that does not occur for every choice of abelian subgroup U: for example, (1.3) contains no information at all when U is the trivial group. This reflects the tension that – even if one somehow relaxes the requirement that U be abelian – the larger U is, the easier it may be to pin down functions satisfying (1.5) and a relevant differential equation, but the more complicated U may become. Of course (1.3) fails to hold without the assumption that U is abelian: in general the sum on the right-hand side of (1.3) must be augmented by other terms coming from higher dimensional representations of U, even though each F_χ from (1.4) still makes sense (see (2.1)).

As a manifestation of this tension, an expansion like (1.2) is hard to come by in most situations. A famous theorem, proven independently by Piatetski-Shapiro [28] and Shalika [37], gives a type of Fourier expansion for cusp forms on $GL(n)$ by an inductive argument involving abelian subgroups of the maximal unipotent subgroup $N = N_n = \{n \times n \text{ unit upper triangular matrices}\}$. Their result is quite general but is somewhat cumbersome to state for congruence subgroups and number fields. For that reason let us consider a cusp form F for $GL(n, \mathbb{Z}) \setminus GL(n, \mathbb{R})$, and let $N' = [N, N]$ denote the derived subgroup of N. All characters of $N(\mathbb{R})$ are trivial on $N'(\mathbb{R})$, and so

$$V(g) = \int_{N'(\mathbb{Z}) \setminus N'(\mathbb{R})} F(ng)\, dn \tag{1.6}$$

represents the sum (1.3) over all characters of $N_n(\mathbb{R})$ that are trivial on $N_n(\mathbb{Z})$. This Fourier expansion is entirely analogous to the $GL(2)$ expansion (1.2), but with coefficients indexed by $n - 1$ integral parameters and a different special function (a "Whittaker" function) which we shall say more about in Section 2.1. As such, $V(g)$ can be thought of as the contribution of the "abelian" terms in the Fourier expansion of $F(g)$ – the ones that come from the abelianization of N. However, unlike the case of $n = 2$, in general the integration in (1.6) loses information about $F(g)$. Piatetski-Shapiro and Shalika proved that nevertheless $F(g)$ can be reconstructed as the sum of translates of V,

$$F(g) = \sum_{\gamma \in N_{n-1}(\mathbb{Z}) \setminus GL(n-1, \mathbb{Z})} V\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right). \tag{1.7}$$

Section 2.1 contains more details about the proof of this formula and its relation to Whittaker functions.

The Piatetski-Shapiro/Shalika expansion has been extremely useful in the analytic theory of automorphic forms on $GL(n)$, perhaps most famously because it allows one to reconstruct a form in terms of its abelian Fourier coefficients – in particular, coefficients which have a direct arithmetic nature, e.g., are directly related to L-function data. Such a result is a particularly friendly special feature of the general linear group that is absent in the general situation; this is because their argument relies on an essential special fact about character variety orbits in $GL(n)$. However, even though their statement does not generalize and the orbit structure becomes considerably more complicated, one can still derive important pieces of the Fourier expansion of an automorphic form from their approach. In this paper we give a generalization of the Piatetski-Shapiro/Shalika expansion to automorphic forms

on arbitrary reductive algebraic groups. This generalization is rarely as precise as (1.7), but we indicate some conditions (on both the group and the cusp form) under which it simplifies to have a comparable form, or at least a useful enough form for some applications. More detailed Fourier expansions have been given in a number of particular examples by earlier authors, which we try to review in Section 2. The study of Fourier coefficients of automorphic forms is a very rich subject which is too broad to recount here. Our main goal is to say something general for a broad range of groups, including details for exceptional groups. We would like to mention the recent paper [9], which takes a complementary approach on exceptional groups.

In a different direction, we apply results of Matumoto [25] to show that many of the Fourier coefficients F_χ from (1.4) vanish if the archimedean component of its automorphic representation is "small" in the sense of having a small wavefront set. These results are analogs of a related nonarchimedean vanishing theorem of Mœglin and Waldspurger [27]. We give a detailed analysis for maximal parabolic subgroups of exceptional groups in Section 5. These results are in turn used in [12, §6] to verify string-theoretic conjectures about the vanishing of certain Fourier coefficients of automorphic forms (without having to explicitly compute them). Indeed, providing background results for the investigation in [12] was a primary motivation for writing this paper. However, we also pursue some more general statements that are perhaps of wider interest internally to automorphic forms. For example, our methods show the following:

Theorem 1. *Let G denote a split Chevalley group of type E_6 or E_7, and let π be an (adelic) automorphic representation of G over a number field k for which at least one component π_v is a minimal representation of $G(k_v)$ – that is, the wavefront set of π_v is the closure of the smallest nontrivial coadjoint nilpotent orbit. Fix a choice of positive roots and let N be the maximal unipotent k-subgroup of G generated by their root vectors. Then the vectors F in the representation space for π are completely determined by the degenerate Whittaker integrals*

$$\int_{N(k)\backslash N(\mathbb{A}_k)} F(ng)\psi(n)^{-1}\,dn, \tag{1.8}$$

in which $\psi : N(\mathbb{A}_k) \to \mathbb{C}$ is trivial on $N(k)$ and on all but at most a single one-parameter subgroup corresponding to a simple positive root.

In fact, the argument gives a formula for F analogous to (1.7) for these automorphic realizations of minimal representations (see [20] for a different formula, which also extends to E_8). If the non-archimedean multiplicity one results from [34] were generalized to archimedean fields, it furthermore would give a global multiplicity one theorem for these automorphic minimal representations. Each character ψ in the statement of the theorem is trivial on the unipotent radical of a proper parabolic subgroup, namely one which contains N and whose Levi component contains the one-parameter subgroup that ψ does not vanish on. The integration in (1.8) then factors over this unipotent radical; by definition, it vanishes when F is a cusp form. Therefore we conclude:

Corollary 1. *There are no cuspidal automorphic representations of Chevalley groups of type E_6 or E_7 which have a minimal local component. In particular, the Gelfand–Kirillov dimension (which is half the dimension of the wavefront set) of any component of a cuspidal automorphic representation of a Chevalley group of type E_6 or E_7 must be at least 16 or 26, respectively.*

Piatetski-Shapiro raised the question as to what the smallest Gelfand–Kirillov dimension can be for a cuspidal automorphic representation of a given group (see [22] for results in the symplectic case). A folklore conjecture asserts that the wavefront set of any component must be the closure of a *distinguished* orbit: one which does not intersect any proper Levi subgroup. This would replace the lower bounds in the corollary by 21 and 33, respectively. A related analysis can be provided for slightly larger representations, but with weaker conclusions.

One of the main ingredients in our results is the full complex orbit structure of the adjoint action of the Levi component of a maximal parabolic subgroup on the Lie algebra of its nilradical. These Levi actions are known as *internal Chevalley modules* and were classified completely when the unipotent radical is abelian [30] or Heisenberg [32]. They are well known for classical groups through classical rank theory. For the general exceptional group case, we used computer programs of Littelmann [23] and de Graaf [11] to arrive at the following:

Theorem 2. *Let $\mathfrak{g}$ be a complex exceptional simple Lie algebra, G a complex Lie group with Lie algebra $\mathfrak{g}$, and P a maximal parabolic subgroup with unipotent radical U. Decompose the Lie algebra $\mathfrak{u}$ of U as a direct sum $\mathfrak{u} = \bigoplus \mathfrak{u}_i$ of irreducible subspaces for the adjoint action of a Levi component L of P. Then all complex orbits of L on $\mathfrak{u}$ are explicitly listed in the tables of Section 5, along with the adjoint nilpotent orbits of G in $\mathfrak{g}$ in which they are contained.*

Section 2 gives a general framework for Fourier expansions, and shows how several known results can be seen as specializations. In Section 2.4 we explain local results of Matumoto and Mœglin and Waldspurger on the vanishing of Fourier coefficients for certain types of representations. Theorem 1 is proven in Section 4, along with information from the tables in Section 5. Section 5 also contains the list of orbits in Theorem 2.

2. Character expansions

Let G_{lin} denote the real points of a connected reductive linear algebraic group defined over $\mathbb{Q}$, and let G be a finite cover of G_{lin} which is a central extension of G_{lin} by a finite abelian group. Let $\Gamma \subset G$ be an arithmetic subgroup, meaning that it is commensurate with the inverse image of $G_{\text{lin}} \cap GL(N, \mathbb{Z})$ in G, where G_{lin} is realized as a subgroup of $GL(N)$ (compatibly with its natural $\mathbb{Q}$-structure). Likewise, let $G_{\mathbb{Q}}$ denote the inverse image of $G_{\text{lin}}(\mathbb{Q}) = G_{\text{lin}} \cap GL(N, \mathbb{Q})$ in G. All unipotent subgroups of G_{lin} split over the finite cover and so can be identified with subgroups of G. If U is defined over $\mathbb{Q}$, as we shall now assume, then the arithmetic subgroup Γ intersects both U and its derived subgroup $U' = [U, U]$ in cocompact lattices.

Let $\mathfrak{C}(\Gamma \cap U)$ denote the group of characters of U which are trivial on $\Gamma \cap U$, which we refer to as the integral points of the *character variety* of U. Any function $F \in C^\infty(\Gamma \setminus G)$ has the Fourier expansion

$$\Pi_U F(g) = \sum_{\chi \in \mathfrak{C}(\Gamma \cap U)} F_\chi(g) \tag{2.1}$$

generalizing (1.3), where $F_\chi(g)$ is defined exactly as in (1.4) and $\Pi_U : C^\infty((\Gamma \cap U) \setminus G) \to C^\infty(((\Gamma \cap U) \cdot U') \setminus G)$ denotes the projection operator

$$\Pi_U F(g) = \int_{\Gamma \cap U' \setminus U'} F(ug)\, du. \tag{2.2}$$

This is because the Fourier series expansion is valid on functions on the quotient $U/[U, U]$, the abelianization of U.

Let H denote the normalizer of U in G,

$$H = \{g \in G \mid gU = Ug\}. \tag{2.3}$$

Then H is defined over $\mathbb{Q}$ and $\Gamma \cap H$ is an arithmetic subgroup of H. The group $\Gamma \cap H$ acts on U by conjugation, and dually on $\mathfrak{C}(\Gamma \cap U)$ by

$$h : \chi(u) \mapsto \chi(h^{-1}uh). \tag{2.4}$$

Let $\mathcal{O}(\Gamma \cap H, \Gamma \cap U)$ denote the set of orbits of this action, and $\mathcal{C}(\Gamma \cap H, \Gamma \cap U)$ denote a fixed set of base points for these orbits. Note that if χ and $\chi'(u) = \chi(h^{-1}uh)$ are in the same orbit, then

$$
\begin{aligned}
F_{\chi'}(g) &= \int\limits_{\Gamma \cap U \backslash U} F(ug)\chi(h^{-1}uh)^{-1}\,du \\
&= \int\limits_{\Gamma \cap U \backslash U} F(huh^{-1}g)\chi(u)^{-1}\,du \\
&= F_\chi(h^{-1}g).
\end{aligned} \tag{2.5}
$$

Here we have used the invariance of F under $h \in \Gamma \cap H$, as well as the fact that conjugation by an arithmetic subgroup of H leaves the du invariant. In particular, this computation shows that $F_\chi(g)$ is automorphic under $(\Gamma \cap H)_\chi$, the stabilizer of χ in $\Gamma \cap H$. Furthermore, the stabilizers

$$H_\chi = \{ h \in H \mid \chi(u) = \chi(h^{-1}uh) \} \subset H \tag{2.6}$$

of characters $\chi \in \mathfrak{C}(\Gamma \cap U)$ are also defined over $\mathbb{Q}$, and have $(\Gamma \cap H)_\chi$ as arithmetic subgroups.

We may hence rewrite (2.1) as

$$\Pi_U F(g) = \sum_{\chi \in \mathcal{C}(\Gamma \cap H, \Gamma \cap U)} \sum_{h \in (\Gamma \cap H)_\chi \backslash (\Gamma \cap H)} F_\chi(hg). \tag{2.7}$$

Let $\mathcal{O}(H, \Gamma \cap U)$ denote the equivalence classes of characters in $\mathcal{O}(\Gamma \cap H, \Gamma \cap U)$ under the action (2.4) of the complexification $H(\mathbb{C})$ of H. We refer to $\mathcal{O}(H, \Gamma \cap U)$ as *complex orbits* and $\mathcal{O}(\Gamma \cap H, \Gamma \cap U)$ as *integral orbits*. A complex orbit groups integral orbits into characters which have a similar algebraic nature, even though they may not be equivalent under the action of the discrete subgroup $\Gamma \cap H$. The expression

$$\Pi_U F(g) = \sum_{\mathfrak{o} \in \mathcal{O}(H, \Gamma \cap U)} \sum_{\chi \in \mathfrak{o}} \sum_{h \in (\Gamma \cap H)_\chi \backslash (\Gamma \cap H)} F_\chi(hg) \tag{2.8}$$

packages the terms more usefully, because certain properties of the Fourier coefficients (e.g., vanishing) are often controlled by the complex orbits $\mathcal{O}(H, \Gamma \cap U)$ rather than the individual orbits $\mathfrak{o}$. Furthermore, the Fourier coefficients within an orbit $\mathfrak{o} \in \mathcal{O}(H, \Gamma \cap U)$ are sometimes related by an external mechanism, such as the action of Hecke operators. The complex orbits $\mathcal{O}(H, \Gamma \cap U)$ have been classified in many cases, and are often finite in number; this is in particular the case in the important example where U is the unipotent radical of a maximal parabolic subgroup of a split Chevalley group. This classification is well known for classical groups and given in Section 5 for the five exceptional groups.

The derivation of formula (2.8) can be iterated further by using the fact that $F_\chi(hg)$ is an automorphic function on $(\Gamma \cap H)_\chi \backslash H_\chi$. This gives a further refinement, though it will necessarily lose some information if nonabelian unipotent subgroups of H are used. In the rest of this section we describe some important instances of (2.8) and this iterative refinement that have appeared in the literature, in particular covering the rank 2 Chevalley groups.

2.1. Example: The Piatetski-Shapiro/Shalika expansion on GL(n)

We begin by stating a motivating geometric fact: for any field K and integer $m \geqslant 1$,

$$GL(m, K) \text{ acts on } K^m \text{ with two orbits, namely } \{0\} \text{ and } K^m - \{0\}. \tag{2.9}$$

In more pedestrian terms, any nonzero vector can be extended to a basis of K^m. As we shall see, this furnishes a rare situation where the set of complex orbits $\mathcal{O}(H, \Gamma \cap U)$ has only two elements, which makes for an elegant Fourier expansion on $GL(n)$. The Fourier expansion on a general group often includes similar terms in addition to more complicated ones not present here.

We shall now explain how Piatetski-Shapiro and Shalika derived (1.7) from an iteration of the principle behind (2.8). Let $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$. Let $P = P_n$ be the standard parabolic subgroup of G of type $(n - 1, 1)$ and $U = U_n \simeq \mathbb{R}^{n-1}$ its unipotent radical

$$U = \left\{ u = \begin{pmatrix} & & & u_1 \\ & & & u_2 \\ I_{n-1} & & & \vdots \\ & & & u_{n-1} \\ 0 & & & 1 \end{pmatrix} \middle| u_1, \ldots, u_{n-1} \in \mathbb{R} \right\}. \tag{2.10}$$

The normalizer H of U coincides with P. Since P factors as LU, where $L \simeq GL(n - 1, \mathbb{R})$ is its Levi component, $(\Gamma \cap H)_\chi \setminus (\Gamma \cap H)$ is in bijective correspondence with $L_\chi \setminus L$, where L_χ is the stabilizer of χ in L. Since $\mathcal{C}(\Gamma \cap U)$ is isomorphic to the lattice $\mathbb{Z}^m$, the orbit statement (2.9) indicates that the complexification of L, $GL(n - 1, \mathbb{C})$, breaks up $\mathcal{O}(H, \Gamma \cap U)$ into two complex orbits: the orbit consisting only of the trivial character, and its complement. The $\mathbb{Z}$-structure in this case is also easy to work out using the theory of elementary divisors – $GL(m, \mathbb{Z})$ acts on $\mathbb{Z}^m$ with orbits of the form $\{(v_1, \ldots, v_m) \in \mathbb{Z}^m \mid \gcd(v_1, \ldots, v_m) = d\}$, indexed by $d \in \mathbb{Z}_{\geqslant 0}$. Putting this together we see that $\mathcal{O}(\Gamma \cap H, \Gamma \cap U)$ has orbit representatives given by the characters $\chi_k(u) = e^{2\pi i k u_{n-1}}$ (in terms of the parameterization (2.10)), one for each $k \in \mathbb{Z}_{\geqslant 0}$.

The character χ_0 is trivial and $L_{\chi_0} = L$. However, for $k \neq 0$ the stabilizer of L_χ is isomorphic to the quotient of the parabolic subgroup $P_{n-2} \subset GL(n - 1, \mathbb{R})$ by its center. Then (2.8) specializes to

$$F(g) = \int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} F(ug) \, du + \sum_{k=1}^{\infty} \sum_{\gamma \in P_{n-2}(\mathbb{Z}) \backslash P_{n-1}(\mathbb{Z})} V_k(\gamma g), \tag{2.11}$$

where $P_{n-2}(\mathbb{Z})$ is embedded in the top left corner of matrices in $P_{n-1}(\mathbb{Z})$ and $V_k(g)$ is the period

$$V_k(g) = \int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} F(ug) \chi_k(u)^{-1} \, du. \tag{2.12}$$

Note that the first term on the right-hand side of (2.11), which equals $V_0(g)$, vanishes by definition when F is *cuspidal*. The second summand on the right-hand side of (2.11) corresponds to the large orbit of $GL(n - 1, \mathbb{Q})$ on $\mathbb{Q}^n$ from (2.9), which is responsible for the tautologically equivalent phrasing of (2.11) in the adelic terminology originally used by Piatetski-Shapiro [28] and Shalika [37].

Of course $V_k(hg)$ too is an automorphic function, under $P_{n-2}(\mathbb{Z})$. Hence we may repeat this discussion with n replaced by $n-1$ and get a nested expansion, involving sums of translated periods over $N = \{\text{unit upper triangular matrices}\} \subset G$ of the form

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} F\left(\begin{pmatrix} 1 & x_1 & \star & \star & \star \\ 0 & 1 & x_2 & \star & \star \\ 0 & 0 & 1 & \ddots & \star \\ \vdots & \vdots & \ddots & \ddots & x_{n-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} g\right) e(-k_1 x_1 - k_2 x_2 - \cdots - k_{n-1} x_{n-1})\, dn \quad (2.13)$$

with $k_1, \dots, k_{n-1} \in \mathbb{Z}_{\geqslant 0}$. These latter integrals are Whittaker integrals (in particular, "degenerate" Whittaker integrals if some $k_i = 0$), and have been widely studied. Assume now that F is a cusp form, so that the degenerate Whittaker integrals vanish automatically. When F is an eigenfunction of the ring of invariant differential operators and of moderate growth, a uniqueness principle allows one to write the nondegenerate Whittaker integrals in terms of multiples of a special function $W_{(k_1,\dots,k_{n-1})}(g)$. The uniqueness principle further relates these to each other by the formula

$$W_{(k_1,\dots,k_{n-1})}(g) = W_{(1,1,\dots,1)}(\Delta_k g),$$

$$\text{with } \Delta_k = \begin{pmatrix} k_1 k_2 \cdots k_{n-1} \\ & k_2 \cdots k_{n-1} \\ & & k_3 \cdots k_{n-1} \\ & & & \ddots \\ & & & & k_{n-1} \\ & & & & & 1 \end{pmatrix} \quad (2.14)$$

(both sides transform identically under left translation by N). Thus we can write $V(g)$ from (1.6) as

$$V(g) = \sum_{k_1,\dots,k_{n-1} \in \mathbb{Z}_{>0}} a_{(k_1,\dots,k_{n-1})} W_{(k_1,\dots,k_{n-1})}(g), \quad a_{(k_1,\dots,k_{n-1})} \in \mathbb{C}, \quad (2.15)$$

a relation entirely analogous to (1.2). The left translations by $\gamma \in P_{n-1}(\mathbb{Z})$ that came from (2.11) and its nested descendants result in the Piatetski-Shapiro/Shalika expansion (1.7).

It should be noted that in the particular case of $n = 3$, the iteration stage of the argument becomes much simpler because $P_{n-2}(\mathbb{Z})$ contains $N(\mathbb{Z})$ as a finite index subgroup. The remaining invariance under the subgroup $\left\{\begin{pmatrix} 1 & \star & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$ leads to a Fourier expansion in the starred entry. This is also the situation for the final step of the iteration for general n. Though the Fourier expansion (1.7) is most frequently used for cusp forms, its analog for Eisenstein series – and in particular the degenerate Whittaker coefficients contained therein – is still important for a number of applications. See [6, §7] for complete details of the Fourier expansions for Borel Eisenstein series on $SL(3)$.

2.2. Example: Jiang's expansion on $Sp(4)$

Let us now consider the split Lie group $G = Sp(4, \mathbb{R})$, defined as

$$G = \{g \in SL(4, \mathbb{R}) \mid gJg^t = J\}, \quad J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.16)$$

The root system of G has 8 roots. With respect to the fixed maximal torus

$$T = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \middle| t_1, t_2 \neq 0 \right\}, \tag{2.17}$$

root vectors with respect to the four positive roots are given by

$$X_{\alpha_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad X_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad X_{\alpha_1+\alpha_2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$X_{2\alpha_1+\alpha_2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.18}$$

where α_1 is the short simple positive root and α_2 is the long simple positive root. A root vector for the negative of any of these roots is given by the transpose of the corresponding matrix.

Let P denote the standard "Klingen" parabolic subgroup of G,

$$P = \left\{ \begin{pmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix} \in G \right\}. \tag{2.19}$$

Its unipotent radical U is a 3-dimensional Heisenberg group, with Lie algebra $\mathfrak{u}$ spanned by X_{α_1}, $X_{\alpha_1+\alpha_2}$, and $X_{2\alpha_1+\alpha_2}$. Its center is $[U, U] = \{e^{t X_{2\alpha_1+\alpha_2}} \mid t \in \mathbb{R}\}$. The Levi factor of this parabolic has semisimple part $SL(2, \mathbb{R})$, and acts on $U/[U, U] \simeq \mathbb{R}^2$ by the same action as in the previous $SL(3, \mathbb{R})$ example.[3] Hence the projection $\Pi_U F(g)$ has an expansion essentially identical to the one there. The resulting formula was discovered by Dihua Jiang [17, Lemma 2.1.1], who used it as an important tool in deriving an integral representation for the degree 16 tensor product L-function on $GSp(4) \times GSp(4)$. A very similar expansion exists for the split rank 2 group $SO(3, 2; \mathbb{R})$, which is a quotient of $Sp(4, \mathbb{R})$.

2.3. Example: Siegel's expansion on $Sp(4)$

Instead of the Klingen parabolic (2.19), consider the standard "Siegel" parabolic subgroup of $G = Sp(4, \mathbb{R})$,

$$P = \left\{ \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix} \in G \right\}. \tag{2.20}$$

Its unipotent radical U is now a 3-dimensional abelian group, whose Lie algebra $\mathfrak{u}$ is spanned by X_{α_2}, $X_{\alpha_1+\alpha_2}$, and $X_{2\alpha_1+\alpha_2}$. The Levi component of P is isomorphic to $GL(2, \mathbb{R})$, and acts by the 3-dimensional symmetric square action, or alternatively the similarity action on the upper right 2×2

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[3] In fact, both this and the $SL(3, \mathbb{R})$ expansion can be proven directly via harmonic analysis of Heisenberg groups (see [26, §3]), without the translation and induction argument just presented.

matrix block in U. The complex orbits $\mathcal{O}(H, \Gamma \cap U)$ can be classified in terms of the rank of this block (see Section 5.3).

The arithmetic structure of the integral orbits in $\mathcal{C}(\Gamma \cap H, \Gamma \cap U)$ is quite subtle, as it is described in terms of ideal classes and class numbers in quadratic fields. This is the basis for classical expansions of genus 2 holomorphic forms. The generalization of the Siegel parabolic to $Sp(2n, \mathbb{R})$ or $O(n, n; \mathbb{R})$ consists of matrices whose lower left $n \times n$ block vanishes. Since its unipotent radical is abelian, this type of expansion naturally generalizes to give a Fourier expansion of an automorphic form F on either of these two groups.

2.4. Expansions on exceptional groups

Gan, Gross and Savin [8] give a theory of Fourier expansions for particular types of automorphic forms on $G =$ the split real form of G_2, namely those whose archimedean component is a quaternionic discrete series representation. They consider a maximal parabolic subgroup $P = LU$ whose unipotent radical U is a 5-dimensional Heisenberg group; the semisimple part of its Levi component L is an $SL(2)$-subgroup determined by a short root. The restriction to quaternionic discrete series representations is made to apply a uniqueness principle that pins down their coefficients as scalar multiples of particular special functions (as in (1.2)), as well as to avoid coefficients from smaller orbits. However, some aspects of their theory apply to more general representations. In this regard it is similar to Siegel's study of holomorphic forms for $Sp(2n, \mathbb{R})$, in which the coefficients from (2.3) must be augmented by Whittaker coefficients for generic representations.

Brandon Bate [3] considers the general automorphic form on G, and in particular a maximal parabolic subgroup determined by a long root. He finds an explicit version of (2.1) similar to the Piatetski-Shapiro/Shalika and Jiang expansions, and applies it to obtain the functional equation of the degree 7 L-function on G_2 (the first explicit functional equation on this group, because the Langlands–Shahidi method only applies to groups whose Dynkin diagram is part of a larger Dynkin diagram). Hence all rank 2 groups have an essentially identical piece of their Fourier expansions of the same type: an average over an embedded $SL(2)$ determined by a long root.

The maximal parabolics of the split real forms of larger exceptional groups have particularly rich structures. We give a listing of the complex orbits in Section 5. We remark that classifying the integral orbits can be extremely subtle, as it is already in the case of G_2 (see, for example, the Fourier expansion in [18]). Recently Bhargava [4], Krutelevich [21], and Savin and Woodbury [35] have made major progress on some of these group actions. This subtlety is apparently more striking for groups of uneven root length than it is for the simply laced groups such as E_6, E_7, and E_8, where it is nevertheless very intricate.

3. Vanishing of coefficients for certain small automorphic representations

From now until the end of the paper we take G_{lin} to be the real points of a Chevalley group, defined compatibly with the Chevalley $\mathbb{Z}$-basis. We shall also suppose that the automorphic form $F \in C^\infty(\Gamma \setminus G)$ is a smooth vector for an automorphic representation.

After fixing a maximal torus and choice of positive root system for G with respect to this torus, let Σ^+ denote the positive simple roots. Let S be an arbitrary subset of Σ^+ and $P = P(S)$ the standard parabolic subgroup associated to S: P contains the one-parameter subgroups generated by root vectors X_α of all positive roots, as well as the negative roots such that $-\alpha \in S$. It has a Levi decomposition $P = LU$, where L is a maximal reductive subgroup of P (containing the one-parameter subgroups generated by the root vectors X_α, $X_{-\alpha}$ of roots $\alpha \in S$), and U is its unipotent radical (containing all one-parameter subgroups generated by root vectors for positive roots, aside from the ones that are contained in L). Since Γ is arithmetic, $\Gamma \cap [L, L]$ and $\Gamma \cap U$ are arithmetic subgroups of $[L, L]$ and U, respectively.

The Lie algebra $\mathfrak{u}$ of the unipotent radical U decomposes as the direct sum

$$\mathfrak{u} = \bigoplus_{i=1}^{m} \mathfrak{u}_i, \tag{3.1}$$

where $\mathfrak{u}_i$ is the span of the root vectors for all positive roots $\sum_{\alpha \in \Sigma^+} n_\alpha \alpha$ such that $\sum_{\alpha \in S} n_\alpha = i$. In the case that P is a maximal parabolic (that is, S has exactly one element), the adjoint action of L acts irreducibly on each $\mathfrak{u}_i$, actions that are known as *internal Chevalley modules*. It is furthermore known that each of these actions has finitely many complex orbits. We shall give an enumeration of these later in Section 5.

Let $U \supset U^{(1)} \supset U^{(2)} \supset \cdots$ be the lower central series of U, i.e., $U^{(j)} := [U, U^{(j-1)}]$. Notice that the Lie algebra of $U^{(j-1)}$ of U is just $\mathfrak{u}^{(j-1)} = \bigoplus_{i \geqslant j} \mathfrak{u}_i$. Suppose now that λ is a nontrivial linear functional on some $\mathfrak{u}_j$, which we extend trivially to the rest of $\mathfrak{u}^{(j-1)}$, and then exponentiate to a character χ of $U^{(j-1)}$. We can now consider the Fourier expansion of the type (2.8), but with the subgroup U replaced by $U^{(j-1)}$ and the projection operator Π_U from (2.2) replaced by integration over $\Gamma \cap U^{(j)} \backslash U^{(j)}$. Consider the linear functional on an automorphic representation which maps an automorphic form F to the Fourier coefficient F_χ defined in (1.4). This map commutes with the automorphic representation's right translation by the adele group of G, and thus gives a global linear functional which is χ-equivariant with respect to $U^{(j-1)}$. By restriction, it thus gives nonvanishing χ-equivariant local linear functionals for each completion of the ground field.

For an intricately defined character χ and a relatively simple automorphic form F, the Fourier coefficient $F_\chi(g)$ from (1.4) may vanish identically in g; for example, this happens if F is constant, but χ is not. The following theorem of Matumoto gives a condition that often ensures this vanishing for all automorphic forms in an automorphic representation with archimedean component π. It is the archimedean analog of a more well-known theorem of Mœglin and Waldspurger [27]. Both results work with the nonvanishing equivariant local linear functionals of the previous paragraph. The statement involves the complexified wavefront set $WF(\pi)_{\mathbb{C}}$ in the dual Lie algebra $\mathfrak{g}^\star$ to $\mathfrak{g}$, which has several different definitions: for example, it is the associated variety of the annihilator ideal of π, and it can also be computed in terms of the support of the Fourier transform of the character of π. It is always the closure of a unique coadjoint nilpotent orbit in $\mathfrak{g}$ [5,19].

Theorem 3. *(See Matumoto [25].) Consider λ as an element of $\mathfrak{g}^\star$ by trivially extending it to the rest of $\mathfrak{g}$, and assume that $\lambda \notin WF(\pi)_{\mathbb{C}}$. Then $F_\chi \equiv 0$ for all vectors F in any automorphic representation that has π as an archimedean component.*

The paper [27] by Mœglin and Waldspurger contains the same assertion, but with π a non-archimedean representation.[4]

4. Abelian unipotent radicals and small representations

Let us now consider a standard maximal parabolic subgroup $P = P(\{\alpha\}) = LU$ with abelian unipotent radical U. This is the case precisely when no root has a coefficient of α greater than one when expanded in terms of the basis of positive simple roots.

According to the tables in Section 5, the action of the complexification $L(\mathbb{C})$ on U will in general have more than two orbits. In general the smallest orbit is always the trivial orbit, while the next biggest orbit corresponds to a character which is sensitive to a single root vector in the Lie algebra $\mathfrak{u}$ of U. An orbit representative can be furnished by restricting the generic character ψ to U; recall ψ is the character of the unipotent radical N of the minimal parabolic $P(\Sigma^+)$ which satisfies

$$\psi(e^{tX_\alpha}) = e^{2\pi i t} \quad \text{for all } \alpha \in \Sigma^+, \tag{4.1}$$

and which is trivial on the one-parameter subgroups generated by all other positive root vectors. Not only is χ $L(\mathbb{C})$-equivalent to $\psi|_U$, but it is furthermore $L(k)$-equivalent if χ is defined over the number field k and G is simply laced (see [30] and [35, pp. 759–760]).

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[4] Though [27] makes a restriction that the field have residual characteristic greater than 2, the authors have informed us that its use on pp. 429 and 431 of their paper can be avoided, and hence the restriction removed.

Now suppose that the archimedean component π of an automorphic representation is a minimal representation. According to Matumoto's theorem 3 and the tables in Section 5, functions F in this automorphic representation will have nonzero Fourier coefficients F_χ only for χ in these smallest two orbits – i.e., χ must be trivial, or $L(\mathbb{C})$-equivalent to $\psi|_U$. As a result, the Fourier expansion for automorphic realizations of minimal representations behaves very similarly to the $GL(n)$ case, with a formula analogous to (1.7). Since this logic breaks down when U is nonabelian, we restrict to the cases in Theorem 1 (where this complication does not occur).

Lemma 1. *Consider the E_n Dynkin diagrams as numbered in Fig. 2 and the chain of Levi components of maximal parabolics $P(\{\alpha_n\})$ of types*

$$E_7 \supset E_6 \supset D_5 \supset A_4 \supset A_2 \times A_1 \supset A_1 \times A_1 \supset A_1 \tag{4.2}$$

formed by successively deleting the highest numbered node. The unipotent radical U of each of these parabolics is abelian. Decompose the Lie algebra $\mathfrak{n}$ of N as the direct sum of $\mathfrak{n}'$ and the Lie algebra $\mathfrak{u}$ of U, where $\mathfrak{n}'$ is spanned by root vectors X_α of positive roots whose coefficient of α_n is zero. Then if X is a nonzero element of $\mathfrak{n}'$, the sum $X + X_{\alpha_n}$ cannot lie in the minimal adjoint nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$.

Proof. The unipotent radicals are abelian because the highest root of each of these root systems has coefficient 1 of the last simple root. For the second statement, we note that X_{α_n} is an element of $\mathcal{O}$. In the minimal (56-dimensional) representation of $\mathfrak{e}_7$, X_{α_7} is a matrix which squares to zero, and therefore $p(x) = x^2$ is the minimal polynomial of any element of $\mathcal{O}$ in this representation. However, when one writes $X = \sum c_\alpha X_\alpha$ as a linear combination of the 36 root vectors $X_\alpha \in \mathfrak{n}'$ (corresponding to the positive roots of the embedded $\mathfrak{e}_6$ in (4.2)), the condition that $p(X + X_{\alpha_n}) = 0$ forces each of the 36 coefficients c_α to vanish. This proves the lemma for $n = 7$. The analogous argument applies to the case of $n = 6$ in its 27-dimensional representation, but with a linear combination of 20 root vectors (corresponding to the positive roots of the embedded $\mathfrak{so}(5, 5)$).

The case of $n = 5$ is slightly different: in the standard 10-dimensional representation of $\mathfrak{so}(5, 5)$ the above argument does not rule out $c_\alpha \neq 0$ for the roots $\alpha = \alpha_1$, α_2, or $\alpha_1 + \alpha_2$. However, the matrices for $c_{\alpha_1} X_{\alpha_1} + c_{\alpha_2} X_{\alpha_2} + c_{\alpha_1 + \alpha_2} X_{\alpha_1 + \alpha_2} + X_{\alpha_5}$ have rank at least 4 unless $c_{\alpha_1} = c_{\alpha_2} = c_{\alpha_1 + \alpha_2} = 0$. The matrix for X_{α_5} has rank 2 and hence so must all elements of $\mathcal{O}$, proving the lemma for $n = 5$. The cases of smaller n can be handled directly in terms of Jordan canonical form. □

Proof of Theorem 1. Let G be a split Chevalley group of type E_7 and let $P = P(\{\alpha_7\})$ denote the standard maximal parabolic subgroup of G associated with the last node (in the numbering of Fig. 2). Its unipotent radical is 27-dimensional, with four complex character variety orbits of dimensions 0, 17, 26, and 27 (see Section 5.8.7). According to the table there, the 26- and 27-dimensional orbits lie in coadjoint nilpotent orbits strictly larger than the minimal coadjoint nilpotent orbit in the usual closure ordering. Hence Matumoto's and Mœglin–Waldspurger's theorems imply that F can be written as a sum (1.3) in which the only characters which contribute are either trivial or $L(k)$-equivalent to the restriction of ψ to U. Since the adelic automorphic form F is left invariant under $G(k) \supset L(k)$, in considering these Fourier coefficients we may furthermore assume that either χ is trivial or equal to $\psi|_U$.

We now separately examine these two types of contributions. First, assume that $\chi = \psi|_U$. The coefficients F_χ are automorphic under the stabilizer of χ inside the Levi component L. This stabilizer is a maximal k-parabolic of L associated with node 6 in the E_6 Dynkin diagram, and contains $N \cap L$. In particular it has a 16-dimensional unipotent radical V in which we may take a Fourier expansion of F_χ. Together V and U generate the unipotent radical of the nonmaximal parabolic subgroup $P(\{\alpha_6, \alpha_7\})$ in G. However, Lemma 1 and Matumoto's and Mœglin–Waldspurger's theorems imply no nontrivial characters on V can contribute to this expansion (essentially because the minimal orbit has already been "used up" by the nontrivial character χ on U). Thus F_χ is trivial under left translation by V, and is consequently automorphic on the Levi component of the stabilizer, of type D_5.

In the case that χ is trivial, F_χ is automorphic on $\Gamma \cap L$ by dint of the fact that the stabilizer of the trivial character is the full group. Thus it is automorphic on a Chevalley group of type E_6. An

automorphic function on E_6, be it F_χ or an automorphic form for a Chevalley group of type E_6, can then be expanded in the parabolic $P(\{\alpha_6\})$ as above. Thus in all cases we can look at a further Fourier expansion of an automorphic function on a smaller group in the chain (4.2). Proceeding downwards and using the fact that the unipotent radicals in Lemma 1 are always abelian, we see that F is a sum of translates of Fourier coefficients of the form (1.8). $\square$

5. Orbit structure of internal Chevalley modules

In this section, we list the full complex orbit structure for all internal Chevalley modules of maximal parabolic subgroups. Recall that these are the actions of the Levi component on the individual graded pieces $\mathfrak{u}_i$ from (3.1). We also give examples for some low rank classical groups, noting that those with rank $\leqslant 2$ have been discussed earlier in Section 2. Papers [30–33] give a complete discussion in a number of important cases; see also [7,13,15,29] for some historically important examples. Peter Littelmann's computer software[5] [23] computes the orbits in first graded piece $\mathfrak{u}_1$. Because of the observation (stated precisely in each case below) that each higher graded piece $\mathfrak{u}_i$, $i > 1$, occurs as the first graded piece of another internal Chevalley module, the software thus handles all cases over $\mathbb{C}$. This observation was previously used in [36] as part of an induction that establishes analytic properties in the Langlands–Shahidi method.

A few comments are in order about covers. First of all, the orbit structure of internal Chevalley modules is unaffected by taking a central extension: this is because the center acts trivially on the Lie algebra under the adjoint action. Therefore in working out the examples for the Lie algebras of various types below, it is sufficient to calculate with a particular semisimple Lie group having that Lie algebra. Furthermore, the action of the Levi L is essentially pinned down by that of $[L, L]$, since the action of the center of L on each $\mathfrak{u}_i$ can be easily described in terms of the structure of the root system. The action of $[L, L]$ can itself be identified using the Weyl character formula, which is slightly more difficult but still straightforward.

Though the tables here compute the orbits on the graded pieces $\mathfrak{u}_i$ only for $i > 0$, the orbits for $\mathfrak{u}_{-i}$ are related using the Cartan involution. In particular, the character variety $\mathfrak{u}_{-1}$'s orbits are identical to those of $\mathfrak{u}_1$.

For the sake of compact notation, we often say that a subalgebra of a Lie algebra "contains a root" α when it contains a root vector X_α for α; likewise, we apply this same terminology to a subgroup that contains the one-parameter subgroup generated by X_α. We shall also sometimes write a nonsimple root by stringing together its coefficients when expanded a sum of the positive simple roots (for example, the root $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ of D_4 could be more concisely written as 1211). Furthermore, we will indicate a basepoint of an orbit is a sum of root vectors by formally adding these abbreviated labels of the respective roots. (In each case, linear combinations with nontrivial coefficients of these root vectors gives a basepoint of the same orbit, so omitting coefficients is harmless.) We also write the basepoint of the trivial orbit as $00\cdots00$. In the tables below we list basepoints for each orbit, their dimensions, and the adjoint nilpotent orbit they are contained in. Since these are isomorphic to coadjoint nilpotent orbits, we will label its column as such (in deference to the commonly recognized terminology). Such orbits will be listed in terms of their marked Dynkin diagrams, which are also strings of nonnegative integers.

Before giving the list of orbits we shall first make some remarks pertinent to the classical group cases. As we remarked in Section 2.1, the orbit structure for groups of type A_n crucially depends on the fact that any linearly independent set of vectors can be extended to a basis. We now describe the analogous vector space statements needed for the other classical groups. These are somewhat more complicated and involve Witt's theorem (which we recall below). We shall keep the statements here flexible enough to cover the case of type A_n, which we will be able to study directly using elementary linear algebra.

Suppose that V is a complex *bilinear* space, that is, a vector space equipped with a symmetric or skew-symmetric bilinear form $\langle .,. \rangle_V$. Let r_V and n_V denote the rank and nullity of $\langle .,. \rangle_V$, respectively, and let $\epsilon_V = 1, -1, 0$ according as $\langle .,. \rangle_V$ is symmetric, skew-symmetric, or both (i.e., identically 0). These three cases correspond to groups of type B_n or D_n, groups of type C_n, and groups of type A_n.

Two complex bilinear spaces V and W are *isometric* when there exists an *isometry* between them, i.e., an invertible linear map $s : V \to W$ satisfying $\langle sv_1, sv_2 \rangle_W = \langle v_1, v_2 \rangle_V$ for all $v_1, v_2 \in V$. Witt's theorem [1, Theorem 3.9] asserts that this is the case precisely when $(r_V, n_V, \varepsilon_V) = (r_W, n_W, \varepsilon_W)$. Moreover, when this condition holds and furthermore the nullities $n_V = n_W = 0$, any isometry between subspaces of V and W extends to one between all of V and W.

The set of isometries from V to itself constitutes its isometry group $I(V) \subset GL(V)$. Witt's theorem implies that two linear maps a, b from a vector space X into a bilinear space V have isometric images if and only if b can be written as $s^{-1}ac$ for some $c \in GL(X)$ and $s \in I(V)$.

We now record the translations of these statements in terms of matrices, via the bilinear form on $\mathbb{C}^n$ defined from an $n \times n$ matrix by the formula $\langle v_1, v_2 \rangle = v_1^t M v_2$. This form is symmetric or skew-symmetric according as M is symmetric or skew-symmetric, and its rank and nullity are the rank and nullity of M, respectively. An isometry between bilinear spaces corresponding to $n \times n$ matrices M_1, M_2 is a matrix $S \in GL(n, \mathbb{C})$ such that $S^t M_1 S = M_2$. The isometry group for the bilinear space is

$$I(M) = \left\{ S \in GL(n, \mathbb{C}) \mid S^t M S = M \right\},$$

and the rank of the column space of an $n \times m$ matrix A is $\mathrm{rank}(A^t M A)$. The consequences of Witt's theorem mentioned above can be restated as follows:

Lemma 2.

(1) *Let M_1 and M_2 be two complex $n \times n$ matrices, both symmetric or both skew-symmetric. Then $\mathrm{rank}(M_1) = \mathrm{rank}(M_2)$ if and only if there exists a matrix $S \in GL(n, \mathbb{C})$ such that $S^t M_1 S = M_2$.*

(2) *Let M be a complex $m \times m$ nonsingular symmetric or skew-symmetric matrix, and let A, B be complex $m \times n$ matrices. Then*

$$\mathrm{rank}(A) = \mathrm{rank}(B) \quad \text{and} \quad \mathrm{rank}\left(A^t M A\right) = \mathrm{rank}\left(B^t M B\right)$$

if and only if there exist $C \in GL(n, \mathbb{C})$ and $S \in I(M)$ such that $B = S^{-1} A C$.

5.1. Type A_n: $SL(n+1)$

Owing to the Dynkin diagram symmetry, there are essentially $\lceil \frac{n}{2} \rceil$ cases here. A standard maximal parabolic subgroup is block upper triangular according to a decomposition $n_1 + n_2 = n + 1$, with Levi component L isomorphic to the subgroup of $GL(n_1) \times GL(n_2)$ defined by $\{(g, g') \in GL(n_1) \times GL(n_2) \mid \det(g) \det(g') = 1\}$. The unipotent radical is abelian in each case, so $\mathfrak{u} = \mathfrak{u}_1$ is isomorphic to $n_1 \times n_2$ matrices. The action of L on $\mathfrak{u}$ is the tensor product action of the standard representation of $GL(n_1)$ on n_1-dimensional vectors, with the contragredient representation of $GL(n_2)$ on n_2-dimensional column vectors.

This action has been classically studied and has $\min(n_1, n_2) + 1$ complex orbits, classified by the rank of the $n_1 \times n_2$ matrix. Indeed, the general orbit classification for unipotent radicals in classical groups goes by the name of "classical rank theory" because of its similarity to this prototypical case. Representatives, accordingly, are given by matrices which are zero except for an $r \times r$ identity matrix in, say, their top right-hand corner, $0 \leqslant r \leqslant \min(n_1, n_2)$.

Besides Theorem 3, a very complete analysis of vanishing Fourier coefficients for representations of $GL(n)$ is given in [10].

5.2. Type B_n: $SO(n+1,n)$

Consider the standard parabolic subgroup P_{α_k} of B_n, where $1 \leqslant k \leqslant n$. In this case the Levi factor L has $[L, L]$ of type $A_{k-1} \times B_{n-k}$. The unipotent radical is abelian when $k = 1$, but is otherwise a two-step nilpotent group. Here the action of L on $\mathfrak{u}_1$ is the tensor product of the standard (vector) actions of $SL(k)$ and $SO(n - k + 1, n - k)$. The action on $\mathfrak{u}_2$ is the exterior square action of $SL(k)$ on antisymmetric n-tensors and arises for D_k node k; its orbits are described in Section 5.6.2.

The action on $\mathfrak{u}_1$ can be thought of as $SL(k) \times SO(n - k + 1, n - k)$ acting on $k \times (2n - 2k + 1)$ matrices A. Here $\mathbb{C}^{2n-2k+1}$ is viewed as a bilinear space, equipped with a symmetric bilinear form given by a $(2n - 2k + 1) \times (2n - 2k + 1)$ matrix J that defines the orthogonal group. According to Lemma 2, orbits of this action have a fixed value of the rank of A, and a fixed value of the rank of $A^t J A$. Because $\ker(A^t J A) \supset \ker(A)$, the ranks must also satisfy the inequality

$$\operatorname{rank}(A^t J A) \leqslant \operatorname{rank}(A). \tag{5.1}$$

Furthermore,

$$2 \operatorname{rank}(A) - (2n - 2k + 1) \leqslant \operatorname{rank}(A^t J A). \tag{5.2}$$

To see this, let $X = \ker(A^t J A)$ and consider the subspace $AX \subset \mathbb{C}^{2n-2k+1}$, which is the radical of the image of A when thought of as a bilinear subspace. The orthogonal complement of AX in $\mathbb{C}^{2n-2k+1}$ contains the image of A. Because J is nondegenerate, $\mathbb{C}^{2n-2k+1}$ is the direct sum of AX and its orthogonal complement, which gives the inequality $\dim(AX) \leqslant 2n - 2k + 1 - \operatorname{rank}(A)$. At the same time, $\dim(AX) = \operatorname{rank}(A) - \operatorname{rank}(A^t J A)$ because of the "rank plus nullity equals dimension" formula for the image of A. Inequality (5.2) immediately follows.

Finally,

$$\operatorname{rank}(A^t J A), \operatorname{rank}(A) \leqslant \min\{k, 2n - 2k + 1\} \tag{5.3}$$

because the rank of a matrix is always bounded by its row and column size.

The complex orbits on $\mathfrak{u}_1$ not only satisfy (5.1)–(5.3), but exist for each possibility. This can be directly seen by listing $k \times (2n - 2k + 1)$ matrix representatives of the form

$$A = A_{s,p} = \begin{pmatrix} w_1 + i w_2 \\ w_3 + i w_4 \\ \vdots \\ w_{2s-1} + i w_s \\ w_{2s+1} \\ w_{2s+2} \\ \vdots \\ w_{2s+p} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.4}$$

where $\{w_1, \ldots, w_{2n-2k+1}\}$ is a basis of $\mathbb{C}^n$ satisfying $w_j^t J w_\ell = \delta_{j=\ell}$. Indeed, $A_{s,p}$ has rank $p + s$ and $(A_{s,p})^t J A_{s,p}$ has rank p. Part (2) of Lemma 2 states that the matrices $A_{s,p}$, where $0 \leqslant p \leqslant \min\{k, 2n - 2k + 1\}$, $0 \leqslant p + s \leqslant \min\{k, 2n - 2k + 1\}$, $2s + p \leqslant k$, and $2(p + s) - (2n - 2k + 1) \leqslant p$, furnish a complete set of orbit representatives for the action of $GL(k, \mathbb{C}) \times O(n - k + 1, n - k)$. A short computation with stabilizers shows that they are furthermore a complete set of basepoints for the complex orbits of L on $\mathfrak{u}_1$.

5.3. Type C_n: $Sp(2n)$

Because most of the statements in Section 5.2 did not distinguish between symmetric and skew-symmetric bilinear forms, Witt's theorem again applies in a very similar way. The main difference is a distinction between the standard maximal parabolic subgroups P_{α_k} for $k < n$ and $k = n$.

5.3.1. C_n node $k < n$

Here the semisimple part of L has type $A_{k-1} \times C_{n-k}$ and the unipotent radical U is a two-step nilpotent group with $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$. The action on $\mathfrak{u}_1$ is the tensor product of the standard actions of $SL(k)$ and $Sp(2n - 2k)$, while the action on $\mathfrak{u}_2$ is the symmetric square action of $SL(k)$ on symmetric k-tensors. The latter occurs for C_k node k and is described in Section 5.3.2.

Witt's theorem applies here nearly exactly as it does for B_n in Section 5.2. The main difference is that the rank of $A^t J A$ must be even because J is now skew-symmetric. Aside from this, the conditions (5.1)–(5.3) stand after replacing $2n - 2k + 1$ with $2n - 2k$. Fix a basis $\{w_1, \ldots, w_{n-k}, z_1, \ldots, z_{n-k}\}$ of $\mathbb{C}^{2n-2k}$ satisfying

$$\langle x_i, y_j \rangle = \delta_{ij}, \qquad \langle x_i, x_j \rangle = \langle y_i, y_j \rangle = 0. \tag{5.5}$$

Let $A_{p,s}$ denote the $k \times (2n - 2k)$ matrix whose first $2p + s$ rows are $x_1, \ldots, x_p, y_1, \ldots, y_{p+s}$, and the rest all zeros. Then $A_{p,s}$ has rank $2p + s$ while $(A_{p,s})^t J A_{p,s}$ has rank $2p$. This shows that each rank inequality is met. Another application of Lemma 2 and a short stabilizer computation show that the matrices $A_{p,s}$ for p, q satisfying $0 \leqslant 2p \leqslant \min\{k, 2n - 2k\}$, $0 \leqslant 2p + s \leqslant \min\{k, 2n - 2k\}$, and $p + s \leqslant n - k$ then furnish a complete set of orbit representatives for $L(\mathbb{C})$ on $\mathfrak{u}_1$.

5.3.2. C_n node n

Here $[L, L]$ is $SL(n)$ and U is abelian of dimension $\frac{n(n+1)}{2}$. The action on $\mathfrak{u}$ is the symmetric square action of $SL(n)$ on symmetric n-tensors. This can be naturally viewed as the action of $g \in SL(n)$ on symmetric $n \times n$ matrices X given by $g : X \mapsto gXg^t$. Because symmetric matrices can be orthogonally diagonalized, this action has $n + 1$ orbits, each of which is represented by a matrix which has zero entries except for precisely k ones on its diagonal, where k ranges from 0 to n (see Lemma 2).

5.4. Type D_4: $SO(4, 4)$
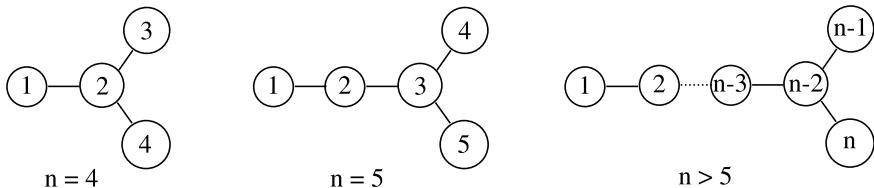
Fig. 1. D_n Dynkin diagrams.

Before giving the general theory, we give some detailed examples for special cases that were important in [12] and in Section 4.

The triality makes D_4 exceptional among classical groups. Let us first consider a maximal parabolic $P = LU$ associated to one of its 3 terminal nodes (numbered 1, 3, or 4 in Fig. 1). Then L is of type A_3 and U is 6-dimensional. Because the largest root of the D_4 root system is $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$, U is abelian and $\mathfrak{u} = \mathfrak{u}_1$. The action of L on $\mathfrak{u}$ is the 6-dimensional representation of $SL(4)$ on antisymmetric tensors. The general theory of this action is well understood and described in comments below in Section 5.6.2. The action of L carves the 6-dimensional vector space into 3 orbits: a zero orbit, one whose basepoint is a root vector for any positive simple root in $\mathfrak{u}$, and an open dense one whose basepoint is a sum of root vectors for $\alpha_1 + \alpha_2 + \alpha_4$ and $\alpha_1 + \alpha_2 + \alpha_3$.

Next, let us consider $P = LU$ associated to the central node (numbered 2). Then the semisimple part of L has type $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ and U is 9-dimensional, breaking up as $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$,

where $\mathfrak{u}_1$ is the 8-dimensional triple tensor product representation of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ and $\mathfrak{u}_2$ is the one-dimensional span of X_{1211}. The action on $\mathfrak{u}_1$ breaks up into 7 orbits under $L(\mathbb{C})$:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$0101 + 1110$	8	0200
$0111 + 1101 + 1110$	7	1011
$0111 + 1101$	5	0002
$1101 + 1110$	5	2000
$0111 + 1110$	5	0020
1111	4	0100
0000	0	0000

Since $\mathfrak{u}_2$ is a line, L acts by multiplication by a scalar on it. Hence there are two orbits: zero and nonzero elements.

5.5. Type D_5: $SO(5, 5)$

There are essentially 4 different cases here, owing to the fact that the two spinor nodes 4 and 5 are interchanged by a diagram automorphism. The behavior for the maximal parabolics associated to the terminal nodes 1, 4, and 5 is similar to that described for D_4, but the internal Chevalley modules for nodes 2 and 3 are new. The actions for all nodes are listed in the following table. The actions are described by the highest weight of the representation of the semisimple group $[L, L]$. For uniformity, the numbering of the fundamental weights $\varpi_1, \varpi_2, \ldots$ corresponds the D_5 Dynkin diagram, not a standard numbering scheme of the individual Levi factors. We use the same convention for higher rank groups whenever such an ambiguity arises.

Node	Type of $[L, L]$	$i = 1$	$i = 2$
1	$SO(4, 4)$	Spin representation	
	dim $\mathfrak{u}_i$	8	
	action	ϖ_2	
2	$SL(2) \times SO(3, 3)$	Standard $\otimes$ Standard	Trivial
	dim $\mathfrak{u}_i$	12	1
	action	$\varpi_1 + \varpi_3$	
3	$SL(3) \times SL(2) \times SL(2)$	Tensor product	Standard
	dim $\mathfrak{u}_i$	12	3
	action	$\varpi_1 + \varpi_4 + \varpi_5$	ϖ_2
4	$SL(5)$	Exterior square	
	dim $\mathfrak{u}_i$	10	
	action	ϖ_2	
5	$SL(5)$	Exterior square	
	dim $\mathfrak{u}_i$	10	
	action	ϖ_2	

5.5.1. D_5 node 1

The parabolic $P = LU$ associated to node 1 has $[L, L]$ of type $SO(4, 4)$, and U an 8-dimensional abelian group. The action on $\mathfrak{u} = \mathfrak{u}_1$ is the 8-dimensional vector representation and has 3 distinct complex orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$11101 + 11110$	8	20000
12211	7	01000
00000	0	00000

5.5.2. D_5 node 2

The parabolic $P = LU$ associated to node 2 has $[L, L]$ of type $SL(2) \times SL(4)$, and U a 13-dimensional Heisenberg group. Thus $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$ where $\mathfrak{u}_1$ is 12-dimensional and $\mathfrak{u}_2$ is 1-dimensional. The action on $\mathfrak{u}_1$ is the tensor product of the standard representation of $SL(2)$ with the exterior square representation of $SL(4)$ (that occurred in Section 5.4). It breaks up into 6 distinct complex orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$01101 + 11110$	12	02000
$01211 + 11101 + 11110$	11	10100
$01211 + 11111$	9	00011
$11101 + 11110$	7	20000
11211	6	01000
00000	0	00000

As before, L acts on the 1-dimensional subspace $\mathfrak{u}_2$ with two orbits: zero and nonzero.

5.5.3. D_5 node 3

The parabolic $P = LU$ associated to node 3 has $[L, L]$ of type $SL(3) \times SL(2) \times SL(2)$, while U is a 15-dimensional two-step nilpotent group such that $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with $\mathfrak{u}_1$ the 12-dimensional representation of $[L, L]$ from the tensor product action of the three Levi factors, and $\mathfrak{u}_2$ a 3-dimensional vector space with a standard $SL(3)$ action. There are 9 complex orbits of L on $\mathfrak{u}_1$:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$00111 + 01101 + 01110 + 11100$	12	00200
$00111 + 01101 + 11110$	11	01011
$01101 + 11110$	10	02000
$01111 + 11101 + 11110$	9	10100
$01111 + 11101$	7	00011
$01111 + 11110$	7	00011
$11101 + 11110$	6	20000
11111	5	01000
00000	0	00000

Since the action on $\mathfrak{u}_2 = \mathbb{C}X_{01211} + \mathbb{C}X_{11211} + \mathbb{C}X_{12211}$ is the standard action of $SL(3)$, it has 2 orbits: zero and nonzero. It occurs for the lower rank group $SL(4)$ and node 3. A representative for the larger orbit is X_{12211}, which intersects the minimal coadjoint orbit, 01000.

5.5.4. D_5 nodes 4 and 5

In the case of the spinor nodes 4 and 5, the action breaks up into three orbits, of dimensions 10, 7, and 0. This is the 10-dimensional exterior square representation of $SL(5)$. These orbits intersect the coadjoint nilpotent orbits with weighted Dynkin diagrams 00011, 01000, and 00000, respectively.

Orbit basepoint	Dimension	Coadjoint orbit intersected
$01211 + 11111$	10	00011
12211	7	01000
00000	0	00000

5.6. Type D_n: $SO(n, n)$

We limit the discussion here to $n > 5$, since the lower rank cases have already been discussed.

5.6.1. D_n node $k < n-1$

The Levi component of the standard maximal parabolic subgroup P_{α_k} is of type $A_{k-1} \times D_{n-k}$. The unipotent radical is nonabelian except for $k = 1$. The action on $\mathfrak{u}_2$ is the symmetric square action of $SL(k)$ on symmetric k-tensors and arises for D_k node k; its orbits are described in Section 5.6.2. We thus focus on $\mathfrak{u}_1$ here, on which L acts by the tensor product of the standard representation of $SL(k)$ with the vector representation of $SO(n-k, n-k)$.

The theory here strongly resembles the case of B_n from Section 5.2 because Witt's theorem again applies nearly verbatim. The rank restriction (5.1) applies directly, while (5.2) and (5.3) need only to be adjusted by replacing $2n - 2k + 1$ by $2n - 2k$. Using the matrices $A_{s,p}$ defined in (5.4), but instead with a basis of dimension $2n - 2k$ of course, we obtain distinct orbit representatives $A_{s,p}$, where $0 \leqslant p \leqslant \min\{k, 2n - 2k\}$, $0 \leqslant p + s \leqslant \min\{k, 2n - 2k\}$, $2s + p \leqslant k$, and $2(p + s) - (2n - 2k) \leqslant p$, for each of the possible configuration of ranks satisfying the inequalities. However, there is an additional wrinkle in this case: Lemma 2 is an assertion about orbit representatives of $GL(k) \times O(n-k, n-k)$. The Levi is connected, and thus its action has a second orbit having $\mathrm{rank}(A) = n - k$ and $A^t J A = 0$ besides the one generated by $A_{n-k,0}$. A representative for this orbit can be given by replacing $w_1 + iw_2$ in (5.4) with $w_1 - iw_2$. This matrix, when combined with the $A_{s,p}$ just listed, comprises a full set of orbit representatives for the complex Levi action on $\mathfrak{u}_1$. This is the only time this phenomenon comes up directly for actions on $\mathfrak{u}_1$, though note that the action on $\mathfrak{u}_2$ in Section 5.9.5 is equivalent to a D_8 action; the second and third of its 18-dimensional orbits are similarly related.

5.6.2. D_n node $k = n - 1$ or n

The two cases here are related by a Dynkin diagram symmetry. The Levi component has $[L, L]$ of type $SL(n)$ and the unipotent radical is abelian of dimension $\frac{n(n-1)}{2}$. The action of L on $\mathfrak{u}$ is the exterior square action of $SL(n)$ on antisymmetric n-tensors. Analogously to the situation in Section 5.3.2, this can be naturally viewed as the action of $g \in SL(n)$ on antisymmetric $n \times n$ matrices X given by $g : X \mapsto gXg^t$. Lemma 2 applies here, and shows that the action has $\lfloor \frac{n}{2} \rfloor + 1$ orbits, given by even rank matrices of the form

$$A_p = \begin{pmatrix} & & J_p \\ & I_{n-2p} & \\ -J_p & & \end{pmatrix}, \quad 0 \leqslant 2p \leqslant n, \tag{5.6}$$

where J_p is the reverse $p \times p$ identity matrix.

5.7. Type E_6
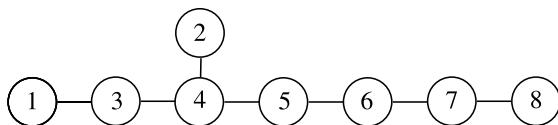
Fig. 2. E_n Dynkin diagrams. The diagram for E_8 is shown here, while the diagrams for E_7 and E_6 are given by removing nodes 8 and then 7, respectively.

Recall from Fig. 2 that nodes 5 and 6 are equivalent, respectively, to nodes 3 and 1, so it suffices to discuss nodes 1, 2, 3, and 4 here. The internal Chevalley modules are described by the table below. In this and the analogous tables for other groups we will abbreviate the types of some semisimple groups below to their Cartan labels, as well as some of the descriptions of the representations. The weights will again be listed using the numbering of the ambient E_6 Dynkin diagram.

Node	Type of $[L, L]$	$i = 1$	$i = 2$	$i = 3$
1	$SO(5, 5)$	Spin representation		
	$\dim \mathfrak{u}_i$	16		
	action	ϖ_2		

2	$SL(6)$	Exterior cube		
	dim $\mathfrak{u}_i$	20	1	
	action	ϖ_4		
3	$SL(2) \times SL(5)$	Standard ⊗ Exterior square		
	dim $\mathfrak{u}_i$	20	5	
	action	$\varpi_1 + \varpi_5$	ϖ_2	
4	$A_2 \times A_1 \times A_2$	Tensor product		
	dim $\mathfrak{u}_i$	18	9	2
	action	$\varpi_1 + \varpi_2 + \varpi_6$	$\varpi_3 + \varpi_5$	ϖ_2
5	$SL(5) \times SL(2)$	Standard ⊗ Exterior square		
	dim $\mathfrak{u}_i$	20	5	
	action	$\varpi_3 + \varpi_6$	ϖ_2	
6	$SO(5,5)$	Spin		
	dim $\mathfrak{u}_i$	16		
	action	ϖ_2		

5.7.1. E_6 node 1

Here $P = LU$ has $[L, L]$ of type $SO(5,5)$ and U is 16-dimensional and abelian. The action of L on $\mathfrak{u} = \mathfrak{u}_1$ is the 16-dimensional spin representation. It has 3 orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$111221 + 112211$	16	100001
122321	11	010000
000000	0	000000

5.7.2. E_6 node 2

In this case $[L, L]$ is of type $SL(6)$ and U is a 21-dimensional Heisenberg group. The action on the 20-dimensional space $\mathfrak{u}_1$ is the exterior cube representation, and breaks up into 5 orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$010111 + 112210$	20	020000
$011221 + 111211 + 112210$	19	000100
$111221 + 112211$	15	100001
112321	10	010000
000000	0	000000

The action on the one-dimensional piece $\mathfrak{u}_2$ has two orbits: zero and nonzero.

5.7.3. E_6 node 3

In this case $[L, L]$ is of type $SL(2) \times SL(5)$ and U is a 25-dimensional two-step unipotent group. The Lie algebra decomposes as $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$ with dim $\mathfrak{u}_1 = 20$ and dim $\mathfrak{u}_2 = 5$. The action on $\mathfrak{u}_1$ is the tensor product of the standard representation of $SL(2)$ with the 10-dimensional exterior square representation of $SL(5)$, and has 8 orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$011111 + 011210 + 101111 + 111110$	20	001010
$011111 + 101111 + 111210$	18	110001
$011111 + 111210$	16	020000
$011221 + 111111 + 111210$	15	000100
$011221 + 111211$	12	100001
$111111 + 111210$	11	100001
111221	8	010000
000000	0	000000

The action on $\mathfrak{u}_2$ is the 5-dimensional action of $SL(5)$, and breaks up into 2 orbits: zero and nonzero. A representative for the big orbit is the highest root 122321, which lies in the minimal coadjoint nilpotent orbit 010000.

5.7.4. E_6 node 4

This is the first case with a 3-step nilpotent group. We have $P = LU$ where $[L, L]$ is of type $SL(3) \times SL(3) \times SL(2)$, and U is 29-dimensional with

$$\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3, \qquad \dim \mathfrak{u}_1 = 18, \qquad \dim \mathfrak{u}_2 = 9, \quad \text{and} \quad \dim \mathfrak{u}_3 = 2. \qquad (5.7)$$

The action on the 18-dimensional piece $\mathfrak{u}_1$ is the tensor product of standard representations of the three factors, and has 18 orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$000111 + 010111 + 011110 + 101100$	18	000200
$001111 + 010110 + 101110 + 111100$	17	011010
$001111 + 010111 + 011110 + 101110 + 111100$	16	100101
$010110 + 011100 + 101111$	15	120001
$001111 + 010111 + 101110 + 111100$	14	200002
$001111 + 010111 + 011110 + 101110$	14	001010
$001111 + 011110 + 101110 + 111100$	14	001010
$010111 + 011110 + 101111 + 111100$	14	001010
$001111 + 010111 + 111110$	13	110001
$011111 + 101110 + 111100$	13	110001
$001111 + 111110$	12	020000
$011111 + 101111 + 111110$	11	000100
$010111 + 011110 + 111100$	10	000100
$011111 + 111110$	9	100001
$011111 + 101111$	8	100001
$101111 + 111110$	8	100001
111111	6	010000
000000	0	000000

The action on the 9-dimensional piece $\mathfrak{u}_2$ occurs for $SL(6)$, node 3, and has 4 orbits that are parameterized by rank. The nontrivial orbits there have dimensions 9, 8, and 5, with basepoints $00111 + 01110 + 11100, 01111 + 11110$, and 11111. Hence the orbits on $\mathfrak{u}_2$ are given as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$112210 + 111211 + 011221$	9	000100
$111221 + 112211$	8	100001
112221	5	010000
000000	0	000000

The action on $\mathfrak{u}_3$ is the standard action of $SL(2)$, and breaks up into 2 orbits: zero and nonzero. A representative for the big orbit is the highest root 122321, which lies in the minimal coadjoint nilpotent orbit 010000.

5.8. Type E_7

The internal Chevalley actions for maximal parabolics are given as follows, with the same labeling conventions used for D_5 and E_6.

Node	Type of $[L, L]$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
1	$SO(6, 6)$	Spin			
	$\dim \mathfrak{u}_i$	32	1		
	action	ϖ_3	0		

2	$SL(7)$	Ext. cube			
	dim $\mathfrak{u}_i$	35	7		
	action	ϖ_5	ϖ_1		
3	$SL(2) \times SL(6)$	Stan. $\otimes$ Ext. sq.			
	dim $\mathfrak{u}_i$	30	15	2	
	action	$\varpi_1 + \varpi_6$	ϖ_4	ϖ_1	
4	$A_2 \times A_1 \times A_3$	Tensor			
	dim $\mathfrak{u}_i$	24	18	8	3
	action	$\varpi_1 + \varpi_2 + \varpi_7$	$\varpi_3 + \varpi_6$	$\varpi_2 + \varpi_5$	ϖ_1
5	$SL(5) \times SL(3)$	Ext. sq. $\otimes$ Stan.			
	dim $\mathfrak{u}_i$	30	15	5	
	action	$\varpi_3 + \varpi_7$	$\varpi_2 + \varpi_6$	ϖ_1	
6	$D_5 \times A_1$	Spin $\otimes$ Stan.	Vector		
	dim $\mathfrak{u}_i$	32	10		
	action	$\varpi_2 + \varpi_7$	ϖ_1		
7	E_6	Stan.			
	dim $\mathfrak{u}_i$	27			
	action	ϖ_1			

5.8.1. E_7 node 1

In this case $P = LU$ where U is a 33-dimensional Heisenberg group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with dim $\mathfrak{u}_1 = 32$ and dim $\mathfrak{u}_2 = 1$. The semisimple part $[L, L]$ of L has type $SO(6, 6)$, and acts on $\mathfrak{u}_1$ by the spin representation with 5 orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$1011111 + 1223210$	32	2000000
$1122221 + 1123211 + 1223210$	31	0010000
$1123321 + 1223221$	25	0000010
1234321	16	1000000
0000000	0	0000000

The action on the one-dimensional piece $\mathfrak{u}_2$ has two orbits: zero and nonzero.

5.8.2. E_7 node 2

In this case $P = LU$ where U is a 35-dimensional 2-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with dim $\mathfrak{u}_1 = 35$ and dim $\mathfrak{u}_2 = 7$. The semisimple part $[L, L]$ of L has type $SL(7)$, and acts on $\mathfrak{u}_1$ as it does on antisymmetric 3-tensors:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$0112111 + 0112210 + 1111111 + 1112110 + 1122100$	35	0200000
$0112211 + 1112111 + 1112210 + 1122110$	34	0001000
$0112221 + 1111111 + 1123210$	31	1000010
$0112221 + 1112211 + 1122111 + 1123210$	28	0100001
$1111111 + 1123210$	26	2000000
$1112221 + 1122211 + 1123210$	25	0010000
$0112221 + 1112211 + 1122111$	21	0000002
$1122221 + 1123211$	20	0000010
1123321	13	1000000
0000000	0	0000000

The action on the 7-dimensional piece $\mathfrak{u}_2$ is the standard action of $GL(7)$, and has 2 orbits: zero and nonzero.

5.8.3. E_7 node 3

In this case $P = LU$ where U is a 47-dimensional 3-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3$, with dim $\mathfrak{u}_1 = 30$, dim $\mathfrak{u}_2 = 15$, and dim $\mathfrak{u}_3 = 2$. The semisimple part $[L, L]$ of L is of type $SL(2) \times SL(6)$. Its

action on $\mathfrak{u}_1$ is the tensor product of the standard action of the $SL(2)$ with the 15-dimensional action of the $SL(6)$ on antisymmetric 2-tensors:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$0011111 + 0111110 + 1011111 + 1112100$	30	0020000
$0112111 + 0112210 + 1011110 + 1111100$	29	1001000
$0112111 + 0112210 + 1011111 + 1111110 + 1112100$	28	0010010
$0111111 + 0112210 + 1011111 + 1112110$	26	0000020
$0112221 + 1011100 + 1111000$	25	2000010
$0112111 + 0112210 + 1111111 + 1112110$	25	0001000
$0112221 + 1011111 + 1111110 + 1112100$	24	0001000
$0112111 + 1111111 + 1112210$	23	1000010
$0112111 + 1112210$	20	2000000
$0112221 + 1112111 + 1112210$	19	0010000
$1011111 + 1111110 + 1112100$	16	0010000
$1112111 + 1112210$	15	0000010
$0112221 + 1112211$	15	0000010
1112221	10	1000000
0000000	0	0000000

The action on the 15-dimensional $\mathfrak{u}_2$ is the exterior square action of $GL(6)$, which arises for $SO(6,6)$, node 6. This latter action has 4 orbits, of dimensions 15, 14, 9, and 0, and basepoints for the nontrivial orbits there are $001211 + 011111 + 111101$, $012211 + 111211$, and 122211, respectively, and correspond to the following orbits here:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$1122221 + 1123211 + 1223210$	15	0010000
$1123321 + 1223221$	14	0000010
1224321	9	1000000
0000000	0	0000000

The action on $\mathfrak{u}_3$ is the standard action of $SL(2)$, and has 2 orbits: zero and nonzero. A representative for the big orbit is the highest root 2234321, which lies in the minimal coadjoint nilpotent orbit 1000000.

5.8.4. E_7 node 4

In this case $P = LU$ where U is a 53-dimensional 4-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3 \oplus \mathfrak{u}_4$, with $\dim \mathfrak{u}_1 = 24$, $\dim \mathfrak{u}_2 = 18$, $\dim \mathfrak{u}_3 = 8$, and $\dim \mathfrak{u}_4 = 3$. The semisimple part $[L, L]$ of L is of type $SL(3) \times SL(2) \times SL(4)$. Its action on $\mathfrak{u}_1$ is the tensor product of the standard representations of its three factors.

Orbit basepoint	Dimension	Coadjoint orbit intersected
$0001111 + 0011110 + 0101110 + 0111100 + 1011100 + 1111000$	24	0002000
$0011111 + 0101110 + 0111100 + 1011100 + 1111000$	23	1001010
$0001111 + 0111110 + 1011100 + 1111000$	22	2000020
$0001111 + 0101111 + 0111110 + 1011100$	21	0020000
$0011111 + 0101111 + 0111100 + 1011110 + 1111000$	21	0001010
$0011111 + 0101110 + 1011110 + 1111100$	20	1001000
$0101110 + 0111100 + 1011111 + 1111000$	19	1001000
$0011111 + 0101111 + 0111110 + 1011110 + 1111100$	19	0010010
$0011111 + 0111110 + 1011100 + 1111000$	18	0000020
$0101110 + 0111100 + 1011111$	18	2000010
$0011111 + 0101111 + 1011110 + 1111100$	17	0000020
$0011111 + 0111110 + 1011110 + 1111100$	17	0001000
$0101111 + 0111110 + 1011111 + 1111100$	17	0001000
$0011111 + 0101111 + 0111110 + 1011110$	16	0001000
$0111110 + 1011111 + 1111100$	16	1000010
$0011111 + 0101111 + 1111110$	15	1000010

0011111 + 1111110	14	2000000
0101111 + 0111110 + 1111100	13	0010000
0111111 + 1011111 + 1111110	13	0010000
0111111 + 1111110	11	0000010
1011111 + 1111110	10	0000010
0111111 + 1011111	9	0000010
1111111	7	1000000
0000000	0	0000000

The action on the 18-dimensional $\mathfrak{u}_2$ is the tensor product action of the standard action of $SL(3)$ factor with the exterior square representation of $SL(4)$ factor. It arises for $SO(6, 6)$, node 3, and has 11 orbits. The nontrivial ones have dimensions 18, 17, 15, 14, 13, 12, 12, 11, 8, 7, with respective basepoints $001111 + 011101 + 011110 + 111100$, $001211 + 011101 + 111110$, $001211 + 011111 + 111101 + 111110$, $011101 + 111110$, $011211 + 111101 + 111110$, $001211 + 011111 + 111101$, $001211 + 011111 + 111110$, $011211 + 111111$, $111101 + 111110$, 111211 there. The basepoints and orbits here on $\mathfrak{u}_2$ are thus given by the following table:

Orbit basepoint	Dimension	Coadjoint orbit intersected
0112211 + 1112111 + 1112210 + 1122110	18	0001000
1112221 + 1112111 + 1122210	17	2000000
0112221 + 1112211 + 1122111 + 1122210	15	0100001
1112111 + 1122210	14	2000000
1112221 + 1122111 + 1122210	13	0010000
0112221 + 1112211 + 1122111	12	0000002
0112221 + 1112211 + 1122210	12	0010000
1112221 + 1122211	11	0000010
1122111 + 1122210	8	0000010
1122221	7	1000000
0000000	0	0000000

The action on the 8-dimensional $\mathfrak{u}_3$ is the tensor product of the standard representations of the $SL(2)$ and $SL(4)$ factors, and arises for $SL(6)$, node 2. It thus has 3 orbits, classified by rank. These have dimensions 8, 5, and 0, with basepoints $01111 + 11110$, 11111, and 00000 there, respectively. The orbits here on $\mathfrak{u}_3$ are given as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
1123321 + 1223221	8	0000010
1223321	5	1000000
0000000	0	0000000

The action on $\mathfrak{u}_4$ is the standard representation of $GL(3)$ and has 2 orbits: zero and nonzero. A representative for the big orbit is the highest root 2234321, which lies in the minimal coadjoint nilpotent orbit 1000000.

5.8.5. E_7 node 5

In this case $P = LU$ where U is a 50-dimensional 3-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3$, with $\dim \mathfrak{u}_1 = 30$, $\dim \mathfrak{u}_2 = 15$, and $\dim \mathfrak{u}_3 = 5$. The semisimple part $[L, L]$ of L is of type $SL(5) \times SL(3)$. Its action on $\mathfrak{u}_1$ is the tensor product of the exterior square representation of the $SL(5)$ factor with the standard representation of the $SL(3)$ factor, and has the following orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
0011111 + 0101111 + 0111110 + 0112100 + 1011110 + 1111100	30	0000200
0011111 + 0101111 + 0111110 + 1011110 + 1112100	29	0001010
0011111 + 0101111 + 0111110 + 1011111 + 1112100	28	0110001
0011111 + 0111110 + 1011111 + 1112100	27	0020000
0011111 + 0101111 + 0112110 + 1011110 + 1122100	27	1000101

$0111111 + 0112110 + 1011110 + 1112100$	26	1001000
$0101111 + 0112110 + 1011111 + 1111110 + 1122100$	25	0010010
$0011111 + 0101111 + 1011110 + 1122100$	24	2000002
$0112111 + 1011110 + 1111100$	23	2000010
$0011111 + 0101111 + 1112110 + 1122100$	23	0000020
$0111111 + 0112110 + 1011111 + 1111110 + 1122100$	23	0200000
$0101111 + 0112110 + 1111110 + 1122100$	22	0001000
$0111111 + 0112110 + 1011111 + 1111110$	22	0001000
$0111111 + 1011111 + 1112110 + 1122100$	22	0001000
$0111111 + 1112110 + 1122100$	21	1000010
$0101111 + 1011111 + 1122110$	20	1000010
$0112111 + 1111111 + 1112110 + 1122100$	19	0100001
$0101111 + 1122110$	18	2000000
$0112111 + 1111111 + 1122110$	17	0010000
$1111111 + 1112110 + 1122100$	16	0010000
$0112111 + 1112110 + 1122100$	15	0000002
$1112111 + 1122110$	14	0000010
$0112111 + 1111111$	12	0000010
1122111	9	1000000
0000000	0	0000000

The action on the 5-dimensional piece $\mathfrak{u}_2$ is the tensor product of the standard representations of the two factors, and occurs for $SL(8)$, node 3. It has 4 orbits, classified by rank, having dimensions 15, 12, 7, and 0 with respective basepoints $0011111 + 0111110 + 1111100$, $0111111 + 1111110$, 1111111, and 0000000 there. Thus the orbits here on $\mathfrak{u}_2$ are given by

Orbit basepoint	Dimension	Coadjoint orbit intersected
$1223210 + 1123211 + 1122221$	15	0010000
$1223211 + 1123221$	12	0000010
1223221	7	1000000
0000000	0	0000000

The action on the 15-dimensional piece $\mathfrak{u}_3$ is the standard action of $SL(5)$, and has 2 orbits: zero and nonzero. A representative for the big orbit is the highest root 2234321, which lies in the minimal coadjoint nilpotent orbit 1000000.

5.8.6. E_7 node 6

In this case $P = LU$ where U is a 42-dimensional 2-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with $\dim \mathfrak{u}_1 = 32$ and $\dim \mathfrak{u}_2 = 10$. The semisimple part $[L, L]$ of L is of type $SO(5, 5) \times SL(2)$. Its action on $\mathfrak{u}_1$ is the tensor product of the spin representation of the $SO(5, 5)$ factor with the standard representation of the $SL(2)$ factor, and has the following orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$0011111 + 0101111 + 1112210 + 1122110$	32	0000020
$0112211 + 1112111 + 1112210 + 1122110$	31	0001000
$0112211 + 1011111 + 1223210$	28	1000010
$1011111 + 1223210$	24	2000000
$1112211 + 1122111 + 1223210$	23	0010000
$1123211 + 1223210$	19	0000010
$1112211 + 1122111$	17	0000010
1223211	12	1000000
0000000	0	0000000

The action on $\mathfrak{u}_2$ is the 10-dimensional vector realization of $SO(5, 5)$, and occurs for $SO(6, 6)$, node 1. It has 2 nontrivial orbits, of dimensions 10 and 9 with basepoints $111101 + 111110$ and 122211, respectively in $\mathfrak{so}(6, 6)$. The orbits are given as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
0112221 + 2234321	10	0000010
2234321	9	1000000
0000000	0	0000000

5.8.7. E_7 node 7

This is the only situation where P has an abelian unipotent radical U, which in this case is 27-dimensional. The semisimple part $[L, L]$ of L is of type E_6, and acts on $\mathfrak{u} = \mathfrak{u}_1$ by the minimal, 27-dimensional representation. It has 3 orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
0112221 + 1112211 + 1122111	27	0000002
1123321 + 1223221	26	0000010
2234321	17	1000000
0000000	0	0000000

5.9. Type E_8

The following table lists the internal Chevalley modules, and also for which smaller groups and parabolics the higher graded ones also occur (aside from the standard actions of $SL(n)$). The minimal representations of E_6 and E_7 are written as **27** and **56**, respectively. The same labeling conventions used for D_5, E_6, and E_7 remain in effect here.

Node	Type of $[L, L]$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
1	$SO(7, 7)$	Spin	D_8 node 1				
	dim $\mathfrak{u}_i$	64	14				
	action	ϖ_2	ϖ_8				
2	$SL(8)$	Ext. cube	D_8 node 8				
	dim $\mathfrak{u}_i$	56	28	8			
	action	ϖ_6	ϖ_3	ϖ_8			
3	$SL(2) \times SL(7)$	Stan. $\otimes$ Ext. sq.	E_7 node 2	A_8 node 7			
	dim $\mathfrak{u}_i$	42	35	14	7		
	action	$\varpi_1 + \varpi_7$	ϖ_5	$\varpi_1 + \varpi_2$	ϖ_8		
4	$A_2 \times A_1 \times A_4$	Stan. $\otimes$ Stan. $\otimes$ Stan.	E_7 node 5	E_6 node 3	A_7 node 3	A_4 node 2	
	dim $\mathfrak{u}_i$	30	30	20	15	6	5
	action	$\varpi_1 + \varpi_2 + \varpi_8$	$\varpi_3 + \varpi_7$	$\varpi_2 + \varpi_6$	$\varpi_1 + \varpi_5$	$\varpi_2 + \varpi_3$	ϖ_8
5	$SL(5) \times SL(4)$	Ext. sq. $\otimes$ Stan.	D_8 node 5	A_8 node 4	D_5 node 5		
	dim $\mathfrak{u}_i$	40	30	20	10	4	
	action	$\varpi_3 + \varpi_8$	$\varpi_2 + \varpi_7$	$\varpi_1 + \varpi_6$	ϖ_4	ϖ_8	
6	$SO(5, 5) \times SL(3)$	Spin $\otimes$ Stan.	D_8 node 3	E_6 node 1			
	dim $\mathfrak{u}_i$	48	30	16	3		
	action	$\varpi_2 + \varpi_8$	$\varpi_1 + \varpi_7$	ϖ_5	ϖ_8		
7	$E_6 \times SL(2)$	**27** $\otimes$ Stan.	E_7 node 7				
	dim $\mathfrak{u}_i$	54	27	2			
	action	$\varpi_1 + \varpi_8$	ϖ_6	ϖ_8			
8	E_7	**56**					
	dim $\mathfrak{u}_i$	56	1				
	action	ϖ_7					

5.9.1. E_8 node 1

In this case $P = LU$ where U is a 78-dimensional 2-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with dim $\mathfrak{u}_1 = 64$ and dim $\mathfrak{u}_2 = 14$. The semisimple part $[L, L]$ of L is of type $SO(7, 7)$, which acts on $\mathfrak{u}_1$ by the spin representation with the following orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
11122111 + 11221111 + 11233210 + 12232210	64	20000000
11222221 + 11233211 + 11233210 + 12232111 + 12232210	63	00100000
11122211 + 11233211 + 12232211 + 12343210	59	00000100
11222221 + 12243211 + 12343210	54	10000001
11233321 + 12233221 + 12243211 + 12343210	50	01000000
11122221 + 12343211	44	00000002
12233321 + 12243221 + 12343211	43	00000010
12244321 + 12343321	35	10000000
13354321	22	00000001
00000000	0	00000000

The action on the 14-dimensional $\mathfrak{u}_2$ is the vector representation of $SO(7,7)$, and occurs for $SO(8,8)$, node 1. It has 3 orbits there, of dimensions 14, 13, and 0 and respective basepoints $11111101 + 11111110$, 12222211, and 00000000 (in $\mathfrak{so}(8,8)$). This translates into the following orbits in $\mathfrak{u}_2$:

Orbit basepoint	Dimension	Coadjoint orbit intersected
23354321 + 22454321	14	10000000
23465432	13	00000001
00000000	0	00000000

5.9.2. E_8 node 2

In this case $P = LU$ where U is a 92-dimensional 3-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3$, with $\dim \mathfrak{u}_1 = 56$, $\dim \mathfrak{u}_2 = 28$, and $\dim \mathfrak{u}_3 = 8$. The semisimple part $[L, L]$ of L is of type $SL(8)$, which acts on $\mathfrak{u}_1$ as it does on antisymmetric 3-tensors. It has the following orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
01121111 + 01122111 + 01122210 + 11111111 + 11122110 + 11221100	56	02000000
01122111 + 01122210 + 11111111 + 11122110 + 11221110 + 11222100	55	00010000
01122211 + 11111111 + 11122110 + 11221110 + 11232100	53	10000100
01122210 + 01122211 + 11122111 + 11221110 + 11232100	52	01000010
01122221 + 11111111 + 11122210 + 11222110 + 11232100	50	00100001
01122210 + 01122211 + 11122111 + 11221110	48	00000020
01122221 + 11122111 + 11122210 + 11221111 + 11222110 + 11232100	48	00001000
01122211 + 11111111 + 11222210 + 11232110	47	00000101
01122211 + 11122111 + 11221111 + 11222210 + 11232110	46	10000010
01122211 + 11122111 + 11222210 + 11232110	44	20000000
01122211 + 11122111 + 11222111 + 11222210 + 11232110	43	00100000
11122111 + 11122210 + 11221111 + 11222110 + 11232100	42	00100000
01121111 + 11111111 + 11233210	41	10000002
11122211 + 11222111 + 11222210 + 11232110	41	00000100
01122221 + 11122111 + 11221111 + 11233210	40	00000100
11122221 + 11221111 + 11233210	38	10000001
11122221 + 11222211 + 11232111 + 11233210	35	01000000
11221111 + 11233210	32	00000002
11222221 + 11233211 + 11233210	31	00000010
11122221 + 11222211 + 11232111	28	00000010
11232221 + 11233211	25	10000000
11233321	16	00000001
00000000	0	00000000

The 28-dimensional action of $SL(8)$ on $\mathfrak{u}_2$ is the exterior square action, which arises for $SO(8,8)$, node 8. The action there has 5 orbits, of dimensions 28, 27, 22, 13, and 0, with respective basepoints $00012211 + 00111211 + 01111111 + 11111101$, $00122211 + 01112211 + 11111211$, $01222211 + 11122211$, 12222211, and 00000000 (in $\mathfrak{so}(8,8)$). The orbits here are given by

Orbit basepoint	Dimension	Coadjoint orbit intersected
12233321 + 12243221 + 12343211 + 22343210	28	01000000
12244321 + 12343321 + 22343221	27	00000010
12354321 + 22344321	22	10000000
22454321	13	00000001
00000000	0	00000000

The action of $SL(8)$ on $\mathfrak{u}_3$ is its standard action, and has 2 orbits: zero and nonzero.

5.9.3. E_8 node 3

In this case $P = LU$ where U is a 98-dimensional 4-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3 \oplus \mathfrak{u}_4$, with $\dim \mathfrak{u}_1 = 42$, $\dim \mathfrak{u}_2 = 35$, $\dim \mathfrak{u}_3 = 14$, and $\dim \mathfrak{u}_4 = 7$. The semisimple part $[L, L]$ of L is of type $SL(2) \times SL(7)$, which acts on $\mathfrak{u}_1$ as the tensor product of the standard representation of the $SL(2)$ factor with the exterior square representation of the $SL(7)$ factor. It has the following orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
01111111 + 01121110 + 01122100 + 10111111 + 11111110 + 11121100	42	00000200
01111111 + 01121110 + 10111111 + 11111110 + 11122100	40	10000101
01111111 + 01121110 + 10111111 + 11122100	38	20000002
01122111 + 01122210 + 10111111 + 11111110 + 11121100	37	10000100
01111111 + 01121110 + 11111111 + 11122100	36	00000020
01122111 + 01122210 + 11111110 + 11121100	35	00000101
01122111 + 01122210 + 11111111 + 11121110 + 11122100	34	10000010
01122221 + 10111111 + 11111100 + 11121000	33	00000101
01121111 + 01122210 + 11111111 + 11122110	32	20000000
01122221 + 11111100 + 11121000	31	10000002
01122111 + 01122210 + 11121111 + 11122110	30	00000100
01122221 + 11111111 + 11121110 + 11122100	30	00000100
01122111 + 11121111 + 11122210	28	10000001
01122111 + 11122210	24	00000002
01122221 + 11122111 + 11122210	23	00000010
11111111 + 11121110 + 11122100	22	00000010
11122111 + 11122210	19	10000000
01122221 + 11122211	18	10000000
11122221	12	00000001
00000000	0	00000000

The action of $SL(7)$ on the 35-dimensional piece $\mathfrak{u}_2$ is the exterior cube action from E_7, node 2, whose orbits were listed in Section 5.8.2. Its orbits are as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
12232210 + 11233210 + 12232111 + 11232211 + 11222221	35	00100000
12233210 + 12232211 + 11233211 + 11232221	34	00000100
12243210 + 12232111 + 11233321	31	10000001
12243210 + 12233211 + 12232221 + 11233321	28	01000000
12232111 + 11233321	26	00000002
12243211 + 12233221 + 11233321	25	00000010
12243210 + 12233211 + 12232221	21	00000010
12243221 + 12233321	20	10000000
12244321	13	00000001
00000000	0	0000000

The 14-dimensional tensor product action of $SL(2) \times SL(7)$ on $\mathfrak{u}_3$ occurs for $SL(9)$, node 7, and has 3 orbits, classified by rank. They have dimensions 14, 8, and 0, with respective basepoints $01111111 + 11111110$, 11111111, and 00000000 (in $\mathfrak{sl}(9)$). The orbits here are

Orbit basepoint	Dimension	Coadjoint orbit intersected
13354321 + 22354321	14	10000000
23354321	8	00000001
00000000	0	00000000

The 7-dimensional action of $SL(7)$ on $\mathfrak{u}_4$ is its standard action, and has 2 nonzero orbits: zero and nonzero.

5.9.4. E_8 node 4

This is the most intricate configuration in that it has the deepest grading. Here $P = LU$ where U is a 116-dimensional 6-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3 \oplus \mathfrak{u}_4 \oplus \mathfrak{u}_5 \oplus \mathfrak{u}_6$, with $\dim \mathfrak{u}_1 = 30$, $\dim \mathfrak{u}_2 = 30$, $\dim \mathfrak{u}_3 = 20$, $\dim \mathfrak{u}_4 = 15$, $\dim \mathfrak{u}_5 = 6$, and $\dim \mathfrak{u}_6 = 5$. The semisimple part $[L, L]$ of L is of type $SL(3) \times SL(2) \times SL(5)$, which acts on $\mathfrak{u}_1$ as the tensor product of the standard representations of each factor. The orbits are given as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
00011111 + 00111110 + 01011110 + 01111100 + 10111000 + 11110000	30	20000200
00111110 + 01011111 + 01111000 + 10111100 + 11110000	28	20000101
00011111 + 00111110 + 01011110 + 01111100 + 10111100 + 11111000	28	00000200
00111110 + 01011111 + 01111100 + 10111100 + 11111000	27	10000101
00011111 + 01111110 + 10111100 + 11111000	26	20000002
00111111 + 01011111 + 01111100 + 10111110 + 11111000	25	10000100
00011111 + 01011111 + 01111110 + 10111110	24	00000020
00111111 + 01011110 + 10111110 + 11111100	23	00000101
01011110 + 01111100 + 10111111 + 11111000	23	00000101
00111111 + 01111100 + 10111110 + 11111000	22	20000000
00111111 + 01011111 + 01111110 + 0111110 + 11111100	22	10000010
01011110 + 01111100 + 10111111	21	10000002
00111111 + 01011111 + 10111110 + 11111100	20	20000000
00111111 + 01111110 + 10111110 + 11111100	20	00000100
01011111 + 01111110 + 10111111 + 11111100	20	00000100
01111110 + 10111111 + 11111100	19	10000001
00111111 + 01011111 + 01111110 + 10111110	18	00000100
00111111 + 01011111 + 11111110	17	10000001
00111111 + 11111110	16	00000002
01011111 + 01111110 + 11111100	16	00000010
01111111 + 10111111 + 11111110	15	00000010
01111111 + 11111110	13	10000000
10111111 + 11111110	12	10000000
01111111 + 10111111	10	10000000
11111111	8	00000001
00000000	0	00000000

The action on $\mathfrak{u}_2$ is the tensor product of the standard action of $SL(3)$ with the exterior square representation of $SL(5)$. It occurs for E_7, node 5, whose orbits were listed in Section 5.8.5. The orbits from there translate here to the following:

Orbit basepoint	Dimension	Coadjoint orbit intersected
11221110 + 11222100 + 11122110 + 01122210 + 11121111 + 01122111	30	00010000
11221110 + 11222100 + 11122110 + 1121111 + 01122211	29	10000100
11221110 + 11222100 + 11122110 + 11221111 + 01122211	28	01000010
11221110 + 11122110 + 11221111 + 01122211	27	00000020
11221110 + 11222100 + 11122210 + 1121111 + 01122221	27	00100001
11222110 + 11122210 + 11121111 + 01122211	26	00000101
11222100 + 11122210 + 11221111 + 11122111 + 01122221	25	10000010
11221110 + 11222100 + 11121111 + 01122221	24	00000101
11222210 + 11121111 + 01122111	23	10000002
11221110 + 11222100 + 11122211 + 01122221	23	20000000
11222110 + 11122210 + 11221111 + 11122111 + 01122221	23	00100000

11222100 + 11122210 + 11122111 + 01122221	22	00000100
11222110 + 11122210 + 11221111 + 11122111	22	00000100
11222110 + 11221111 + 11122211 + 01122221	22	00000100
11222110 + 11122211 + 01122221	21	10000001
11222100 + 11221111 + 11122221	20	10000001
11222210 + 11222111 + 11122211 + 01122221	19	01000000
11222100 + 11122221	18	00000002
11222210 + 11222111 + 11122221	17	00000010
11222111 + 11122211 + 01122221	16	00000010
11222210 + 11122211 + 01122221	15	00000010
11222211 + 11122221	14	10000000
11222210 + 11222111	12	10000000
11222221	9	00000001
00000000	0	0000000

The action on $\mathfrak{u}_3$ is the tensor product of the standard action of $SL(2)$ with the exterior square representation of $SL(5)$. It occurs for E_6, node 3, whose orbits were listed in Section 5.8.3. The orbits here are the following:

Orbit basepoint	Dimension	Coadjoint orbit intersected
11233211 + 11232221 + 12233210 + 12232211	20	00000100
11233211 + 12233210 + 12232221	18	10000001
11233211 + 12232221	16	00000002
11233321 + 12233211 + 12232221	15	00000010
11233321 + 12233221	12	10000000
12233211 + 12232221	11	10000000
12233321	8	00000001
00000000	0	00000000

The action on $\mathfrak{u}_4$ is the tensor product of the standard actions of $SL(3)$ and $SL(5)$, and arises for $SL(8)$, node 3. It has 4 orbits, classified by rank. They have dimensions 15, 12, 7, and 0, with respective basepoints $0011111 + 0111110 + 1111100$, $0111111 + 1111110$, 1111111, and 0000000 (in $\mathfrak{a}_7$). It has the following orbits in $\mathfrak{u}_4$:

Orbit basepoint	Dimension	Coadjoint orbit intersected
22343221 + 12343321 + 12244321	15	00000010
22343321 + 12344321	12	10000000
22344321	7	00000001
00000000	0	00000000

The action of $\mathfrak{u}_5$ is the tensor product of the standard actions of $SL(3)$ and $SL(2)$, and occurs for $SL(5)$, node 2. It has 3 orbits, classified by rank. They have dimensions 6, 4, and 0, with respective basepoints $0111 + 1110$, 1111, and 0000 there. The orbits here are as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
23354321 + 22454321	6	10000000
23454321	4	00000001
00000000	0	00000000

The action of $\mathfrak{u}_6$ is the standard action of $GL(5)$, and has two orbits: zero and nonzero.

5.9.5. E_8 node 5

Here $P = LU$ where U is a 104-dimensional 5-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3 \oplus \mathfrak{u}_4 \oplus \mathfrak{u}_5$, with $\dim \mathfrak{u}_1 = 40$, $\dim \mathfrak{u}_2 = 30$, $\dim \mathfrak{u}_3 = 20$, $\dim \mathfrak{u}_4 = 10$, and $\dim \mathfrak{u}_5 = 4$. The semisimple part $[L, L]$ of L is of type $SL(5) \times SL(4)$, which acts on $\mathfrak{u}_1$ as the tensor product of the exterior square representation of the $SL(5)$ factor with the standard representation of the $SL(4)$ factor. The orbits are given as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
00001000 + 00011110 + 01011111 + 01111110 + 01121100 + 10111100 + 11121000 + 11221111	40	00002000
00011110 + 00111100 + 01011111 + 01111110 + 10111110 + 11111100 + 11221000	39	00010100
00011111 + 00111110 + 01011110 + 01111111 + 01121100 + 11111100 + 11221000	38	10001010
00111111 + 01011110 + 01121000 + 10111110 + 11111100 + 11121000	38	01100010
00011110 + 01011110 + 01111110 + 01121100 + 10111110 + 11111100 + 11221000	37	00100101
00111111 + 01011111 + 01111110 + 01121100 + 10111100 + 11121000	37	10010001
00011111 + 00111110 + 01011110 + 01111111 + 11111100 + 11221000	36	20000020
00011111 + 01111110 + 01121100 + 10111110 + 11111100 + 11221000	36	02000002
00111111 + 01011111 + 01111110 + 01121100 + 10111110 + 11111100 + 11121000	36	00010010
00111111 + 01011111 + 01111110 + 01121100 + 10111110 + 11111100 + 11221000	35	00100100
00011111 + 01011110 + 01111110 + 01121100 + 11111100 + 11221000	35	00010002
00111111 + 01011111 + 01121110 + 10111100 + 11111000	35	20000101
00001111 + 01121110 + 10111110 + 11111100 + 11121100 + 11221000	35	00010002
00001111 + 01121110 + 11111110 + 11121100 + 11221000	34	10000102
00111111 + 01111110 + 01121100 + 10111110 + 11111100 + 11121000	34	00000200
00111111 + 01011111 + 01121100 + 10111110 + 11111100 + 11221000	34	00000200
00111111 + 01011111 + 01111110 + 01121100 + 10111110 + 11221000	34	00010001
01011111 + 01111110 + 01121100 + 10111111 + 11111100 + 11121000	34	00010001
00111111 + 01011111 + 01121100 + 11111110 + 11121000	33	10000101
01111110 + 01121100 + 10111111 + 11111100 + 11121000	33	10000101
00111111 + 01011111 + 01111110 + 10111110 + 11121100 + 11221000	33	10001000
00111111 + 01011111 + 01111110 + 01121100 + 10111110 + 11111100	33	00010000
00001111 + 01121110 + 11111110 + 11121100 + 11221000	32	01000012
00111111 + 01121100 + 11111110 + 11121000	32	20000002
00111111 + 01011111 + 01111110 + 10111111 + 11121100 + 11221000	32	02000000
00111111 + 01011111 + 01111110 + 10111110 + 11121100	32	10000100
00111111 + 01011111 + 01111110 + 10111111 + 11121100	31	01000010
00111111 + 01011110 + 10111111 + 11121100 + 11221000	31	01000010
00111111 + 01011111 + 01121100 + 11111110 + 11121100 + 11221000	31	00010000
01011111 + 01121110 + 10111111 + 11111100 + 11221000	31	10000100
00111111 + 01011111 + 01121110 + 10111110 + 11221100	30	00100001
00111111 + 01111110 + 10111111 + 11121100	30	00000020
01111111 + 01121110 + 10111111 + 11121100 + 11221000	30	00100001
01111111 + 01121110 + 10111110 + 11121100	29	00000101
00111111 + 01011111 + 11111110 + 11121100 + 11221000	29	10000100
01111111 + 01121110 + 10111111 + 11111110 + 11121100 + 11221000	29	00001000
01011111 + 01121110 + 10111111 + 11111110 + 11221100	28	10000010
01011111 + 01111110 + 11121100 + 11221000	28	20000000
01121111 + 10111110 + 11111100 + 11221000	28	00000101
00001111 + 01121110 + 11121100 + 11221000	28	00000022
00111111 + 01011111 + 10111110 + 11221100	27	00000101
01111111 + 01121110 + 10111111 + 11121100 + 11221000	27	10000010
01111111 + 01121110 + 11111110 + 11121100 + 11221000	27	00100000
00111111 + 01011111 + 11121110 + 11221100	26	20000000
01121111 + 10111110 + 11111100	26	10000002
01111111 + 01121110 + 10111111 + 11111110 + 11221100	26	00100000
01121111 + 10111111 + 11111110 + 11121100 + 11221000	26	00100000
01011111 + 01121110 + 11111110 + 11221100	25	00000100
01121111 + 11111110 + 11121100 + 11221000	25	00000100
01111111 + 10111111 + 11121110 + 11221100	25	00000100
01111111 + 01121110 + 10111111 + 11111110	24	00000100
01111111 + 11121110 + 11221100	24	10000000
01011111 + 10111111 + 11221110	22	10000001
01121111 + 11111111 + 11121110 + 11221100	22	01000000
01011111 + 11221110	20	00000002
10111111 + 11111110 + 11121100 + 11221000	20	01000000
11111111 + 11121110 + 11221100	19	00000010
01121111 + 11111111 + 11221110	19	00000010
01121111 + 11121110 + 11221100	18	00000010
11121111 + 11221110	16	10000000
01121111 + 11111111	13	10000000
11221111	10	00000001
00000000	0	00000000

The 30-dimensional action of $SL(5) \times SL(4)$ on $\mathfrak{u}_2$ is the tensor product of the standard action on $SL(5)$ with the exterior square action of $SL(4)$, and occurs for $SO(8,8)$, node 5. It has 16 orbits, of dimensions 30, 29, 28, 27, 25, 24, 23, 21, 18, 18, 18, 17, 15, 10, 9, and 0, with respective basepoints

00001211 + 00011111 + 00111101 + 00111110 + 01111100 + 11111000, 00001211 + 00011111 + 00111101 + 01111110 + 11111100, 00011111 + 00111101 + 01111110 + 11111100, 00001211 + 00111111 + 01111101 + 01111110 + 11111100, 00011211 + 00111111 + 01111101 + 11111110, 00111111 + 01111101 + 01111110 + 11111100, 00111211 + 01111101 + 11111110, 00111211 + 01111111 + 11111101 + 11111110, 01111101 + 11111110, 00111211 + 01111111 + 11111101, 00111211 + 01111111 + 11111110, 01111211 + 11111101 + 11111110, 01111211 + 11111111, 11111101 + 11111110, and 11111211, 00000000 (in $\mathfrak{so}(8,8)$). The orbits here are given as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
01122221 + 11122211 + 11222210 + 11222111 + 11232110 + 12232100	30	00001000
01122221 + 11122211 + 11222210 + 11232111 + 12232110	29	10000010
11122211 + 11222210 + 11232111 + 12232110	28	20000000
11122221 + 11222211 + 11232210 + 11232111 + 12232110	27	00100000
11122221 + 11222211 + 11232210 + 12232111	25	00000100
11222211 + 11232210 + 11232111 + 12232110	24	00000100
11222221 + 11232210 + 12232111	23	10000001
11222221 + 11232211 + 12232210 + 12232111	21	01000000
11232210 + 12232111	18	00000002
11222221 + 11232211 + 12232210	18	00000010
11222221 + 11232211 + 12232111	18	00000010
11232221 + 12232210 + 12232111	17	00000010
11232221 + 12232211	15	10000000
12232210 + 12232111	10	10000000
12232221	9	00000001
00000000	0	00000000

The action on $\mathfrak{u}_3$ is the tensor product of the standard representations of $SL(5)$ and $SL(4)$, and arises for $SL(9)$, node 4. It has 5 orbits, classified by rank, with dimensions 20, 18, 14, 8, and 0, and respective basepoints 00011111 + 00111110 + 01111100 + 11111000, 00111111 + 01111110 + 11111100, 01111111 + 11111110, 11111111, and 00000000, respectively (in $\mathfrak{a}_8$). The orbits here are as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
12233321 + 12243221 + 12343211 + 22343210	20	01000000
12243321 + 12343211 + 22343211	18	00000010
12343321 + 22343221	14	10000000
22343321	8	00000001
00000000	0	00000000

The action on $\mathfrak{u}_4$ is the exterior square representation of $SL(5)$, and arises for $SO(5,5)$, node 5 (see Section 5.5.4). It has three orbits, of dimensions 10, 7, and 0, with respective basepoints 01211 + 11111, 12211, and 00000 there. The orbits here have the following basepoints:

Orbit basepoint	Dimension	Coadjoint orbit intersected
22454321 + 23354321	10	10000000
23464321	7	00000001
00000000	0	00000000

The action on $\mathfrak{u}_5$ is the standard representation of $SL(4)$, and has two orbits: zero and nonzero.

5.9.6. E_8 node 6

Here $P = LU$ where U is a 97-dimensional 4-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3 \oplus \mathfrak{u}_4$, with $\dim \mathfrak{u}_1 = 48$, $\dim \mathfrak{u}_2 = 30$, $\dim \mathfrak{u}_3 = 16$, and $\dim \mathfrak{u}_4 = 3$. The semisimple part $[L, L]$ of L is of type $SO(5,5) \times SL(3)$, which acts on $\mathfrak{u}_1$ as the tensor product of the spin representation of the $SO(5,5)$ factor with the standard representation of the $SL(3)$ factor. The orbits are given as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
00111111 + 01011111 + 01122110 + 10111110 + 11122100 + 11221100	48	00000200
01011111 + 01122110 + 10111111 + 11121110 + 11122100 + 11221100	47	00010001
00111111 + 01011111 + 01122110 + 10111110 + 12232100	45	10000101
01121111 + 01122110 + 10111111 + 11111110 + 11122100 + 11221100	45	10001000
01111111 + 01121111 + 01122110 + 10111111 + 11121110 + 11222100	43	02000000
01121111 + 01122110 + 11111111 + 11121110 + 11122100 + 11221100	43	00010000
00111111 + 01011111 + 10111110 + 12232100	42	20000002
01121111 + 01122110 + 11111111 + 11121110 + 11222100	42	10000100
01011111 + 01122110 + 10111111 + 11221110 + 12232100	41	10000100
01121111 + 01122111 + 11111111 + 11122110 + 11221100	41	01000010
01121111 + 01122110 + 11111111 + 11221110 + 11232100	40	00100001
01121111 + 01122111 + 11122110 + 11221100	39	00000020
01122110 + 11121111 + 11221110 + 11222100	38	00000101
01121111 + 01122110 + 11111111 + 11232100	37	00000101
01122111 + 10111111 + 11122110 + 11221110 + 12232100	37	10000010
01111111 + 10111111 + 11232110 + 12232100	35	20000000
01122111 + 11121111 + 11122110 + 11221110 + 12232100	35	00100000
00111111 + 01011111 + 11122110 + 11221110	34	20000000
01122110 + 11122100 + 11221111	34	10000002
01122111 + 11121111 + 11122110 + 11221110	33	00000100
01122111 + 11111111 + 11232110 + 12232100	33	00000010
10111111 + 11122110 + 11221110 + 12232100	32	00000100
11111111 + 11232110 + 12232100	31	10000001
01122111 + 10111111 + 12232110	30	10000001
11122111 + 11221111 + 11232110 + 12232100	29	01000000
10111111 + 12232110	26	00000002
11122111 + 11221111 + 12232110	25	00000010
11222111 + 11232110 + 12232100	25	00000010
11232111 + 12232110	21	10000000
11122111 + 11221111	18	10000000
12232111	13	00000001
00000000	0	00000000

The action on $\mathfrak{u}_2$ is the tensor product of the 10-dimensional vector representation of the $SO(5, 5)$ factor with the standard representation of $SL(3)$. It occurs for $SO(8, 8)$ node 3, and has orbits of dimensions 30, 29, 27, 24, 22, 21, 19, 12, 11, and 0, with respective basepoints 00111111 + 01111101 + 01111110 + 11111100, 00122211 + 01111101 + 11111110, 00122211 + 01112211 + 11111101 + 11111110, 00122211 + 01112211 + 11111211, 01111101 + 11111110, 01122211 + 11111101 + 11111110, 01122211 + 11112211, 11111101 + 11111110, 11122211, and 00000000 (in $\mathfrak{so}(8, 8)$). The orbits here are as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
12233210 + 11233211 + 12232211 + 11232221	30	00000100
22343210 + 11233211 + 12232221	29	10000001
22343211 + 12343211 + 11233221 + 12232221	27	01000000
22343210 + 12343211 + 12243221	24	00000010
11233211 + 12232221	22	00000002
22343211 + 11233221 + 12232221	21	00000010
22343211 + 12343221	19	10000000
11233221 + 12232221	12	10000000
22343221	11	00000001
00000000	0	00000000

The action on $\mathfrak{u}_3$ is the 16-dimensional spin representation of $SO(5, 5)$, and occurs earlier for E_6, node 1 (see Section 5.7.1). It has the following orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
22454321 + 23354321	16	10000000
23465321	11	00000001
00000000	0	00000000

The action on $\mathfrak{u}_4$ is the 3-dimensional standard representation of $GL(3)$, and has two orbits: zero and nonzero.

5.9.7. E_8 node 7

Here $P = LU$ where U is an 83-dimensional 3-step nilpotent group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3$, with $\dim \mathfrak{u}_1 = 54$, $\dim \mathfrak{u}_2 = 27$, and $\dim \mathfrak{u}_3 = 2$. The semisimple part $[L, L]$ of L is of type $E_6 \times SL(2)$, which acts on $\mathfrak{u}_1$ as the tensor product of the (minimal) 27-dimensional representation of the E_6 factor with the standard representation of the $SL(2)$ factor. The orbits are given as follows:

Orbit basepoint	Dimension	Coadjoint orbit intersected
01122210 + 01122211 + 11122111 + 11221110	54	00000020
01122111 + 10111111 + 11233210 + 12232210	53	00000101
01122211 + 11122111 + 11221111 + 11233210 + 12232210	52	10000010
11122111 + 11221111 + 11233210 + 12232210	50	20000000
11232211 + 11233210 + 12232111 + 12232210	47	00000100
00111111 + 01011111 + 22343210	45	10000002
01122211 + 11122111 + 11221111 + 22343210	44	00000100
01122211 + 12232111 + 22343210	43	10000001
01122211 + 22343210	36	00000002
11233211 + 12232211 + 22343210	35	00000010
12343211 + 22343210	29	10000000
01122211 + 11122111 + 11221111	28	00000010
11233211 + 12232211	27	10000000
22343211	18	00000001
00000000	0	00000000

The action on $\mathfrak{u}_2$ is the 27-dimensional representation of E_6, which occurs for E_7, node 7 (see Section 5.8.7). It has the following orbits here:

| Orbit basepoint	Dimension	Coadjoint orbit intersected
22343221 + 12343321 + 12244321	27	00000010
22454321 + 23354321	26	10000000
23465421	17	00000001
00000000	0	00000000

The action on $\mathfrak{u}_3$ is the standard action of $GL(2)$, and has 2 orbits: zero and nonzero.

5.9.8. E_8 node 8

Here $P = LU$ where U is a 57-dimensional Heisenberg group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with $\dim \mathfrak{u}_1 = 56$ and $\dim \mathfrak{u}_2 = 1$. The semisimple part $[L, L]$ of L is of type E_7, which acts on $\mathfrak{u}_1$ as its (minimal) 56-dimensional representation.

Orbit basepoint	Dimension	Coadjoint orbit intersected
01122221 + 22343211	56	00000002
12244321 + 12343321 + 22343221	55	00000010
22454321 + 23354321	45	10000000
23465431	28	00000001
00000000	0	00000000

The action on the one-dimensional piece $\mathfrak{u}_2$ has two orbits: zero and nonzero.
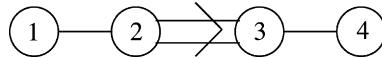
Fig. 3. F_4 Dynkin diagram.

5.10. Type F_4

The following table lists the internal Chevalley modules that arise for maximal parabolic subgroups of F_4. We also indicate where the higher graded pieces arise earlier, except for those which are the standard actions of $SL(n)$. We write the 3rd fundamental representation of $Sp(6)$ as **14** (though caution the reader that the 2nd fundamental representation has the same dimension). The numbering and labeling conventions for the analogous charts for D_5, E_6, E_7, and E_8 remain in effect. (See Fig. 3.)

Node	Type of $[L, L]$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
1	$Sp(6)$	**14**	Trivial		
	dim $\mathfrak{u}_i$	14	1		
	action	ϖ_2			
2	$SL(2) \times SL(3)$	Standard $\otimes$ Sym^2	C_3 node 3		
	dim $\mathfrak{u}_i$	12	6	2	
	action	$\varpi_1 + 2\varpi_4$	$2\varpi_3$	ϖ	
3	$SL(3) \times SL(2)$	Tensor product	B_4 node 3		
	dim $\mathfrak{u}_i$	6	9	2	3
	action	$\varpi_1 + \varpi_4$	$\varpi_2 + 2\varpi_4$	ϖ_4	ϖ_1
4	$SO(7)$	Spin	B_4 node 1		
	dim $\mathfrak{u}_i$	8	7		
	action	ϖ_3	ϖ_1		

5.10.1. F_4 node 1

Here $P = LU$ where U is a 15-dimensional Heisenberg group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with dim $\mathfrak{u}_1 = 14$ and dim $\mathfrak{u}_2 = 1$. The semisimple part $[L, L]$ of L is of type $Sp(6)$ and acts on $\mathfrak{u}_1$ as its 14-dimensional representation 3rd fundamental representation (corresponding to the long root). Its orbits are as follows.

Orbit basepoint	Dimension	Coadjoint orbit intersected
$1122 + 1220$	14	2000
$1222 + 1231$	13	0100
1232	10	0001
1342	7	1000
0000	0	0000

The action on the one-dimensional piece $\mathfrak{u}_2$ has two orbits: zero and nonzero.

5.10.2. F_4 node 2

Here $P = LU$ where U is a 20-dimensional 3-step Heisenberg group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3$, with dim $\mathfrak{u}_1 = 12$, dim $\mathfrak{u}_2 = 6$, and dim $\mathfrak{u}_3 = 2$. The semisimple part $[L, L]$ of L is of type $SL(2) \times SL(3)$ and acts on $\mathfrak{u}_1$ as the tensor product of the standard representation of the $SL(2)$ factor with the 6-dimensional symmetric square representation of the $SL(3)$ factor. It has the following orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
$0100 + 0121 + 1111 + 1120$	12	0200
$0120 + 0122 + 1110$	11	1010
$0122 + 1110$	10	2001
$0121 + 1111 + 1120$	10	0101
$0122 + 1111 + 1120$	9	0010
$0121 + 1111$	8	0002

0122 + 1120	8	2000
1111 + 1120	7	0100
0122 + 1121	7	0100
1121	6	0001
1122	4	1000
0000	0	0000

The symmetric square action of the $SL(3)$ factor on $\mathfrak{u}_2$ comes up earlier for $Sp(6)$, node 3, and has four orbits there of dimensions 6, 5, 3, and 0, with respective basepoints $021 + 111$, 121, 221, and 000. The orbits here are the following:

Orbit basepoint	Dimension	Coadjoint orbit intersected
1222 + 1231	6	0100
1232	5	0001
1242	3	1000
0000	0	0000

The action on $\mathfrak{u}_3$ is the standard action of $SL(2)$ has 2 orbits: zero and nonzero.

5.10.3. F_4 node 3

Here $P = LU$ where U is a 20-dimensional 4-step Heisenberg group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_3 \oplus \mathfrak{u}_4$, with $\dim \mathfrak{u}_1 = 6$, $\dim \mathfrak{u}_2 = 9$, $\dim \mathfrak{u}_3 = 2$, and $\dim \mathfrak{u}_4 = 3$. The semisimple part $[L, L]$ of L is of type $SL(3) \times SL(2)$ and acts on $\mathfrak{u}_1$ as the tensor product of the standard representations of the two factors. It has the following orbits:

Orbit basepoint	Dimension	Coadjoint orbit intersected
0111 + 1110	6	0002
1111	4	0001
0000	0	0000

The action on $\mathfrak{u}_2$ is the tensor product of the standard representation of $SL(3)$ with the symmetric square representation of $SL(2)$, and came up earlier for $SO(5, 4)$, node 3. It has orbits there of dimensions 9, 8, 7, 5, 4, and 0, with respective basepoints $0012 + 0111 + 1110$, $0112 + 1110$, $0112 + 1111$, 1111, 1112, and 0000. It has the following orbits here:

Orbit basepoint	Dimension	Coadjoint orbit intersected
0122 + 1121 + 1220	9	0010
1122 + 1220	8	2000
1122 + 1221	7	0100
1221	5	0001
1222	4	1000
0000	0	0000

The standard actions of $GL(2)$ and $GL(3)$, respectively, on $\mathfrak{u}_3$ and $\mathfrak{u}_4$ have two orbits each: zero and nonzero.

5.10.4. F_4 node 4

Here $P = LU$ where U is a 15-dimensional 2-step Heisenberg group and $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with $\dim \mathfrak{u}_1 = 6$, $\dim \mathfrak{u}_2 = 8$ and $\dim \mathfrak{u}_2 = 7$. The semisimple part $[L, L]$ of L is of type $SO(7)$ and acts on $\mathfrak{u}_1$ as its spin representation, with the following orbits:
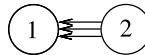
Fig. 4. G_2 Dynkin diagram.

Orbit basepoint	Dimension	Coadjoint orbit intersected
$0121 + 1111$	8	0002
1231	7	0001
0000	0	0000

The action on $\mathfrak{u}_2$ is the 7-dimensional vector action, which came up previously for $SO(5,4)$, node 1. It has orbits there of dimensions 7, 6, and 0, with respective basepoints 1111, 1222, and 0000. Its orbits here are:

Orbit basepoint	Dimension	Coadjoint orbit intersected
1232	7	0001
2342	6	1000
0000	0	0000

5.11. Type G_2

The following table lists the internal Chevalley modules for the two conjugacy classes of maximal parabolic subgroups of G_2. The actions on $\mathfrak{u}_2$ are trivial, while the action for node 1 on $\mathfrak{u}_3$ is the 2-dimensional standard representation of $SL(2)$ (it has two orbits: zero and nonzero). We list the actions below and the orbits on $\mathfrak{u}_1$, which are most interesting in the case of the symmetric cube action for node 2. (See Fig. 4.)

Node	Type of $[L,L]$	$i = 1$	$i = 2$	$i = 3$
1	$SL(2)$	Standard	Trivial	Standard
	dim $\mathfrak{u}_i$	2	1	2
	action	ϖ		($SL(3)$ node 1)
2	$SL(2)$	Symmetric cube	Trivial	
	dim $\mathfrak{u}_i$	4	1	
	action	3ϖ		

Orbit basepoint	Dimension	Coadjoint orbit intersected
Node 1		
11	2	10
00	0	00
Node 2		
$01 + 31$	4	02
21	3	10
31	2	01
00	0	00

Supplementary material

The online version of this article contains additional supplementary material.
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