DEGENERATE WHITTAKER FUNCTIONALS FOR REAL REDUCTIVE GROUPS

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ABSTRACT. In this paper we establish a connection between the associated variety of a representation and the existence of certain *degenerate* Whittaker functionals, for both smooth and K-finite vectors, for all quasi-split real reductive groups, thereby generalizing results of Kostant, Matumoto and others.

1. Introduction

Let G be a real reductive group with Cartan involution θ and maximal compact subgroup $K = G^{\theta}$. Let $\mathcal{M} = \mathcal{M}(G)$ be the category of smooth admissible Fréchet G-representations of moderate growth, and let $\mathcal{HC} = \mathcal{HC}(G) = \mathcal{HC}(\mathfrak{g}, K)$ be the category of Harish-Chandra modules (finitely generated admissible $(\mathfrak{g}, K)$-modules). Here $\mathfrak{g}$ denotes the complexification of the Lie algebra $\mathfrak{g}_0$ of G, and analogous notation will be applied without comment to Lie algebras of other groups below. We will denote a typical representation in $\mathcal{M}(G)$ by (π, W) (or π or W) and a representation in $\mathcal{HC}(G)$ by (σ, M) (or σ or M). By [Wall92, Chapter 11] or [Cas89] or [BK] we have an equivalence of categories

$$(\pi, W) \mapsto \left(\pi^{HC}, W^{HC}\right) : \mathcal{M} \to \mathcal{HC}$$

where $\left(\pi^{HC}, W^{HC}\right)$ denotes the Harish-Chandra module of K-finite vectors in (π, W).

We assume throughout this paper that G is **quasisplit**. We fix a Borel subgroup B with nilradical N and θ-stable maximally split Cartan subgroup $H = TA$, and we define

$$\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}], \mathfrak{v} = \mathfrak{n}/\mathfrak{n}', \Psi = \mathfrak{v}^* \subset \mathfrak{n}^*, \Psi_0 = \{\psi \in \Psi : \psi(x) \in i\mathbb{R} \text{ for } x \in \mathfrak{n}_0\}. \quad (1)$$

Thus Ψ is the space of Lie algebra characters of $\mathfrak{n}$ or equivalently, via the exponential map, group characters of N, while Ψ_0 corresponds to unitary characters of N. We say that ψ is *non-degenerate* if its $H_{\mathbb{C}}$-orbit is open in Ψ. We define $\Psi^{\times}$ to be the set of nondegenerate characters, and set $\Psi_0^{\times} = \Psi^{\times} \cap \Psi_0$.

For $\psi \in \Psi$, $\pi \in \mathcal{M}$ and $\sigma \in \mathcal{HC}$ we define the corresponding Whittaker spaces as follows

$$Wh_{\psi}^{*}(\pi) := \mathrm{Hom}_N^{ct}(\pi, \psi), \Psi(\pi) := \{\psi \in \Psi : Wh_{\psi}^{*}(\pi) \neq 0\}, \quad (2)$$

$$Wh_{\psi}'(\sigma) := \mathrm{Hom}_{\mathfrak{n}}(\sigma, \psi), \Psi(\sigma) := \{\psi \in \Psi : Wh_{\psi}'(\sigma) \neq 0\}, \quad (3)$$

where $\mathrm{Hom}_N^{ct}(\cdot)$ denotes the space of *continuous* N-homomorphisms (functionals). We also define

$$\Psi^{\times}(\pi) = \Psi(\pi) \cap \Psi^{\times}, \Psi_0(\pi) = \Psi(\pi) \cap \Psi_0, \Psi_0^{\times}(\pi) = \Psi(\pi) \cap \Psi_0^{\times}$$
$$\Psi^{\times}(\sigma) = \Psi(\sigma) \cap \Psi^{\times}, \Psi_0(\pi) = \Psi(\sigma) \cap \Psi_0, \Psi_0^{\times}(\sigma) = \Psi(\sigma) \cap \Psi_0^{\times}$$

If $(\pi, W) \in \mathcal{M}(G)$ then W^{HC} is dense in W and thus

$$Wh_{\psi}^{*}(\pi) \subset Wh_{\psi}'\left(\pi^{HC}\right) \text{ and } \Psi(\pi) \subset \Psi(\pi^{HC}).$$

We say that π or σ is *generic* if $\Psi^{\times}(\cdot)$ is not empty. By [CHM, Theorem 8.2] we have $\Psi^{\times}(\pi) = \Psi_0^{\times}(\pi)$. In fact using the same argument one can show $\Psi(\pi) = \Psi_0(\pi)$, but we will not use this result .

Let $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone, and define

$$\mathcal{N}_{\theta} = \mathcal{N} \cap \mathfrak{k}^{\perp}, \mathcal{N}_0 = \mathcal{N} \cap \mathfrak{g}_0^*.$$

To a representation π or σ one can attach invariants such as the annihilator variety, associated variety and wavefront set (see §2.2 below)

$$\mathrm{An}\mathcal{V}(\cdot) \subset \mathcal{N}, \ \mathrm{As}\mathcal{V}(\cdot) \subset \mathcal{N}_{\theta}, \ \mathrm{WF}(\cdot) \subset i\mathcal{N}_0$$

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The dimension of these invariants determines the size (Gelfand-Kirillov dimension) of the representation. We say that π or σ is *large* if its annihilator variety is all of $\mathcal{N}$. A key result of Kostant [Kos78] proves that a representation is large if and only if it is generic. More precisely for $\pi \in \mathcal{M}(G)$ one has

$$\mathrm{An}\mathcal{V}(\pi) = \mathrm{An}\mathcal{V}\left(\pi^{HC}\right) = \mathcal{N} \iff \mathrm{As}\mathcal{V}\left(\pi^{HC}\right) = \mathcal{N}_\theta \iff \mathrm{WF}(\pi) = i\mathcal{N}_0$$
$$\iff \Psi_0^\times(\pi) \neq \emptyset \iff \Psi^\times(\pi) \neq \emptyset \iff \Psi^\times\left(\pi^{HC}\right) \neq \emptyset$$

A number of papers (e.g. [GW80, Mat87, Mat90, Mat88]) provide certain generalizations of [Kos78] to non-generic representations; namely, they consider functionals equivariant with respect to non-degenerate characters of nilradicals of other parabolic subgroups, often referred to as *generalized* Whittaker functionals. In this paper we study a different type of analog: we consider functionals equivariant with respect to possibly degenerate characters of the nilradical of the standard Borel subgroup. Following Zelevinsky [Zel80, §§8.3] we refer to these as *degenerate* Whittaker functionals.

1.1. Main results.

Theorem A. *Let $pr_{\mathfrak{n}^*} : \mathfrak{g}^* \to \mathfrak{n}^*$ denote the restriction to $\mathfrak{n}$, then for $\sigma \in \mathcal{HC}$ we have*

$$\Psi(\sigma) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}(\sigma)) \cap \Psi.$$

This is proved in section 3 below. We now describe the connection between $\Psi_0(\pi)$, $\Psi_0(\pi^{HC})$ and the wavefront set $\mathrm{WF}(\pi)$. Let $H = TA$ be the maximally split Cartan subgroup of G as above, and let $F_G \subset G_{\mathbb{C}}$ be the subgroup of order 2 elements in $\exp i\mathfrak{a}_0$. By [KR71] F_G normalizes G and thus we get an action $\pi \mapsto \pi^a$ on representations of G, and we define

$$\widetilde{\pi} = \bigoplus \{\pi^a : a \in F_G\}$$

We emphasize that the set-theoretic notation means that we only sum over *distinct* transforms π^a of π. Now F_G normalizes $\mathfrak{g}_0$ and $\mathfrak{n}_0$, hence it acts on Ψ_0 and we have

$$\Psi(\pi^a) = a \cdot \Psi(\pi),\ \Psi(\widetilde{\pi}) = F_G \cdot \Psi(\pi) = \bigcup \{a \cdot \Psi(\pi) : a \in F_G\}$$

Theorem B. *Let $\pi \in \mathcal{M}$ and write $\sigma = \pi^{HC}$; then we have*

$$\Psi_0(\pi) \subset \mathrm{WF}(\pi) \cap \Psi \subset \Psi_0(\widetilde{\pi}) = \Psi_0(\sigma). \tag{4}$$

Moreover if $G = GL_n(\mathbb{R})$ or if G is a complex group then $\widetilde{\pi} = \pi$ and we have

$$\Psi_0(\pi) = \mathrm{WF}(\pi) \cap \Psi = \Psi_0(\sigma) = \mathrm{An}\mathcal{V}(\sigma) \cap \Psi_0. \tag{5}$$

If π is generic, then Theorem B follows immediately from [Mat92, Theorem A] and Theorem A. We prove the general result by reduction to the generic case using the Kostant-Sekiguchi correspondence and the coinvariants functor $C_{\mathfrak{u}}$, where $\mathfrak{u}$ is the nilradical of a suitable parabolic subalgebra.

Theorem B implies $\mathrm{An}\mathcal{V}(\sigma) \cap \Psi_0 \supset \Psi_0(\sigma)$ though the reverse inclusion can fail, as shown in section 4.3 for the group $U(2,2)$. We conjecture however that for all quasi-split groups one has the equality

$$\Psi_0(\pi) = \mathrm{WF}(\pi) \cap \Psi, \tag{6}$$

although the proof probably requires additional arguments of an analytic nature.

We prove a stronger result if $G = GL_n(\mathbb{R})$ or if G is a complex classical group, *i.e.* one of the groups

$$GL_n(\mathbb{C}), SL_n(\mathbb{C}), O_n(\mathbb{C}), SO_n(\mathbb{C}), Sp_n(\mathbb{C}). \tag{7}$$

Theorem C. *Let $\pi \in \mathcal{M}(G)$ and suppose one of the following holds:*

(1) $G = GL_n(\mathbb{R}), GL_n(\mathbb{C})$ or $SL_n(\mathbb{C})$;
(2) $G = O_n(\mathbb{C}), SO_n(\mathbb{C})$, or $Sp_n(\mathbb{C})$ and π is irreducible;

then $\Psi_0(\pi)$ and $\mathrm{WF}(\pi)$ determine each other uniquely.

The first case of Theorem C follows easily from Theorem B, since for the groups in this case, every nilpotent orbit intersects Ψ_0. This enables us to strengthen several results from [AGS]. We note that for *unitarizable* π, a weaker version of this theorem follows from [GS12, Theorem A].

For the groups in the second case of Theorem C not every nilpotent orbit intersects Ψ_0, however if π is irreducible then $\mathrm{An}\mathcal{V}(\pi)$ is the closure of a single nilpotent orbit (see [Jos85]), and this allows us to deduce the second case from the following result that may be of independent interest.

Theorem D. *Every nilpotent orbit $\mathcal{O}$ for a complex classical group is uniquely determined by $\overline{\mathcal{O}} \cap \Psi$.*

If π is not irreducible then $\mathrm{WF}(\pi)$ might be the union of several orbit closures, and as shown in (18) such a union is not determined by its intersection with Ψ_0. We also note that Theorem D does not hold for exceptional Lie groups and we describe all the counterexamples in section 6.3.

Remark. *Over p-adic fields, the associated and annihilator varieties are not defined but the notion of wave front set still makes sense (see* [HCh78, How74, Rod75]*). In* [MW87]*, the authors give a very general definition of degenerate Whittaker spaces and prove that the dimensions of "minimally degenerate" Whittaker spaces equal the multiplicities of corresponding coadjoint nilpotent orbits in the wave front set.*

1.2. **The structure of our proof.** In section 2 we give several necessary definitions and preliminary results on filtrations, associated/annihilator varieties, Whittaker functionals, and discuss a version of the Casselman-Jacquet functor.

In section 3 we prove Theorem A. Let (σ, M) be a Harish-Chandra module for G, then every good $\mathfrak{g}$-filtration on M is good as an $\mathfrak{n}$-filtration. This implies that $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) = pr_{\mathfrak{n}^*}\left[\mathrm{As}\mathcal{V}_{\mathfrak{g}}(M)\right]$ where $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(M)$ denotes the associated variety of M as an $\mathfrak{n}$-module (by restriction). We next pass to the commutative Lie algebra $\mathfrak{v} = \mathfrak{n}/\mathfrak{n}'$ by considering the module of coinvariants $C(M) = C_{\mathfrak{n}'}(M) = M/\mathfrak{n}'M$. Since $\mathfrak{v}$ is commutative, $\mathrm{As}\mathcal{V}_{\mathfrak{v}}(CM) = \mathrm{An}\mathcal{V}_{\mathfrak{v}}(CM)$ and we denote both by $\mathcal{V}_{\mathfrak{v}}(CM)$. Then as shown in Lemma 3.0.1,$\mathcal{V}_{\mathfrak{v}}(CM) = \mathrm{Supp}(CM)$, which further coincides with $\Psi(M)$ by the Nakayama Lemma (see §2.1).

If M is any finitely generated $\mathfrak{n}$-module, $\mathcal{V}_{\mathfrak{v}}(CM) \subset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi$, and our task is to prove that

$$\mathcal{V}_{\mathfrak{v}}(CM) \supset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi.$$

This is *not* true for a general $\mathfrak{n}$-module V, indeed CV could even vanish; for example, let $G = GL(3, \mathbb{R})$ and consider the identification of $\mathfrak{n}$ with the Heisenberg Lie algebra $\left\langle x, \frac{d}{dx}, 1\right\rangle$ acting on $V = \mathbb{C}[x]$. However, if M is a Harish-Chandra module it was proven by Casselman that even $M/\mathfrak{n}M$ is non-zero, indeed he proved that $\cap \mathfrak{n}^i M = 0$. This implies that M imbeds (densely) into its $\mathfrak{n}$-adic completion $\widehat{M} := \varprojlim M/\mathfrak{n}^i M$. Following [ENV04] we let $J(M) = \widehat{M}^{\mathfrak{h}\text{-finite}}$ denote the submodule of $\mathfrak{h}$-finite vectors. The functor J can be applied to both M and CM and we prove that

$$\mathcal{V}_{\mathfrak{v}}(CM) = \mathcal{V}_{\mathfrak{v}}(J(CM)) = \mathcal{V}_{\mathfrak{v}}(C(JM)).$$

The first equality follows from the fact that CM and $J(CM)$ are both dense in $(CM)_{[\mathfrak{n}]}$ and hence have the same annihilator, while the second follows from the isomorphism $J(CM) \simeq C(JM)$ proved in Lemma 3.0.2. Moreover $J(M)$ is finitely generated over $\mathfrak{n}$ and glued from lowest weight modules, and hence we get (by Lemma 3.0.3)

$$\mathcal{V}_{\mathfrak{v}}(C(JM)) = \mathcal{V}_{\mathfrak{v}}(JM) \cap \Psi$$

This reduces the problem to showing

$$\mathcal{V}_{\mathfrak{v}}(JM) \cap \Psi \supset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi,$$

which we prove in section 3.3, using the main result of [ENV04] that describes $J(M)$ as a deformation of M. The description is in the language of D-modules, using the Beilinson-Bernstein localization. While it is not true in general that the operation of taking associated variety commutes with limits, but this was proven to be true for holonomic D-modules with regular singularities in [Gin86]. This implies the above containment and finishes the proof of Theorem A.

In section 4 we first prove Theorem B. The special case of large representations follows from [Mat88, Mat92]. To reduce to this case, note that any unitary character ψ of N defines a parabolic subgroup $P = LU$ such that ψ is trivial on $N \cap U$ and non-degenerate on $N \cap L$. Thus we consider the U-coinvariants of π, and we need to know when this space is large as a representation of the Levi subgroup L. For that purpose we use Theorem A. We also use [SV00] that shows that the wave-front set corresponds to the associated variety via the Kostant-Sekiguchi bijection. We next use Theorem B to reduce the proof of Theorem C to Theorem D. In subsection 4.4 we deduce from Theorem B a formula for the wave front set of U-coinvariants of π, where U is the nilradical of a standard parabolic subgroup P of G, whose Levi part is a product of general linear groups.

In section 5 we give several consequences of Theorem C, including applications to the theory of derivatives. In section 6 we prove Theorem D.using basic results on nilpotent orbits from [CoMG93, Car85].

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2. Preliminaries

2.1. **The Nakayama lemma.** The classical Nakayama lemma is a commutative analog of the problems considered in this paper, and in this section we explain this point of view.

Let A be a commutative algebra, finitely generated over complex numbers. The characters of A are ring homomorphisms $A \to \mathbb{C}$. Such a homomorphism is uniquely described by its kernel, which is a maximal ideal in A. Conversely, by Hilbert's Nullstellensatz, every maximal ideal $\mathfrak{m} \in \operatorname{Max} A$ is the kernel of a (unique) ring homomorphism $\phi_{\mathfrak{m}} : A \to \mathbb{C}$. For an A-module M and $\mathfrak{m} \in \operatorname{Max} A$, we have $\operatorname{Hom}_A(M, \phi_{\mathfrak{m}}) \cong M/\mathfrak{m}M$. Thus, we can define

$$\Psi(M) := \{\mathfrak{m} \in \operatorname{Max} A \mid M \neq \mathfrak{m}M\}.$$

On the other hand, the support of M is defined to be

$$\operatorname{Supp}(M) := \operatorname{Var}(\operatorname{Ann} M),$$

where Var denotes the variety of zeroes, and $\operatorname{Ann} M = \{a \in A \mid aM = 0\}$ is the annihilator ideal of M.

Lemma 2.1.1 (Nakayama)**.** *If M is finitely generated over A then* $\operatorname{Supp}(M) = \Psi(M)$.

For completeness, we deduce this result from the version in [AM69].

Proof. We have to show that for any $\mathfrak{m} \in \operatorname{Max} A$ we have $\mathfrak{m}M = M \Leftrightarrow \operatorname{Ann} M \not\subseteq \mathfrak{m}$. But $\operatorname{Ann} M \not\subseteq \mathfrak{m} \Leftrightarrow 1 \in \mathfrak{m} + \operatorname{Ann} M \Leftrightarrow$ there exists an element of $\mathfrak{m}$ that acts on M as identity. The last statement is equivalent to $\mathfrak{m}M = M$ by [AM69, Corollary 2.5]. □

2.2. **Associated variety and annihilator variety.** In this section we let $\mathfrak{q}$ be an arbitrary finite dimensional complex Lie algebra, and let $U = U(\mathfrak{q})$ be its universal enveloping algebra with the usual increasing filtration $U^i, i \geq 0$. By the PBW theorem the associated graded algebra $\overline{U}$ is isomorphic to the symmetric algebra $S(\mathfrak{q})$. For a $\mathfrak{q}$-module V, we define its *annihilator* and *annihilator variety* as follows

$$\operatorname{Ann}(V) = \{u \in U : \forall x \in V \;\; ux = 0\}, \; \operatorname{An}\mathcal{V}(V) := \operatorname{Var}\left(\overline{\operatorname{Ann} V}\right) \subset \mathfrak{q}^*$$

Here $\overline{\operatorname{Ann} V} \subset S(\mathfrak{q})$ denotes the associated graded space of $\operatorname{Ann} V$ under the filtration inherited from U.

For the rest of the section we assume that V is generated by a *finite* dimensional subspace V^0.

In this case we get a filtration $V^i = U^i V^0$. The associated graded space $\overline{V}$ is then an $S(\mathfrak{q})$ module and we define the *associated variety* to be

$$\operatorname{As}\mathcal{V}(V) := \operatorname{Var}(\operatorname{Ann} \overline{V}) = \operatorname{Supp}\left(\overline{V}\right) \subset \mathfrak{q}^*$$

It is standard that $\operatorname{As}\mathcal{V}(V)$ does not depend on the choice of the generating subspace. More generally a filtration on V is called a *good filtration* if the associated graded space is a finitely generated $S(\mathfrak{q})$-module, and any two good filtrations lead to the same associated variety. For a submodule $W \subset V$, a good filtration on V induces good filtrations on W and on V/W by $W^i = W \cap V^i$ and $(V/W)^i = V^i/W^i$.

Lemma 2.2.1. [Ber72] $\operatorname{As}\mathcal{V}(V) = \operatorname{Var}\left(\overline{\operatorname{Ann} V^0}\right)$ *where* $\operatorname{Ann} V^0 = \left\{u \in U : \forall x \in V^0 \;\; ux = 0\right\}$.

Corollary 2.2.2. *We have* $\operatorname{As}\mathcal{V}(V) \subseteq \operatorname{An}\mathcal{V}(V)$*, and equality holds if* $\mathfrak{q}$ *is commutative.*

If there is possibility of confusion we will write $\mathrm{As}\mathcal{V}_{\mathfrak{q}}(V)$ etc. to emphasise dependence on $\mathfrak{q}$.

If $\mathfrak{w}$ is a subalgebra of $\mathfrak{q}$ we define the *coinvariant space* to be the quotient

$$C_{\mathfrak{w}}(V) = V/\mathfrak{w}V$$

If $\mathfrak{w}$ is an ideal then $C_{\mathfrak{w}}V$ is a $\mathfrak{q}$-module and the action descends to the quotient Lie algebra $\mathfrak{r} := \mathfrak{q}/\mathfrak{w}$.

Lemma 2.2.3. *If V is a finitely generated $\mathfrak{q}$-module then $\mathrm{As}\mathcal{V}_{\mathfrak{q}}(C_{\mathfrak{w}}V) = \mathrm{As}\mathcal{V}_{\mathfrak{r}}(C_{\mathfrak{w}}V) \subset \mathrm{As}\mathcal{V}_{\mathfrak{q}}(V) \cap \mathfrak{r}^*$, where $\mathfrak{r}^* \subset \mathfrak{q}^*$ in the usual way.*

Proof. Let Y^0 denote the image of the generating space V^0 under the quotient map $V \to C_{\mathfrak{w}}(V)$. Then Y^0 generates $C_{\mathfrak{w}}(V)$ and $\mathrm{Ann}_{\mathfrak{q}} Y^0 \supset \mathrm{Ann}_{\mathfrak{q}} V^0 + \mathfrak{w}$. The result now follows from Lemma 2.2.1. □

As was discussed in the introduction, the converse statement is not true in general.

2.3. **Nilpotent orbits and wavefront sets.** In this section we assume that G is a real reductive group. Let $\mathcal{N} \subset \mathfrak{g}^*$ denote the null cone, with $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{k}^\perp$ and $\mathcal{N}_0 = \mathcal{N} \cap \mathfrak{g}_0^*$ as before. The groups $G_{\mathbb{C}}, K_{\mathbb{C}}$ and G act with finitely many orbits on $\mathcal{N}$, $\mathcal{N}_\theta$ and $\mathcal{N}_0$ respectively. We write $\mathcal{O}' \leq \mathcal{O}$ if $\mathcal{O}'$ is contained in the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$, and we refer to $\leq$ as the closure order.

Theorem 2.3.1. *There is an order preserving bijection between G-orbits on $\mathcal{N}_0$ and $K_{\mathbb{C}}$-orbits on $\mathcal{N}_\theta$.*

The bijection, which we denote by $\mathcal{O} \mapsto \mathrm{KS}(\mathcal{O}) = \mathrm{KS}_G(\mathcal{O})$, is called the Kostant-Sekiguchi correspondence [Sek87] – the order property is proved in [BS98] (see also [Ohta81]). The group $F_G \subset G_{\mathbb{C}}$ acts on both $\mathcal{N}_\theta$ and $\mathcal{N}_0$, and we have

Proposition 2.3.2. *If $\mathcal{O}$ is a G-orbit on $\mathcal{N}_0$ and $L \subset G$ is a θ-stable Levi subgroup, then we have*

$$F_G \cdot \mathrm{KS}(\mathcal{O}) = \mathrm{KS}(F_G \cdot \mathcal{O}), \mathrm{KS}_L(\mathcal{O} \cap \mathfrak{l}^*) \subset \mathrm{KS}_G(\mathcal{O}).$$

If $(\pi, W) \in \mathcal{M}(G)$ then W^{HC} is dense in W and we can choose a finite dimensional K-invariant generating subspace of W^{HC}. It follows that we have

$$\mathrm{An}\mathcal{V}(\pi) = \mathrm{An}\mathcal{V}(\pi^{HC}) \subset \mathcal{N},\ \mathrm{As}\mathcal{V}(\pi^{HC}) \subset \mathcal{N}_\theta$$

and the two varieties are unions of $G_{\mathbb{C}}$-orbits and $K_{\mathbb{C}}$-orbits respectively.

There is a further invariant of π called the *wavefront set*, which was defined in [How81] in terms of the global character of π. This is a G-invariant set

$$\mathrm{WF}(\pi) \subset i\mathcal{N}_0$$

which, by [Ros95a, Ros95b], coincides with the *asymptotic support* of π introduced in [BV78]. As conjectured in [BV78] and proved in [SV00] one also has

Theorem 2.3.3. *If $(\pi, W) \in \mathcal{M}(G)$ then $\mathrm{WF}(\pi) = i\,\mathrm{KS}\left(\mathrm{As}\mathcal{V}\left(\pi^{HC}\right)\right)$.*

It follows that for all $\pi \in \mathcal{M}(G)$ we have

$$\mathrm{An}\mathcal{V}(\pi) = G_{\mathbb{C}} \cdot \mathrm{As}\mathcal{V}(\pi) = G_{\mathbb{C}} \cdot \mathrm{WF}(\pi) \tag{8}$$

Now suppose that G is a complex reductive group, regarded as a real group. Then the real Lie algebra $\mathfrak{g}_0$ is already a complex Lie algebra, and we have $\mathfrak{g} \cong \mathfrak{g}_0 \times \mathfrak{g}_0$, and $\mathfrak{g}_0$ is diagonally embedded into $\mathfrak{g}$. The Lie algebra $\mathfrak{k}$ is also isomorphic to $\mathfrak{g}_0$, and is embedded into $\mathfrak{g}$ by $X \mapsto (X, \theta(X))$. For a nilpotent orbit $\mathcal{O} \subset \mathcal{N}(\mathfrak{g})$ we have $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$ where $\mathcal{O}_i \subset \mathcal{N}(\mathfrak{g}_0)$. However, if $\mathcal{O}$ intersects $i\mathfrak{g}_0^* \subset \mathfrak{g}^*$ or $\mathfrak{k}^\perp \subset \mathfrak{g}^*$ then $\mathcal{O}_1 = \mathcal{O}_2$, and thus $\mathcal{O}$ is defined by a single nilpotent orbit in $\mathfrak{g}_0$. By [Vog91, Theorem 8.4], only orbits intersecting $\mathfrak{k}^\perp$ can be open orbits in the annihilator variety of an admissible representation $\pi \in \mathcal{M}(G)$, and thus we will be only interested in such orbits.

2.4. **Restricted roots and parabolic subgroups.** Recall that $H = TA$ denotes our fixed θ-stable maximally split Cartan subgroup. Let Σ and Σ_0 denote the root systems of $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathfrak{a}_0$ in $\mathfrak{g}_0$ respectively, and let $\mathfrak{g}^\alpha \subset \mathfrak{g}$ and $\mathfrak{g}_0^\beta \subset \mathfrak{g}_0$ denote the root spaces for $\alpha \in \Sigma$ and $\beta \in \Sigma_0$. For $\alpha \in \Sigma$ let $\widetilde{\alpha}$ denote the restriction of α to $\mathfrak{a}_0$ then either $\widetilde{\alpha} = 0$ or else $\widetilde{\alpha} \in \Sigma_0$. Moreover for any $\beta \in \Sigma_0$ we have

$$\dim_{\mathbb{R}} \left(\mathfrak{g}_0^\beta\right) = \left|\{\alpha \in \Sigma : \widetilde{\alpha} = \beta\}\right|$$

Every $\alpha \in \Sigma$ is real-valued on $\mathfrak{a}_0$ and imaginary-valued on $\mathfrak{t}_0$. The involution θ acts naturally on Σ and if $\alpha' = -\theta\alpha$ then we have

$$\alpha'|_{\mathfrak{a}_0} = \alpha|_{\mathfrak{a}_0}, \alpha'|_{\mathfrak{t}_0} = -\left(\alpha'|_{\mathfrak{t}_0}\right)$$

Lemma 2.4.1. *Let G be a real reductive group then the following are equivalent:*

(1) G is quasi-split.
(2) For all $\alpha \in \Sigma$ we have $\widetilde{\alpha} \neq 0$.
(3) For all $\beta \in \Sigma_0$, we have $\dim_{\mathbb{R}} \left(\mathfrak{g}_0^\beta\right) \leq 2$.

Proof. Since G is quasi-split iff $\mathfrak{g}_0$ has a Borel subalgebra, the lemma depends only on the Lie algebra $\mathfrak{g}_0$. Moreover it suffices to prove the lemma for simple factors of $\mathfrak{g}_0$. The result is obvious for split and complex factors, and by [He08] the other possible simple quasi-split factors are of the form

$\mathfrak{g}_0$	$\mathfrak{su}_{l,l}$	$\mathfrak{su}_{l,l+1}$	$\mathfrak{so}_{l,l+2}$	$\mathfrak{e}_{6(2)}$
Label	$AIII\,(r = 2l - 1)$	$AIII\,(r = 2l)$	$DI\,(r = l + 1)$	EII

Now the lemma can be checked using Table VI of [He08], where (2) means that there are no black dots in the Satake diagram, and (3) means that each of the multiplicities m_λ and $m_{2\lambda}$ is at most 2. □

Since in this paper we suppose that G is quasi-split, we obtain

Corollary 2.4.2. *If $\alpha \in \Sigma$ satisfies $\dim_{\mathbb{R}} \left(\mathfrak{g}_0^{\widetilde{\alpha}}\right) = 2$, then $\alpha|_{\mathfrak{t}_0} \neq 0$ and $\mathfrak{g}^\alpha \cap \mathfrak{g}_0^{\widetilde{\alpha}} = \{0\}$.*

Proof. Suppose by way of contradiction that $\alpha|_{\mathfrak{t}_0} = 0$. Since $\dim_{\mathbb{R}} \left(\mathfrak{g}_0^{\widetilde{\alpha}}\right) = 2$ there is a root $\alpha_1 \neq \alpha$ such that $\alpha|_{\mathfrak{a}_0} = \alpha_1|_{\mathfrak{a}_0}$. Since $\alpha|_{\mathfrak{t}_0} = 0$, α_1 must be nonzero on $\mathfrak{t}_0$, and thus α, α_1 and $\alpha_2 = -\theta\alpha_1$ are *three* distinct roots with the same restriction $\widetilde{\alpha}$, contrary to assumption. Hence $\alpha|_{\mathfrak{t}_0} \neq 0$.

Since $\alpha|_{\mathfrak{t}_0}$ is imaginary valued we may choose $X \in \mathfrak{t}_0$ such that $\alpha(X) = i$. Now suppose $v \in \mathfrak{g}^\alpha \cap \mathfrak{g}_0^{\widetilde{\alpha}}$. Then we have $[X, v] \in [\mathfrak{t}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$ while on the other hand $[X, v] = \alpha(X) v = iv \in i\mathfrak{g}_0$. Thus we get $iv = 0$, and since v was arbitrary we conclude that $\mathfrak{g}^\alpha \cap \mathfrak{g}_0^{\widetilde{\alpha}} = \{0\}$. □

Our choice of B determines simple roots $\Pi \subset \Sigma$ and $\Pi_0 \subset \Sigma_0$, and the restriction $\widetilde{\alpha}$ is simple if α is simple. Let $\mathfrak{v}, \Psi, \Psi_0$ be as before and define $\mathfrak{v}_0 = \mathfrak{n}_0 / [\mathfrak{n}_0, \mathfrak{n}_0]$ so that $\mathfrak{v} = (\mathfrak{v}_0)_{\mathbb{C}}$ and $\Psi_0 = i\mathfrak{v}_0^*$, where $\mathfrak{v}_0^*$ denotes the space of $\mathbb{R}$-linear fiunctionals on $\mathfrak{v}_0$. The natural projection $\mathfrak{n} \to \mathfrak{v}$ restricts to isomorphisms

$$\bigoplus_{\alpha \in \mathfrak{k}} (\mathfrak{g}^\alpha) \approx \mathfrak{v}, \bigoplus_{\beta \in \mathfrak{k}_o} \left(\mathfrak{g}_0^\beta\right) \approx \mathfrak{v}_0$$

We write $\mathfrak{z} \subset \mathfrak{h}$ for the center of $\mathfrak{g}$, and $\mathfrak{z} \subset \mathfrak{s}_\psi \subset \mathfrak{h}$ for the stabilizer of $\psi \in \Psi$. We recall that ψ is said to be nondegenerate if its $H_{\mathbb{C}}$-orbit is open.

Lemma 2.4.3. *For $\psi \in \Psi$ the following are equivalent*

(1) ψ is non-degenerate.
(2) $\mathfrak{s}_\psi = \mathfrak{z}$
(3) $\psi|_{\mathfrak{g}^\alpha} \neq 0$ for all $\alpha \in \Pi$.

Proof. We note that $\dim(\Psi) = \dim(\mathfrak{h}/\mathfrak{z}) = |\Pi|$ is the semisimple rank of G, while the dimension of the $H_{\mathbb{C}}$-orbit of ψ is $\dim \left(\mathfrak{h}/\mathfrak{s}_\psi\right)$; thus (1) is equivalent to (2). Also we have $X \in \mathfrak{z}$ iff $\alpha(X) = 0$ for all $\alpha \in \Pi$, while $X \in \mathfrak{s}_\psi$ iff $\alpha(X) = 0$ whenever $\psi|_{\mathfrak{g}^\alpha} \neq 0$; thus (2) is equivalent to (3). □

We now prove an analogous characterization for $\psi \in \Psi_0$, using the following elementary result.

Lemma 2.4.4. *Let W_0 be a two-dimensional real vector space with complexification W, and let ω be a $\mathbb{C}$-linear functional on W that is real valued on W_0. Let $W_1 \subset W$ be a two dimensional real subspace such that $W_0 \cap W_1 = 0$, then we have*

$$\omega|_{W_0} = 0 \iff \omega = 0 \iff \omega|_{W_1} = 0$$

Proof. The first equivalence holds since ω is $\mathbb{C}$-linear. Also clearly $\omega = 0 \implies \omega|_{W_1} = 0$. Conversely suppose $\omega|_{W_1} = 0$. Since ω is real-valued on W_0 and $\dim_{\mathbb{R}} W_0 = 2$ we have $\ker \omega \cap W_0 \neq 0$. Since $W_0 \cap W_1 = 0$, this forces $\ker \omega \supsetneq W_1$. Since $\dim_{\mathbb{C}} W = 2$ we get $\omega = 0$ as desired. □

Proposition 2.4.5. *For $\psi \in \Psi_0$ the following are equivalent*

(1) ψ is non-degenerate
(2) $\psi|_{\mathfrak{g}_0^\beta} \neq 0$ for all $\beta \in \Pi_0$
(3) The H orbit of ψ is open in Ψ_0.
(4) $\mathfrak{s}_\psi \cap \mathfrak{h}_0 = \mathfrak{z} \cap \mathfrak{g}_0$.

Proof. The equivalence of (3) and (4) follows from a dimension argument similar to Lemma 2.4.3. It suffices to show that (4) is equivalent to (2) of Lemma 2.4.3, which is obvious, and that (2) is equivalent to (3) of Lemma 2.4.3. For the latter it is enough to show that if $\alpha \in \Pi$ and $\beta = \widetilde{\alpha}$ then

$$\psi|_{\mathfrak{g}^\alpha} = 0 \iff \psi|_{\mathfrak{g}_0^\beta} = 0 \tag{9}$$

Now (9) is obvious if $\dim \mathfrak{g}_0^\beta = 1$ for then $\mathfrak{g}^\alpha$ is the complexification of $\mathfrak{g}_0^\beta$. Otherwise by Corollary 2.4.2 we have $\mathfrak{g}_0^\beta \cap \mathfrak{g}^\alpha = 0$, and (9) follows from the previous lemma with $W_0 = \mathfrak{g}_0^\beta, W_1 = \mathfrak{g}^\alpha, \omega = i\psi$. □

The standard parabolic subgroups of G are those that contain B, and these correspond bijectively to subsets of Π_0. Indeed every $P \supset B$ admits a Levi decomposition $P = LU$ with θ-stable Levi component $L \supset H$, the group $B \cap L$ is a Borel subgroup of L and the corresponding simple roots for $\mathfrak{a}_0$ in $\mathfrak{l}_0$ give the desired subset of Π_0.

Lemma 2.4.6. *For $\psi \in \Psi_0$ there exists a standard parabolic subgroup $P = LU$ such that ψ vanishes on $\mathfrak{u}_0$ and restricts to a nondegenerate character of $\mathfrak{l}_0 \cap \mathfrak{n}_0$.*

Proof. Let P correspond to the set $\left\{\beta \in \Pi_0 : \psi|_{\mathfrak{g}_0^\beta} \neq 0\right\}$, then the result follows from Proposition 2.4.5. □

2.5. **Nilpotent orbits for complex classical groups.** If $G = GL(d, \mathbb{R})$ or if G is a complex classical group as in (7), then the real nilpotent orbits of G are naturally indexed by partitions, as in [CoMG93]. A *partition* λ of d of length l is a weakly decreasing integer sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ such that $\sum_j \lambda_j = d$. The λ_i are called the parts of λ, and the number of parts of size p is called the multiplicity $m_p(\lambda)$ of p. We write $\mathcal{P}(d)$ for the set of all partitions of d and $\mathcal{P}_1(d)$ (resp. $\mathcal{P}_{-1}(d)$) for the subset such that $m_p(\lambda)$ is even for all even (resp. odd) p. We set $\lambda_j = 0$ if j exceeds the length of λ, and we define a partial order on partitions as follows:

$$\lambda \leq \mu \text{ iff } \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \text{ for all } k.$$

Theorem 2.5.1. *There is an order-preserving bijection between nilpotent G-orbits and the set $\mathcal{P}(G)$ below:*

G	$GL(d,\mathbb{R}), GL(d,\mathbb{C}), SL(d,\mathbb{C})$	$O(d,\mathbb{C})$	$Sp(d,\mathbb{C})$
$\mathcal{P}(G)$	$\mathcal{P}(d)$	$\mathcal{P}_1(d)$	$\mathcal{P}_{-1}(d)$

The case of $SO(d,\mathbb{C})$ is slightly different. We say that $\lambda \in \mathcal{P}_1(d)$ is "very even" if λ has only even parts. Note that each even part must occur with even multiplicity, forcing d to be a multiple of 4.

Theorem 2.5.2. *The nilpotent orbits of $SO(d,\mathbb{C})$ are the same as $O(d,\mathbb{C})$ except that the very even orbits $\mathcal{O}_\lambda$ split into two orbits for $SO(d,\mathbb{C})$, denoted $\mathcal{O}_\lambda^I$ and $\mathcal{O}_\lambda^{II}$.*

For proofs we refer the reader to [CoMG93], especially Chapters 5 and 6.

2.6. **The Osborne lemma.** Let $S(\mathfrak{g})^i$ and $U(\mathfrak{g})^i$ denote the usual filtrations of the symmetric and enveloping algebras of $\mathfrak{g}$ and let $I(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$ and $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ denote the subrings of $\mathfrak{g}$-invariants.

Lemma 2.6.1. [Wall88, §§3.7] *There exist finite dimensional subspaces $E \subset S(\mathfrak{g})$, $F \subset U(\mathfrak{g})$ such that*

$$S(\mathfrak{g})^i \subset S(\mathfrak{n})^i E I(\mathfrak{g}) S(\mathfrak{k}),\ U(\mathfrak{g})^i \subset U(\mathfrak{n})^i F Z(\mathfrak{g}) U(\mathfrak{k})$$

As before let $\mathcal{N} \subset \mathfrak{g}^*$ be the null cone and let $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{k}^\perp$.

Corollary 2.6.2. *The projection $pr_{\mathfrak{n}^*} : \mathcal{N}_\theta \to \mathfrak{n}^*$ is a finite morphism.*

Proof. The maximal ideals $S^{>0}(\mathfrak{k}) \subset S(\mathfrak{k})$ and $I^{>0}(\mathfrak{g}) \subset I(\mathfrak{g})$ vanish on on $\mathfrak{k}^\perp$ and $\mathcal{N}$ respectively, hence both ideals vanish on $\mathcal{N}_\theta$. By Proposition 2.6.1 $\mathbb{C}[\mathcal{N}_\theta]$ is generated by E as a module over $S(\mathfrak{n}) = \mathbb{C}[\mathfrak{n}^*]$. □

Corollary 2.6.3. *If Z is any irreducible component of $\mathcal{N}_\theta$ then $pr_{\mathfrak{n}^*}(Z) = \mathfrak{n}^*$.*

Proof. By Corollary 2.6.2, $pr_{\mathfrak{n}^*}$ is a finite map and thus its image is a closed subset of $\mathfrak{n}^*$ of the same dimension as Z. By [Vog91, 8.4] $\dim Z = 1/2 \dim(\mathcal{N}) = \dim(\mathfrak{n}^*)$, thus $pr_{\mathfrak{n}^*}(Z)$ has full dimension, so $pr_{\mathfrak{n}^*}(Z) = \mathfrak{n}^*$. □

Corollary 2.6.4 (Casselman-Osborne-Gabber). *If $\sigma \in \mathcal{HC}(G)$ then σ is finitely generated as a $U(\mathfrak{n})$-module. Moreover, any good $\mathfrak{g}$-filtration on σ is good as an $\mathfrak{n}$-filtration, and every good $\mathfrak{b}$-filtration on σ is good as an $\mathfrak{n}$-filtration. In particular, $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(\sigma) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}(\sigma))$.*

For proof of the "moreover" part see [Jos81, §7.8.1] or [AGS, Appendix B].

2.7. **The Casselman-Jacquet Functor.** As before let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be the Borel subalgebra of $\mathfrak{g}$. For a $\mathfrak{b}$-module V we define its $\mathfrak{n}$-adic completion and its Jacquet module as follows:

$$\widehat{V} = \widehat{V}_{\mathfrak{n}} = \varprojlim V/\mathfrak{n}^i V, \quad J(V) = J_{\mathfrak{b}}(V) = \left(\widehat{V}_{\mathfrak{n}}\right)^{\mathfrak{h}\text{-finite}}$$

We note that $J(V)$ is different from the Casselman-Jacquet module considered in [Wal]. However it is closely related to the geometric Jacquet functor considered in [ENV04]. (See Theorem 2.7.6 below.)

Let $\mathcal{G}(\mathfrak{b})$ be the category of finitely generated $\mathfrak{b}$-modules for which every good $\mathfrak{b}$-filtration is also good as an $\mathfrak{n}$-filtration. Note that $\mathcal{G}(\mathfrak{b})$ is closed under subquotients. The following result is due to Gabber.

Theorem 2.7.1 ([Jos81], §7). *If $V \in \mathcal{G}(\mathfrak{b})$ then we have $\bigcap_{k\geq 0} \mathfrak{n}^k V = 0$. Hence V embeds into $\widehat{V}$ with dense image.*

By the Artin-Rees theorem for nilpotent Lie algebras ([McC67, Theorem 4.2]) we deduce

Corollary 2.7.2. *$V \mapsto \widehat{V}$ is an exact faithful functor from $\mathcal{G}(\mathfrak{b})$ to the category of $\mathfrak{b}$-modules.*

Lemma 2.7.3 ([Jos85], §3.5). *If $V \in \mathcal{G}(\mathfrak{b})$ then there exists a* finite dimensional *$\mathfrak{h}$-invariant subspace $S_\infty \subset J(V)$, which maps onto $V/\mathfrak{n}V$.*

Since this result plays a key role in the subsequent discussion, we include a proof here.

Proof. Let Ω_j be the set of (generalized) weights of $\mathfrak{h}$ appearing in $\mathfrak{n}^j V/\mathfrak{n}^{j+1} V$. Then there exists i such that $\Omega_j \cap \Omega_0 = \emptyset$ for all $j \geq i$. Let us define

$$\overline{S} = \bigoplus_{\mu \in \Omega_0} (V/\mathfrak{n}^i V)_\mu.$$

Then any (generalized) $\mathfrak{h}$ -eigenvector of $\overline{S}$ can be lifted by successive approximation to a (generalized) $\mathfrak{h}$ -eigenvector of the same weight in $\hat{V}$. In this way we find an $\mathfrak{h}$-invariant finite dimensional subspace $S_\infty \subset \hat{V}$ that maps bijectively to $\overline{S}$ and thus onto $V/\mathfrak{n}V$. □

Lemma 2.7.4. *If $V \in \mathcal{G}(\mathfrak{b})$ and $W \subset J(V)$ is a dense $\mathfrak{h}$-submodule of $\widehat{V}$ then $W = J(V)$.*

Proof. Let $J(V)^\mu = \left(\widehat{V}\right)^\mu$ be the generalized $\mathfrak{h}$-eigenspace for some fixed weight μ. Then for all sufficiently large i we have natural $\mathfrak{h}$-isomorphisms

$$J(V)^\mu \approx \left(\widehat{V}/\mathfrak{n}^i\widehat{V}\right)^\mu \approx \left(V/\mathfrak{n}^i V\right)^\mu \approx W^\mu \tag{10}$$

The first isomorphism follows since $J(V)^\mu \cap \mathfrak{n}^i\widehat{V} = 0$ for all sufficiently large i, the last by the density of W. It now follows from formula (10) that $J(V) = W$. □

Lemma 2.7.5. *$J(V)$ is dense in $\widehat{V}$ for any $V \in \mathcal{G}(\mathfrak{b})$. Moreover $V \mapsto J(V)$ is an exact faithful functor from $\mathcal{G}(\mathfrak{b})$ to $\mathcal{G}(\mathfrak{b})$.*

Proof. Let S_∞ be as in Lemma 2.7.3 and let $V_\infty \subset J(V)$ be the $\mathfrak{n}$-submodule generated by S_∞, then it follows that $V_\infty \in \mathcal{G}(\mathfrak{b})$. Also arguing by induction on i we deduce that V_∞ surjects onto each $\mathfrak{n}^iV/\mathfrak{n}^{i+1}V$ and hence that V_∞ is dense in $\widehat{V}$. By Lemma 2.7.4 $J(V) = V_\infty$, and thus $J(V)$ is dense and belongs to $\mathcal{G}(\mathfrak{b})$.

Corollary 2.7.2 implies that J is left exact. For right exactness, we need to show that if $\phi : V \to V'$ is a surjection then so is $J\phi$; since the image of $J\phi$ is dense in $\widehat{V'}$, this follows from Lemma 2.7.4. Now to prove faithfulness it suffices to show $V \neq 0$ implies $J(V) \neq 0$, but this follows from Corollary 2.7.2 and the density of $J(V)$ in $\widehat{V}$. □

If $M \in \mathcal{HC}(\mathfrak{g}, K)$ then $M \in \mathcal{G}(\mathfrak{b})$ by Corollary 2.6.4 so the above results apply to M, indeed in this case Corollary 2.7.2 is due to Casselman. However one can say more. Let $\bar{B} = \theta(B)$ be the opposite Borel subgroup, and let $\mathcal{C}\left(\mathfrak{g}, \bar{\mathfrak{b}}\right)$ be the category of finitely generated $\mathfrak{g}$-modules, which are $\bar{\mathfrak{b}}$-finite.

Theorem 2.7.6. *If $M \in \mathcal{HC}(\mathfrak{g}, K)$ then $\widehat{M}$ is a $\mathfrak{g}$-module and we have*

(1) $J(M) = \left(\widehat{M}_{\mathfrak{n}}\right)^{\bar{\mathfrak{n}}\text{-finite}}$
(2) $J(M) \in \mathcal{C}\left(\mathfrak{g}, \bar{\mathfrak{b}}\right)$.

Proof. Part (1) follows from [ENV04, Proposition 2.4]. More precisely [ENV04] proves

$$\left(\widehat{M}_{\bar{\mathfrak{n}}}\right)^{\mathfrak{n}\text{-finite}} = \left(\widehat{M}_{\bar{\mathfrak{n}}}\right)^{\mathfrak{h}\text{-finite}}$$

and we get part (1) upon replacing $\mathfrak{n}$ by $\bar{\mathfrak{n}}$. Part (2) is [ENV04, Proposition 2.2]. □

The theorem implies that $\mathrm{An}\mathcal{V}(JM) = \mathrm{An}\mathcal{V}(M)$, and that $\mathrm{As}\mathcal{V}(JM)$ is a union of $\overline{B}$-orbits in $\mathrm{An}\mathcal{V}(M) \cap \overline{\mathfrak{b}}^\perp$. It is known that $\dim \mathrm{An}\mathcal{V}(M) \cap \overline{\mathfrak{b}}^\perp = 1/2 \dim \mathrm{An}\mathcal{V}(M)$, but unfortunately this variety has many irreducible components. We note also that $\mathrm{As}\mathcal{V}_{\mathfrak{g}}(JM) = \mathrm{As}\mathcal{V}_{\mathfrak{n}}(JM)$ and thus from now on we will write just $\mathrm{As}\mathcal{V}(JM)$.

2.8. **Whittaker Functionals.** We recall that a representation in $\mathcal{M}$ or $\mathcal{HC}$ is said to be *large* if its annihilator variety is the nilpotent cone $\mathcal{N}(\mathfrak{g})$, and *generic* if it admits a Whittaker functional for some non-degenerate $\psi \in \Psi$.

Theorem 2.8.1. *For $\pi \in \mathcal{M}$ the following are equivalent:*

$$\pi \textit{ is large} \Leftrightarrow \pi \textit{ is generic} \Leftrightarrow \pi^{HC} \textit{ is generic.} \tag{11}$$

Moreover if π is large and $\psi \in \Psi$ is non-degenerate, then

(1) $Wh'_\psi\left(\pi^{HC}\right) \neq 0$.
(2) If $\psi \in \Psi_0$ then there exists $a \in F_G$ such that $Wh^*_{a\cdot\psi}(\pi) \neq 0$
(3) If $\psi \notin \Psi_0$ then $Wh^*_\psi(\pi) = 0$.

For π irreducible the implications in (11) are in [Kos78, Theorems K and L]. Part (1) follows from [Mat88, Corollary 2.2.2]. Part (2) is [Kos78, Theorem K] and Part (3) is in [CHM, Theorem 8.2]. The case of general π follows from this by exactness of the functors Wh^*_ψ and Wh'_ψ proved in [CHM, Theorem 8.2] and [Kos78, Theorem 4.3] respectively. Using the notion of wave-front set, Part (2) can be strengthened as follows.

Theorem 2.8.2 ([Mat92], Theorem A)**.** *For $\pi \in \mathcal{M}(G)$, we have*

$$\mathrm{WF}(\pi) \cap \Psi_0^{\times} = \Psi_0^{\times}(\pi)$$

3. Proof of Theorem A

Let $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$ and $\mathfrak{v} = \mathfrak{n}/\mathfrak{n}'$ be as in (1), and for an $\mathfrak{n}$-module V, we denote the $\mathfrak{v}$-module of $\mathfrak{n}'$-coinvariants by

$$C(V) = C_{\mathfrak{n}'}(V) = V/\mathfrak{n}'V$$

Since $\mathfrak{v}$ is commutative $\mathrm{An}\mathcal{V}_{\mathfrak{v}}(CV) = \mathrm{As}\mathcal{V}_{\mathfrak{v}}(CV)$ and we simply write $\mathcal{V}_{\mathfrak{v}}(CV)$. We note that

$$V \in \mathcal{G}(\mathfrak{b}) \implies C(V) \in \mathcal{G}(\mathfrak{b}).$$

For the rest of this section let $M \in \mathcal{HC}(G)$ denote a fixed Harish-Chandra module.

Lemma 3.0.1. *We have*

$$\Psi(M) = \mathrm{Supp}_{\mathfrak{v}}(CM) = \mathcal{V}_{\mathfrak{v}}(CM).$$

Proof. The Lie algebra $\mathfrak{h}$ contains an element h that acts by 1 on $\mathfrak{v}$, and by the degree on $S(\mathfrak{v})$. Since the ideal $Ann_{S(\mathfrak{v})}(CM)$ is $\mathfrak{h}$-invariant, it is homogeneous and consequently $\mathrm{Supp}_{\mathfrak{v}}(CM) = \mathcal{V}_{\mathfrak{v}}(CM)$. Finally $\Psi(M) = \mathrm{Supp}_{\mathfrak{v}}(CM)$ by Nakayama's lemma (see §§2.1). □

For the proof of Theorem A we need three further results, which are stated below and proved in sections 3.1,3.2,3.3.

Lemma 3.0.2. *We have a $\mathfrak{b}$-module isomorphism $C(JM) \approx J(CM)$*

Lemma 3.0.3. $\mathcal{V}_{\mathfrak{v}}(C(JM)) = \mathrm{As}\mathcal{V}_{\mathfrak{n}}(JM) \cap \Psi$.

Lemma 3.0.4. $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(JM) \supset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi$.

We now prove Theorem A.

Proof of Theorem A. By Lemma 3.0.1 and Corollary 2.6.4 we have

$$\Psi(M) = \mathcal{V}_{\mathfrak{v}}(CM), \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}_{\mathfrak{g}}(M))$$

By Lemma 2.2.3 we have$\mathcal{V}_{\mathfrak{v}}(CM) \subset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi$, and it remains only to prove

$$\mathcal{V}_{\mathfrak{v}}(CM) \supset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi \tag{12}$$

By Corollary 2.6.4 $M \in \mathcal{G}(\mathfrak{b})$ and hence $C(M) \in \mathcal{G}(\mathfrak{b})$ as well. By Lemma 2.7.5 $J(CM)$ is dense in $\widehat{CM}$, since CM is also dense in $\widehat{CM}$ we get

$$\mathcal{V}_{\mathfrak{v}}(CM) = \mathrm{An}\mathcal{V}_{\mathfrak{v}}\left(\widehat{CM}\right) = \mathcal{V}_{\mathfrak{v}}(J(CM))$$

Now by Lemmas 3.0.2, 3.0.3 and 3.0.4 we get

$$\mathcal{V}_{\mathfrak{v}}(J(CM)) = \mathcal{V}_{\mathfrak{v}}(C(JM)) = \mathrm{As}\mathcal{V}_{\mathfrak{n}}(J(M)) \cap \Psi \supset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi$$

This proves (12) and finishes the proof of Theorem A. □

3.1. **Proof of Lemma 3.0.2.** For $V \in \mathcal{G}(\mathfrak{b})$ we let $\widehat{V}$ denote its $\mathfrak{n}$-adic completion and let $J(V) = \left(\widehat{V}\right)^{\mathfrak{h}\text{-finite}}$ denote the associated Jacquet functor as before. In this section we prove Lemma 3.0.2 in a more general setting. Let $\mathfrak{c} \subset \mathfrak{n}$ be any $\mathfrak{h}$-invariant ideal, and define $C_{\mathfrak{c}}(V) = V/\mathfrak{c}V$.

Lemma 3.1.1. *For $V \in \mathcal{G}(\mathfrak{b})$ we have $C_{\mathfrak{c}}J(V) \approx J(C_{\mathfrak{c}}V)$.*

Proof. By Lemma 2.7.5 J is exact hence it is enough to show that $\mathfrak{c}J(V) = J(\mathfrak{c}V)$ as submodules of $J(V)$. Since V dense in $\widehat{V}$, $\mathfrak{c}\widehat{V}$ is contained in the closure of $\mathfrak{c}V$ in $\widehat{V}$, which by the Artin-Rees theorem ([McC67, Theorem 4.2]) coincides with $\widehat{\mathfrak{c}V}$. Since $\mathfrak{c}\widehat{V}$ contains $\mathfrak{c}V$ we see that $\mathfrak{c}\widehat{V}$ is dense $\widehat{\mathfrak{c}V}$, and since $J(V)$ is dense in $\widehat{V}$ it follows that $\mathfrak{c}J(V)$ is dense in $\widehat{\mathfrak{c}V}$. Evidently $\mathfrak{c}J(V) \subset \left(\widehat{\mathfrak{c}V}\right)^{\mathfrak{h}\text{-finite}} = J(\mathfrak{c}V)$, hence $\mathfrak{c}J(V) = J(\mathfrak{c}V)$ by Lemma 2.7.4. □

3.2. **Proof of Lemma 3.0.3.** We prove Lemma 3.0.3 for a more general class of modules. As before, let

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}, \mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}], \mathfrak{v} = \mathfrak{n}/\mathfrak{n}', \Psi = \mathfrak{v}^*.$$

Let $\mathcal{J}(\mathfrak{b})$ be the category of $\mathfrak{b}$-modules with a finite dimensional $\mathfrak{h}$-invariant generating subspace. Evidently if M is a Harish-Chandra module, or even from category $\mathcal{G}(\mathfrak{b})$, then $J(M) \in \mathcal{J}(\mathfrak{b})$. Therefore Lemma 3.0.3 follows from the next result.

Lemma 3.2.1. *If $V \in \mathcal{J}(\mathfrak{b})$ then we have $\mathcal{V}_{\mathfrak{v}}(C_{\mathfrak{n}'}V) = \mathrm{As}\mathcal{V}_{\mathfrak{n}}(V) \cap \Psi$.*

Proof. By Lemma 2.2.3, we have $\mathcal{V}_{\mathfrak{v}}(C_{\mathfrak{n}'}V) = \mathrm{As}\mathcal{V}_{\mathfrak{n}}(C_{\mathfrak{n}'}V) \subset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(V) \cap \Psi$, and so it suffices to prove the reverse containment. Let E be a finite dimensional $\mathfrak{h}$-invariant generating subspace of V, and let F be its image in $C_{\mathfrak{n}'}(V)$. By Lemma 2.2.1 we have

$$\mathrm{As}\mathcal{V}_{\mathfrak{n}}(C_{\mathfrak{n}'}V) = \mathrm{Var}\left(\overline{J}\right),$$

where J is the annihilator of F in $U = U(\mathfrak{n})$ and $\overline{J} \subset S(\mathfrak{n})$ is its associated graded space under the usual filtration U^i of U. Therefore it is enough to prove that $\overline{J}$ vanishes on $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(V) \cap \Psi$, i.e. that if $u \in J$ then $\overline{u}$ vanishes on $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(V) \cap \Psi$. To prove this we need some additional notation.

We fix $\rho^\vee \in \mathfrak{h}$ satisfying $\alpha(\rho^\vee) = 1$ for every simple root α, and for an $\mathfrak{h}$-module X we consider generalized $\rho^\vee$-weights, which we refer to simply as *weights*. We write X_μ for the μ-weight space for $\mu \in \mathbb{C}$, and if x is a weight vector we write $[x]$ for the real part of its weight; thus $[x] = \mathrm{Re}(\mu)$ for $x \in X_\mu$. This notation will be applied to U, V and to the filtrands U^i and $V^i = U^i E$. We also fix a weight basis $v_1, \dots, v_m$ of E, ordered so that $[v_i] \geq [v_j]$ if $i \geq j$.

Now let $u \in J$ be as before, then $u \in J \cap U^d$ for some d, and since J is $ad(\mathfrak{h})$-stable we may assume $u \in U^d_l$ for some integer l. If $l > d$ then $u \in \mathfrak{n}'U^{d-1}$ and $\overline{u} = 0$ on all of Ψ, therefore we may assume that $l \leq d$. For $1 \leq t \leq m$ let $L^t \subset V$ denote the submodule generated by $v_1, \dots, v_t$. Since V is glued from the subquotients L^t/L^{t-1} we have

$$\mathrm{As}\mathcal{V}_{\mathfrak{n}}(V) = \bigcup_t \mathrm{As}\mathcal{V}_{\mathfrak{n}}(L^t/L^{t-1}).$$

Thus it suffices to show that u vanishes on $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(L^t/L^{t-1})$ for each t, i.e. that

$$uv_t \in L^{t-1} + V^{d-1}.$$

Now we may write $uv_t = \sum_{i=1}^m \left(\sum_j X_{ij} b_{ij} v_i\right)$, where $X_{ij} \in \mathfrak{n}'$ and $b_{ij} \in U$ are weight vectors satisfying

$$[uv_t] = [X_{ij}] + [b_{ij}] + [v_i].$$

We have $[X_{ij}] \geq 2$, $[u] = l \leq d$, and $[v_t] \leq [v_i]$ for $i \geq t$. Thus we get

$$[b_{ij}] = [u] - [X_{ij}] + [v_t] - [v_i] \leq d - 2 \text{ for } i \geq t.$$

It follows that for $i \geq t$ we have $b_{ij} \in U^{d-2}$ and $X_{ij} b_{ij} \in U^{d-1}$. Hence we get

$$uv_t = \sum_{i=1}^{t-1} \sum_j X_{ij} b_{ij} v_i + \sum_{i=t}^{m} \sum_j X_{ij} b_{ij} v_i \in L^{t-1} + V^{d-1}.$$

□

3.3. **Proof of Lemma 3.0.4.** We will use Beilinson-Bernstein localization [BB81], the paper [ENV04] that describes the Casselman-Jacquet functor in geometric terms, and the paper [Gin86] that describes the behavior of the singular support of D-modules under the nearby cycle functor. Let us describe the setting in detail.

Let M be an admissible $(\mathfrak{g}, K)$ module with infinitesimal character χ_λ, with parameter λ chosen to be dominant. Then M is a (U_λ, K)-module, where U_λ is the quotient of $U(\mathfrak{g})$ by the two-sided ideal generated by $z - \chi_\lambda(z)$. Let $\mathcal{D}_\lambda$ denote the λ-twisted sheaf of differential operators on the flag variety X, then $U_\lambda = \Gamma(X, \mathcal{D}_\lambda)$. By a $(\mathcal{D}_\lambda, K)$-module we mean a coherent $\mathcal{D}_\lambda$-module that is K-equivariant. Such a module is necessarily holonomic with regular singularities. By Beilinson-Bernstein ([BB81]) the global sections functor

$$\Gamma : \{(\mathcal{D}_\lambda, K)\text{-modules}\} \to \{(U_\lambda, K)\text{-modules}\}$$

is exact and essentially surjective, a section of Γ is given by the localization functor $\mathcal{D}_\lambda \otimes_{U_\lambda} (\cdot)$. Moreover if λ is regular then Γ is an equivalence of categories. Let $X_1, \ldots, X_n$ be the K-orbits on X, and let $T^*_{X_i}X$ denote the corresponding conormal bundles. If $\mathcal{M}$ is a $(\mathcal{D}_\lambda, K)$-module, then its characteristic cycle (see [Gin86]) is of the form

$$SS(\mathcal{M}) = \sum_{i=1}^{n} m_i T^*_{X_i} X.$$

for some nonnegative integers m_i. The characteristic variety $CV(\mathcal{M})$ is the union of $T^*_{X_i}X$ for which $m_i > 0$. Let us describe the connection between the characteristic cycle of a $\mathcal{D}_\lambda$-module $\mathcal{M}$ and the associated cycle of the Harish-Chandra module $M := \Gamma(\mathcal{M})$. Any point $x \in X$ defines a Borel subalgebra $\mathfrak{b}_x \subset \mathfrak{g}$. The tangent space T_xX can be identified with $\mathfrak{g}/\mathfrak{b}_x$ and the cotangent space with $(\mathfrak{g}/\mathfrak{b}_x)^* = (\mathfrak{b}_x)^\perp \subset \mathfrak{g}^*$. This gives a natural embedding of the cotangent bundle T^*X into the trivial bundle $X \times \mathfrak{g}^*$. The composition of this map with the projection on the second coordinate is called the moment map, denoted by μ. By a result of Borho and Brylinski ([BB85]) we have

$$\mu(CV(\mathcal{M})) = \mathrm{As}\mathcal{V}_{\mathfrak{g}}(M) \tag{13}$$

By Corollary 2.6.4 we have

$$\mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}_{\mathfrak{g}}(M)) \tag{14}$$

The paper [ENV04] gives a precise geometric description of $J(M)$, that we now recall briefly. Actually [ENV04] deals with $J_{\bar{\mathfrak{n}}}(M)$, so the description below is a trivial modification of [ENV04]. Let H be the maximally split torus of G and let $\rho^\vee : \mathbb{G}_m \to H$ be the cocharacter such that $\alpha \circ d\rho^\vee = -Id_{\mathbb{C}}$ for every simple root α. By composing $\rho^\vee$ with the action of G on X, we get an action map $a : \mathbb{G}_m \times X \to X$. Consider now the following diagram

$$X \xleftarrow{a} \mathbb{G}_m \times X \xrightarrow{j} \mathbb{A}^1 \times X \xleftarrow{i} \{0\} \times X \approx X$$

For a $(\mathcal{D}_\lambda, K)$ module $\mathcal{M}$, let $\Phi(\mathcal{M})$ be the $\mathcal{D}_\lambda$-module obtained by applying the nearby cycles functor to $j_* a^*(\mathcal{M})$ along $\{0\} \times X \approx X$.

Theorem 3.3.1. [ENV04] *$\Phi(\mathcal{M})$ is a $\left(\mathcal{D}_\lambda, \bar{N}\right)$-module and one has*

$$\Gamma(\Phi(\mathcal{M})) = J_{\mathfrak{n}}(\Gamma(\mathcal{M})).$$

In view of this theorem $\Phi(\mathcal{M})$ can be regarded as the geometric Casselman-Jacquet functor.

The paper [Gin86] describes the behavior of the characteristic cycle under the nearby cycle functor in the following way. For an algebraic variety Z, and a regular function $f : Z \to \mathbb{C}$ let $U := f^{-1}(\mathbb{C} \setminus \{0\})$. Suppose we have an algebraic family S_t of subvarieties of $Z_t := f^{-1}(t)$ parameterized by $t \in \mathbb{C} \setminus \{0\}$. Let $S \subset U$ denote the total space of this family and let $\overline{S}$ denote the closure of S in Z. Denote by $\lim_{t\to 0} S_t$ the algebraic cycle corresponding to the scheme-theoretic intersection $\overline{S} \cap f^{-1}(0)$ (cf. [Gin86, 1.4]).

Theorem 3.3.2 ([Gin86], Theorem 5.5)**.** *Let $\mathcal{M}$ be a $\mathcal{D}_\lambda$-module over Z, let $\Phi_f\mathcal{M}$ denote the nearby cycle functor and let i_t denote the embedding of $f^{-1}(t)$ into Z. Then*

$$SS(\Phi_f(\mathcal{M})) = \lim_{t\to 0} SS((i_t)_*(i_t)^*\mathcal{M}).$$

Proof of Lemma 3.0.4. From Theorem 3.3.2 we obtain

$$SS(\Phi_f(\mathcal{M})) = \lim_{t\to 0} \rho^\vee(t) SS(\mathcal{M})$$

and passing to characteristic varieties we get

$$CV(\Phi_f(\mathcal{M})) = \lim_{t\to 0} \rho^\vee(t) CV(\mathcal{M}) \tag{15}$$

Identify $\mathfrak{n}^*$ with the subspace of $\mathfrak{g}^*$ consisting of vectors having negative weights under the action of $d\rho^\vee(1)$, $[\mathfrak{n},\mathfrak{n}]^\perp$ with vectors having weights at least -1 and Ψ with those having weight -1. Then by (13) and (14), $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi$ is obtained by intersecting $CV(\mathcal{M})$ with the constant bundle $X \times [\mathfrak{n},\mathfrak{n}]^\perp$, projecting to the second coordinate and then further projecting to Ψ. Denote this operation on subvarieties of T^*X

by p_Ψ. Since the characteristic variety is a conical set (in cotangent directions), $p_\Psi(\rho^\vee(t)CV(\mathcal{M}))$ does not depend on t. Since X is complete we get

$$p_\Psi(\lim_{t\to 0}\rho^\vee(t)CV(\mathcal{M})) \supset p_\Psi(CV(\mathcal{M}))$$

Thus we get

$$p_\Psi(CV(\Phi_f(\mathcal{M}))) \supset p_\Psi(CV(\mathcal{M}))$$

Lemma 3.0.4 follows now from Theorem 3.3.1. □

3.3.1. *A counterexample to a stronger statement.* One might ask whether inclusion holds without intersection with Ψ, i.e. $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \subset \mathrm{As}\mathcal{V}(J(M))$. The answer is no, as shown by the following example.

Let $G = GL(3,\mathbb{R})$ and let $\mathfrak{g}$ be its complexified Lie algebra. Let $\mathfrak{b}$ be the Borel subalgebra of upper-triangular matrices, let $\mathfrak{n}$ be its nilradical, and let $\mathfrak{s}$ be the space of symmetric matrices. Using the trace form, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\bar{\mathfrak{n}}$ with $\mathfrak{n}^*$. Let M be a degenerate principal series representation corresponding to the $(2,1)$ parabolic. Then we have

$$\mathrm{AnV}_{\mathfrak{g}}(M) = \mathcal{R}, \quad \mathrm{AsV}_{\mathfrak{g}}(M) = \mathcal{R} \cap \mathfrak{s}$$

where $\mathcal{R}$ is the set of nilpotent matrices of rank ≤ 1.

For a lower triangular matrix let a, b, c denote its entries as shown

$$\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix}$$

Then we get

$$\mathrm{AsV}_{\mathfrak{n}}(M) = pr_{\bar{\mathfrak{n}}}(\mathcal{R} \cap \mathfrak{s}) = \{a^2b^2 + a^2c^2 + b^2c^2 = 0\}$$
$$\mathrm{AsV}_{\mathfrak{n}}(JM) \subset \mathcal{R} \cap \bar{\mathfrak{n}} = \{ac = 0\}$$

4. Proof of Theorems B, C

4.1. **The Jacquet restriction functor.** As before let B be the fixed Borel subgroup of G. Let $P \supset B$ be a standard parabolic subgroup, fix a Levi decomposition $P = LU$ and let $\mathfrak{u}$ be the complexified Lie algebra of U. For $(\pi, W) \in \mathcal{M}(G)$ we have a natural representation of L on the space $W/\overline{\mathfrak{u}W}$ where $\overline{\mathfrak{u}W}$ denotes the closure of $\mathfrak{u}W$ in W. This representation is usually denoted by $r_P(\pi)$ and referred to as the Jacquet restriction functor, for simplicity we will write $r_P(\pi) = \pi_P$. Its main properties are summarized below.

Theorem 4.1.1. [Wall88, 3.8.2 and 5.2.3]

(1) $\pi_P \in \mathcal{M}(L)$
(2) r_P *is left adjoint to the parabolic induction functor* $\mathcal{M}(L) \to \mathcal{M}(G)$.
(3) $(\pi_P)^{HC} = C_{\mathfrak{u}}(\pi^{HC}) := \pi^{HC}/\mathfrak{u}\pi^{HC}$.

Remark 4.1.2. *By an unpublished result of Casselman,* $\mathfrak{u}W$ *is already closed in* W*, and hence* $W/\overline{\mathfrak{u}W} = W/\mathfrak{u}W =: C_{\mathfrak{u}}(W)$.

Recalling the definition of $\Psi = (\mathfrak{n}/\mathfrak{n}')^*$ etc. from (1) we write Ψ_G to denote its dependence on G. For each standard parabolic $P = LU$ we can regard Ψ_L as a subset of Ψ_G as follows: $\Psi_L \approx \{\psi \in \Psi_G : \psi|_{\mathfrak{u}} = 0\}$. It follows immediately that for $\pi \in \mathcal{M}(G)$ we have

$$\Psi(\pi_P) = \Psi(\pi) \cap \Psi_L$$

Lemma 4.1.3. *For* $\pi \in \mathcal{M}(G)$ *and* $P = LU \supset B$ *we have*

$$\mathrm{As}\mathcal{V}(\pi_P) \subset \mathrm{As}\mathcal{V}(\pi) \cap \mathfrak{l}^*, \mathrm{WF}(\pi_P) \subset \mathrm{WF}(\pi) \cap \mathfrak{l}^*$$

$$\Psi_0(\pi) = \bigcup_P \Psi_0^\times(\pi_P), \Psi_0(\pi^{HC}) = \bigcup_P \Psi_0^\times(\pi_P^{HC}), \Psi_0(\widetilde{\pi}) = \bigcup_P \Psi_0^\times(\widetilde{\pi_P})$$

Proof. The result on As$\mathcal{V}$ follows from Lemma 2.2.3, and by [SV00] this implies the result on WF via the Kostant-Sekiguchi correspondence. The results on $\Psi_0(\pi)$ and $\Psi_0\left(\pi^{HC}\right)$ follow from the definition of π_P and Lemma 2.4.6. For the result on $\Psi_0(\widetilde{\pi}) = \bigcup_{a\in F_G}\Psi_0(\pi^a)$ we recall the definition of $F_G \subset G_{\mathbb{C}}$, namely

$$F_G = \left\{a \in \exp i\mathfrak{a}_0 \mid a^2 = 1\right\}$$

Since $\mathfrak{a}_0 \subset \mathfrak{l}$ we have a surjection $a \mapsto a_P : F_G \to F_L$, such that $(a\cdot\pi)_P = a_P \cdot \pi_P$. Thus we get

$$\Psi_0(\widetilde{\pi}) = \bigcup_{a\in F_G}\Psi_0(a\cdot\pi) = \bigcup_{a,P}{\Psi_0}^{\times}\left((a\cdot\pi)_P\right) = \bigcup_{a,P}{\Psi_0}^{\times}(a_P\cdot\pi_P) = \bigcup_{P}{\Psi_0}^{\times}(\widetilde{\pi_P})$$

□

Lemma 4.1.4. *If $i\lambda \in \Psi_0$ then $\lambda \in pr_{\mathfrak{n}^*}(\overline{\mathrm{KS}(G\cdot\lambda)})$.*

Proof. By Corollary 2.6.3 $pr_{\mathfrak{n}^*}$ projects any irreducible component of $\mathcal{N}_\theta$ onto $\mathfrak{n}^*$. This implies the result if λ is principal nilpotent. For general λ, we choose a standard parabolic LU such that λ vanishes on $\mathfrak{u}$ and is principal nilpotent on $\mathfrak{l}\cap\mathfrak{n}$. The result now follows from Theorem 2.3.1 and Proposition 2.3.2. □

4.2. **Proofs of the theorems.** We first prove Theorem B.

Proof of Theorem B. By Theorems 2.8.1 and 2.8.2, for all P we have

$$\Psi_0^{\times}(\pi_P) \subset \mathrm{WF}(\pi_P)\cap\Psi, \quad \Psi_0^{\times}\left(\pi_P^{HC}\right) = \Psi_0^{\times}(\widetilde{\pi_P}).$$

Taking the union over all $P \supset B$ and using Lemma 4.1.3 we get

$$\Psi_0(\pi) \subset \mathrm{WF}(\pi)\cap\Psi, \quad \Psi_0(\pi^{HC}) = \Psi_0(\widetilde{\pi}). \tag{16}$$

By Lemma 4.1.4, by [SV00], and Theorem A, we get

$$\mathrm{WF}(\pi)\cap\Psi \subset pr_{\mathfrak{n}^*}(\mathrm{KS}(\mathrm{WF}\pi)) = pr_{\mathfrak{n}^*}\left(\mathrm{As}\mathcal{V}(\pi^{HC})\right) = \Psi(\pi^{HC})$$

Since $\mathrm{WF}(\pi) \subset \Psi_0$, it follows that

$$\mathrm{WF}(\pi)\cap\Psi \subset \Psi_0\left(\pi^{HC}\right) \tag{17}$$

Combining (16) and (17) we obtain (4).

Finally if G is a complex group or if $G = GL(n,\mathbb{R})$ then each complex nilpotent orbit contains at most one real orbit. This has two consequences. First by (8) it follows that

$$\mathrm{WF}(\pi)\cap\Psi = \mathrm{An}\mathcal{V}(\pi)\cap\Psi_0$$

Second, since the group F_G permutes the real forms of a complex nilpotent orbit, it acts trivially on orbits and we get $\Psi(\widetilde{\pi}) = \Psi(\pi)$. Thus (5) follows from (4). □

We next prove Theorem C using Theorem D.

Proof of Theorem C. In view of (8) it suffices to show that $\mathrm{WF}(\pi)$ is determined by $\Psi_0(\pi) = \mathrm{WF}(\pi)\cap\Psi_0$.

For the first case, this is straightforward since every G-orbit in $\mathrm{WF}(\pi)$ intersects Ψ_0. In the second case, since π is irreducible then by [Jos85] there is a complex nilpotent orbit $\mathcal{O}$ such that $\mathrm{An}\mathcal{V}(\pi) = \overline{\mathcal{O}}$. Since G is itself a complex group, $\mathcal{O}_0 = \mathcal{O}\cap\mathfrak{g}_0^*$ is a single G-orbit and we have $WF(\pi) = i\overline{\mathcal{O}_0}$. Thus it suffices to show that $\mathcal{O}_0$ is determined by $\overline{\mathcal{O}_0}\cap\Psi_0$, which follows from Theorem D below. □

4.3. **Some remarks on Theorem B.** The action of F_G is not very significant in Theorem B. For instance, let $\psi \in \mathrm{WF}(\pi)\cap\Psi_0$ and choose a parabolic subgroup $P = LU$ such that ψ is a principal nilpotent element in $\mathfrak{l}^*$. Then we have shown that $\mathrm{WF}(\pi_P)$ contains some (real) principal nilpotent orbit. The action of $F_L \subset F_G$ is used to permute the real principal nilpotent orbits, but if G is classical then there are only 2 such orbits (since L is then a product of a classical group with GL_{n_i}).

We next give an example to show that $\Psi_0(\widetilde{\pi}) = \Psi_0(\pi^{HC})$ can be a *proper* subset of $\mathrm{An}\mathcal{V}(\pi)\cap\Psi_0$. Let $P \approx GL(n,\mathbb{C}) \ltimes Herm_n$ be the Siegel-Shilov parabolic subgroup of $U(n,n)$ where $Herm_n$ is the space of $n\times n$ Hermitian matrices. Let π be the corresponding unitary degenerate principal series representation considered by Kashiwara and Vergne in [KV79a, KV79b]. As shown in [KV79a] π decomposes into $n+1$ constituents $\pi_0,\dots,\pi_n$, all of the same Gelfand-Kirillov dimension (π_i is denoted $\pi_{n-i,i}$ in [KV79a]). On the other hand, the complex Richardson orbit $\mathcal{O}_{\mathbb{C}}$ for P contains $n+1$ real orbits $\mathcal{O}_0,\dots,\mathcal{O}_n$ as well, and by a result of Barbasch (see [MT07]) the associated wavefront cycle of π is $\sum\left[\overline{\mathcal{O}_i}\right]$, i.e. all multiplicities are 1. It follows that the wavefront cycle of each π_i is the closure of a *single* real orbit, which after relabeling we may assume to be $\overline{\mathcal{O}_i}$.

For the group $U(2,2)$, $\mathcal{O}_{\mathbb{C}}$ consists of matrices with rank 2 and square 0. It contains three real nilpotent orbits $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2$ whose representatives are the respective block matrices

$$\begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \operatorname{diag}(1,-1) \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -I_2 \\ 0 & 0 \end{bmatrix}$$

The group F_G preserves $\mathcal{O}_1$, and permutes $\mathcal{O}_0$ with $\mathcal{O}_2$. It is easy to see that $\mathcal{O}_1$ intersects Ψ, while $\mathcal{O}_0$ and $\mathcal{O}_2$ do not. Thus we get that

$$\Psi(\pi_0^{HC}) \cap \Psi_0 = \Psi_0(\pi_0^{HC}) = \left(\overline{\mathcal{O}_0} \cup \overline{\mathcal{O}_2}\right) \cap \Psi_0$$

is not equal to $\overline{\mathcal{O}_{\mathbb{C}}} \cap \Psi_0 = \operatorname{An}\mathcal{V}(\pi_0) \cap \Psi_0$. Hence $\Psi_0(\pi_0^{HC})$ is a proper subset of $\operatorname{An}\mathcal{V}(\pi_0) \cap \Psi_0$.

Theorem B determines only $\Psi_0(\pi^{HC})$, not $\Psi(\pi^{HC})$. From Lemma 3.0.1 we see that $\Psi(\pi^{HC})$ is Zariski closed, so one might ask whether $\Psi(\pi^{HC})$ is the Zariski closure of $\Psi_0(\pi^{HC})$. Using the arguments of the above two subsections it can be easily proven for split groups. However, this statement does not generalize to all quasi-split groups. The representation π_1 provides a counter-example for $U(2,2)$, and a degenerate principle series representation (i.e. sections of a line bundle on the projective space $\mathbb{CP}^2$) provide a counter-example for $\mathfrak{sl}_3(\mathbb{C})$. This can be shown using Theorem A.

4.4. **Wave-front set of the Jacquet restriction.** Theorem B implies the following proposition.

Proposition 4.4.1. *Let $\pi \in \mathcal{M}(G)$ and suppose $P = LU$ is a standard parabolic subgroup such that L is a product of several GL_{n_i} factors. Then*

$$\operatorname{WF}(\pi_P) = \operatorname{WF}(\pi) \cap \mathfrak{l}^* \text{ for all } \pi \in \mathcal{M}(G)$$

Proof. If L is a product of GL_{n_i} then F_L acts trivially on representations of L and hence $\Psi(\pi_P) = \Psi(\widetilde{\pi_P})$. Now by Theorem B we get

$$\operatorname{WF}(\pi_P) \cap \Psi = \Psi_0(\pi_P^{HC}) \supset \Psi_0(\pi^{HC}) \cap \mathfrak{l}^* \supset \operatorname{WF}(\pi) \cap \Psi \cap \mathfrak{l}^*$$

Since $\operatorname{WF}(\pi_P)$ and $\operatorname{WF}(\pi) \cap \mathfrak{l}^*$ are L-stable, and every nilpotent L-orbit intersects Ψ, we conclude that $\operatorname{WF}(\pi_P) \supset \operatorname{WF}(\pi) \cap \mathfrak{l}^*$. The reverse containment follows from Lemma 4.1.3. □

It is very interesting for us to know whether this holds without the assumption on L. Obviously, Proposition 4.4.1 cannot have a p-adic analogue.

5. Applications to $GL(n)$

Let $G_n = GL(n, F)$ with $F = \mathbb{R}$ or $\mathbb{C}$, and suppose $\pi \in \mathcal{M}(G_n)$. [AGS] gives several definitions for the derivative of π, inspired by the p-adic notion defined in [BZ77]. Here we will use the following definition. Let $P_n \subset G_n$ be the mirabolic subgroup, consisting of matrices with last row $(0, \dots, 0, 1)$, then $P_n \approx G_{n-1} \ltimes V_n$ where $V_n \approx F^{n-1} \subset P_n$ is imbedded as the last column. If (τ, V) is a representation of $\mathfrak{p}_n$ and ξ is a character of $\mathfrak{v}_n$ we can consider the coinvariants

$$C_\xi(\tau) = V/\operatorname{Span}\{\tau(X)\, v - \xi(X) v : v \in V,\ X \in \mathfrak{v}_n\}.$$

Let ξ_0 be the trivial character of $\mathfrak{v}_n$ and let ξ_1 be the character given by

$$\xi_1(x_1, \dots, x_{n-1}) := \sqrt{-1}\operatorname{Re} x_{n-1}.$$

The normalizers of ξ_0 and ξ_1 in G_{n-1} are G_{n-1} and P_{n-1} respectively, and hence $C_{\xi_0}(\tau)$ and $C_{\xi_1}(\tau)$ are representations of $\mathfrak{g}_{n-1}$ and $\mathfrak{p}_{n-1}$, respectively. We write $\Phi(\tau) = |\det|^{-1/2} \otimes C_{\xi_1}(\tau)$ and we define the k-th derivative of τ to be the following representation of $\mathfrak{g}_{n-k}$

$$B^k(\tau) = C_{\xi_0}\Phi^{k-1}(\tau).$$

By [AGS, Proposition 3.0.3] if $\sigma \in \mathcal{HC}(G_n)$ then $B^k(\sigma) = B^k(\sigma|_{\mathfrak{p}_n}) \in \mathcal{HC}(G_{n-k})$, i.e. $B^k(\sigma)$ is admissible.

Theorem C allows one to calculate the annihilator variety $\operatorname{An}\mathcal{V}\left(B^k(\sigma)\right)$ in terms of $\operatorname{An}\mathcal{V}(\sigma)$. For simplicity we consider the case of $G_n = GL(n, \mathbb{R})$, since the case of $GL(n, \mathbb{C})$ is very similar. Note that

$\mathrm{An}\mathcal{V}(\sigma)$ is a union of complex nilpotent orbits $\mathcal{O}_\lambda \subset \mathfrak{g}_n^* = \mathfrak{gl}(n,\mathbb{C})^*$, which are indexed by partitions λ of n as in section 2.3. Also the nilradical $\mathfrak{n}$ consists of upper triangular matrices and for each partition λ we consider the character

$$\psi_\lambda(X) = \sqrt{-1}\left(\sum\nolimits_{j\notin S_\lambda} X_{j,j+1}\right)$$

where S_λ is the index set of partial sums $\{\lambda_k + \cdots + \lambda_l : 1 \leq k < l\}$ with $l = \mathrm{length}(\lambda)$; then we have $\psi_\lambda \in \mathcal{O}_\lambda$.

Lemma 5.0.2. *Let μ be a partition of $n-k$ and let $\mu \cup k$ be the partition of n obtained by inserting the part k in the appropriate place of μ. Then we have*

$$\psi_\mu \in \Psi\left(B^k\sigma\right) \iff \psi_{\mu\cup k} \in \Psi(\sigma)$$

Proof. Let α be the composition of n obtained by inserting the part k in the end of μ. It is a reordering of $\mu \cup k$ and thus ψ_α and $\psi_{\mu\cup k}$ belong to the same nilpotent orbit. The composition with the natural projection $B^k\sigma \twoheadrightarrow \sigma$ defines an isomorphism $Wh'_{\psi_\mu}(B^k\sigma) \cong Wh'_{\psi_\alpha}(\sigma)$. Thus

$$\psi_\mu \in \Psi\left(B^k\sigma\right) \iff \psi_\alpha \in \Psi(\sigma) \iff \psi_{\mu\cup k} \in \Psi(\sigma)$$

□

If λ is a partition and $k \leq \lambda_1$ then there is a unique i such that $\lambda_i \geq k > \lambda_{i+1}$, and we define

$$B^k(\lambda) := (\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+2},\ldots,\lambda_l) \cup (\lambda_i + \lambda_{i+1} - k), B^k(\mathcal{O}_\lambda) = \mathcal{O}_{B^k(\lambda)}.$$

We extend this definition to unions of orbits, setting $B^k(\mathcal{O}_\lambda) = \emptyset$ if $k > \lambda_1$.

Lemma 5.0.3. *Let λ be a partition of n and let μ be a partition of $n-k$, then $\mu \cup k \leq \lambda$ if and only if $k \leq \lambda_1$ and $\mu \leq B^k(\lambda)$.*

Proof. We use the notion of transposed partition $(\lambda^t)_i = \max\{j|\lambda_j \geq i\}$ and note that
(1) transposition is order-reversing.
(2) $(\mu \cup k)^t$ is obtained from μ^t by adding 1 to each of the first k parts.
(3) $(B^k(\lambda))^t$ is obtained from λ^t by subtracting 1 from each of the first k parts.

The lemma follows. □

Theorem 5.0.4. *If $\sigma \in \mathcal{HC}(G_n)$ then $\mathrm{An}\mathcal{V}(B^k\sigma) = B^k(\mathrm{An}\mathcal{V}(\sigma))$.*

Proof. Since $\psi_\lambda \in \mathcal{O}_\lambda$, by Theorem C and Lemma 5.0.2 we have

$$\mathcal{O}_\mu \subset \mathrm{An}\mathcal{V}(B^k\sigma) \Leftrightarrow \psi_\mu \in \Psi\left(B^k\sigma\right) \iff \psi_{\mu\cup k} \in \Psi(\sigma) \iff \mathcal{O}_{\mu\cup k} \subset \mathrm{An}\mathcal{V}(\sigma)$$

Thus it suffices to show that for any $\mathcal{O}$, $\mathcal{O}_{\mu\cup k} \leq \mathcal{O} \Leftrightarrow \mathcal{O}_\mu \leq B^k(\mathcal{O})$; this follows from Lemma 5.0.3. □

In [AGS] the *depth* of σ is defined to be the maximal rank of a matrix $A \in \mathrm{An}\mathcal{V}(\sigma) \subset \mathfrak{gl}_n(\mathbb{C})$, which is identified with $Mat_{n\times n}(\mathbb{C})$ via the trace form. Note that if $\mathrm{An}\mathcal{V}(\sigma) = \overline{\mathcal{O}_\lambda}$, then $\mathrm{depth}(\sigma) = \lambda_1$.

Corollary 5.0.5. *If $\sigma \in \mathcal{HC}(G_n)$ then $B^k(\sigma) = 0$ if and only if $k > \mathrm{depth}(\sigma)$.*

By [AGS, Corollary 4.2.2], Theorem C implies

Proposition 5.0.6. *Let χ_i be characters of GL_{n_i} and $n := n_1 + \cdots + n_k$. Let $\pi = \chi_1 \times \cdots \times \chi_k \in \mathcal{M}(G_n)$ be the corresponding induced representation. Then π has a unique irreducible subquotient τ with*

$$\mathrm{An}\mathcal{V}(\tau) = \mathrm{An}\mathcal{V}(\pi) = \overline{\mathcal{O}_{(n_1,\ldots,n_k)}}.$$

Moreover, τ occurs in π with multiplicity one.

Proof. Without loss of generality, we can suppose $n_1 \geq \cdots \geq n_k$ and write $\lambda := (n_1,\ldots,n_k)$. Then it is known that $\mathrm{An}\mathcal{V}(\pi) = \overline{\mathcal{O}_\lambda}$ and thus $\mathrm{An}\mathcal{V}(\tau) \subset \overline{\mathcal{O}_\lambda}$ for any subquotient τ of π. Now by Theorem C, $\mathrm{An}\mathcal{V}(\tau) \supset \overline{\mathcal{O}_\lambda}$ iff $\psi_\lambda \in \Psi(\tau)$, and by [AGS, Corollary 4.2.2], π has a unique such constituent. □

6. The case of complex classical groups

For convenience we fix an invariant form $\langle x, y\rangle$ on $\mathfrak{g}_0$ and we identify $i\mathfrak{g}_0^*$ with $\mathfrak{g}_0$ as follows:

$$\psi_x(y) = i\langle x, y\rangle .$$

In this section we prove Theorem D.

Let H, B, Π_0 etc. be as before, then $\Psi_0 \subset \mathfrak{g}_0$ can be identified with the direct sum of negative simple roots spaces

$$\Psi_0 = \bigoplus_{\beta\in\Pi_0} \mathfrak{g}_0^{-\beta}$$

By Proposition 2.4.5 this identifies non-degenerate elements of Ψ_0 with the *principal nilpotent elements* in Ψ_0, namely those for which each of the projections $p_\beta : \Psi_0 \to \mathfrak{g}_0^{-\beta}$ is non-zero. Lemma 2.4.6 gives the following result.

Lemma 6.0.1. *If $e \in \Psi_0$ then there is a standard Levi subalgebra $\mathfrak{l}_0$ such that e is a principal nilpotent element in $\mathfrak{l}_0$.*

We say that a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}_0$ is a PL-orbit if $\mathcal{O} \cap \mathfrak{l}_0$ is a principal nilpotent orbit in some Levi subalgebra $\mathfrak{l}_0$. Let $PL(G)$ denote the set of PL-orbits, and for an arbitrary nilpotent orbit $\mathcal{O}$ we define

$$PL(\mathcal{O}) = \left\{\mathcal{O}' \le \mathcal{O} \mid \mathcal{O}' \in PL(G)\right\}.$$

Lemma 6.0.2. *For each nilpotent orbit $\mathcal{O}$, the sets $\overline{\mathcal{O}} \cap \Psi_0$ and $PL(\mathcal{O})$ determine each other uniquely.*

Proof. Let X denote the union of the orbits in $PL(\mathcal{O})$ and let $Y = \overline{\mathcal{O}} \cap \Psi_0$. Then by the previous lemma we get $X = G\cdot Y$ and $Y = X \cap \Psi_0$. □

Therefore Theorem D reduces to the following statement.

Theorem 6.0.3. *For a complex classical group, every nilpotent orbit $\mathcal{O}$ is determined by $PL(\mathcal{O})$.*

We will prove this in §6.2 after describing the sets $PL(G)$ for classical groups.

6.1. Principal nilpotents in Levi subgroups.

In this section we assume that $G = GL(d,\mathbb{C}), O(d,\mathbb{C})$ or $Sp(d,\mathbb{C})$, and write $GL(d)$ etc. for simplicity. Nilpotent orbits for G are parametrized by partitions of d as in Theorem 2.5.1; we will regard $PL(G)$ as a set of partitions and write $PL(\lambda)$ instead $PL(\mathcal{O}_\lambda)$.

Lemma 6.1.1. *Let $\lambda_{\max}$ be the partition corresponding to a principal nilpotent orbit; then*

$$\lambda_{\max} = \begin{cases} (d-1, 1, 0, \ldots) & \text{if } G = O(d) \text{ with } d \text{ even} \\ (d, 0, 0, \ldots) & \text{otherwise} \end{cases}$$

Proof. The principal nilpotent orbit is maximal with respect to the closure order. The result follows from Theorem 2.5.1 and the easy verification that $\lambda_{\max}$ is the maximal element in $\mathcal{P}(G)$. □

For a partition λ write $OM(\lambda) = \{p > 1 \mid m_p(\lambda) \text{ is odd}\}$ and define

$$\mathcal{X}(G) = \begin{cases} \mathcal{P}(G) = \mathcal{P}(d) & \text{if } G = GL(d) \\ \{\lambda \in \mathcal{P}(G) : |OM(\lambda)| \le 1\} & \text{otherwise} \end{cases}$$

Proposition 6.1.2. *If $G = GL(d), O(d)$ or $Sp(d)$ then $PL(G) = \mathcal{X}(G)$.*

Proof. For $G = GL(d)$ the proposition asserts that every orbit is principal in some Levi subgroup, which follows from the Jordan canonical form.

The Levi subgroups of $O(d)$ and $Sp(d)$ are given as follows: up to conjugacy there is one for each partition κ with $\kappa_1 \ge \cdots \ge \kappa_r$ such that $d' = d - 2(\kappa_1 + \cdots + \kappa_r) \ge 0$. Explicitly

$$L_\kappa = \begin{cases} O(d') \times GL(\kappa_1) \times \cdots \times GL(\kappa_r) & \text{if } G = O(d) \\ Sp(d') \times GL(\kappa_1) \times \cdots \times GL(\kappa_r) & \text{if } G = Sp(d) \end{cases}$$

The principal nilpotent orbit in L_κ can be determined by the previous lemma. In the partition λ_κ for corresponding nilpotent orbit in $\mathfrak{g}_0^*$, each $GL(\kappa_i)$ factor contributes *two* parts of size κ_i. Thus up to decreasing reordering of the parts, we have

$$\lambda_\kappa = \begin{cases} (d'-1, 1, \kappa_1, \kappa_1, \dots, \kappa_r, \kappa_r, 0, 0, \cdots) & \text{if } G = O(d) \text{ with } d \text{ even} \\ (d', \kappa_1, \kappa_1, \dots, \kappa_r, \kappa_r, 0, 0, \cdots) & \text{otherwise} \end{cases}$$

By definition, parts with even multiplicity do not contribute to $OM(\lambda)$, thus

$$OM(\lambda_\kappa) = \begin{cases} OM((d'-1,1)) & \text{if } G = O(d) \text{ with } d \text{ even} \\ OM((d')) & \text{otherwise} \end{cases}$$

Moreover since the part 1 does not contribute to $OM(\lambda)$, we get $|OM(\lambda_\kappa)| \leq 1$. Thus $PL(G) \subseteq \mathcal{X}(G)$.

Conversely suppose $\lambda \in \mathcal{P}(G)$ satisfies $|OM(\lambda)| \leq 1$. Then λ has 0, 1, or 2 parts with odd multiplicity, and in the last case the part 1 has odd multiplicity. Thus the last case can only occur if $G = O(d)$, and since there are exactly two odd parts with odd multiplicity, d must be even. It follows now that λ is of the form λ_κ for some κ. Thus $\mathcal{X}(G) \subseteq PL(G)$. □

We prove Theorem 6.0.3 in the next subsection, using the following lemma.

Lemma 6.1.3. *For each $\lambda \in \mathcal{P}(G)$ and each k there is a partition $\mu = \mu(\lambda, k) \in PL(\lambda)$ such that*

$$\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k$$

Proof. Let j be the largest index such that $\lambda_j = \lambda_k$. If $(\lambda_1, \dots, \lambda_j)$ contains two or more parts p, q with odd multiplicity, then necessarily p, q have the same parity and so $r = (p+q)/2$ is an integer. If ν is obtained from λ by replacing a pair (p,q) by (r,r), then we have $\nu \leq \lambda$ and $\nu_1 + \cdots + \nu_k = \lambda_1 + \cdots + \lambda_k$. Iterating this we may assume that $(\lambda_1, \dots, \lambda_j)$ contains at most one part with odd multiplicity.

Now let $\mu \in \mathcal{P}(G)$ be obtained from λ by replacing the parts $\lambda_{j+1}, \lambda_{j+2} \dots$ by a string of 1's of length $(\lambda_{j+1} + \lambda_{j+2} + \cdots)$ Then $|OM(\mu)| \leq 1$ and hence μ satisfies the condition of the Lemma. □

6.2. **Proof of Theorem 6.0.3.** We now prove Theorem 6.0.3 for all classical groups.

Proof of Theorem 6.0.3. First suppose $G = GL(d), SL(d), O(d)$ or $Sp(d)$. We need to show that each $\lambda \in \mathcal{P}(G)$ is determined by the set $PL(\lambda)$. This is obvious for $G = GL(d)$, $SL(d)$ and therefore we may assume that $G = O(d)$ or $Sp(d)$. By definition of the partial order, for each k we have

$$\mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k \text{ for all } \mu \in PL(\lambda).$$

Moreover by Lemma 6.1.3 equality holds for some μ. Therefore for each k we can recover the sum $\lambda_1 + \cdots + \lambda_k$ as the maximum of $\mu_1 + \cdots + \mu_k$ for $\mu \in PL(\lambda)$, and hence we can determine λ as well.

Finally we consider $G = SO(d)$. If $\mathcal{O} = \mathcal{O}_\lambda$ where λ is not very even, then $\mathcal{O}$ is a single $O(d)$ orbit and so the result follows by the $O(d)$ argument. If $\mathcal{O} = \mathcal{O}_\lambda^I$ or $\mathcal{O}_\lambda^{II}$ for some very even λ, then $\mathcal{O} \cap \Psi_0$ is nonempty, thus $\mathcal{O}$ can be recovered from $\overline{\mathcal{O}} \cap \Psi_0$ in this case as well. □

The theorem does not extend to unions of orbits.

Example 6.2.1. *The table below lists examples of partition triples $[\lambda, \mu, \nu]$ such that $PL(\lambda) = PL(\mu) \cup PL(\nu)$. All orbits are special in the sense of Lusztig-Spaltenstein* [CoMG93, Section 6.3].

(18)

G	λ	μ	ν
$O(11)$	$(7,3,1)$	$(5,5,1)$	$(7,2,2)$
$Sp(10)$	$(6,4)$	$(5,5)$	$(6,2,2)$
$O(8)$	$(5,3)$	$(4,4)$	$(5,1,1,1)$

Remark 6.2.2. *If G is a classical group, we can regard elements of $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ as matrices. For a matrix $X \in \mathcal{O}_\lambda$ its rank and order of nilpotence are given by $n - length(\lambda)$ and λ_1 respectively; we refer to these as the rank and depth of $\mathcal{O}_\lambda$. If $\mathcal{V}$ is a union of orbits we define $rank(\mathcal{V})$ and $depth(\mathcal{V})$ by taking maxima, and the arguments above show that these are uniquely determined by $\mathcal{V} \cap \Psi_0$.*

For $\pi \in \mathcal{M}(G)$ we define $rank(\pi) = rank(\mathrm{WF}\,\pi)$ and $depth(\pi) = depth(\mathrm{WF}\,\pi)$. By [He08] *$rank(\pi)$ coincides with the Howe rank of π, and for $GL(n)$, $depth(\pi)$ coincides with the notion of depth in section*

5. It would be interesting to give a representation-theoretic characterization of depth for other classical groups. This remark shows that for all $\pi \in \mathcal{M}(G)$ the rank and the depth are determined by $\Psi(\pi)$.

6.3. **Exceptional groups.** Theorem D is false for every exceptional complex group and we now describe all counterexamples via the Bala-Carter classification [Car85, §§13.4]. Let us say for simplicity that two nilpotent G-orbits $\mathcal{O}$ and $\mathcal{O}'$ are *related* if $PL(\mathcal{O}) = PL(\mathcal{O}')$.

Proposition 6.3.1. *The following is a complete list of related orbits, with special orbits underlined.*

G	Related orbits	G	Related orbits
E_6	$\underline{E_6(a_1)}$ *and* $\underline{D_5}$	G_2	$\underline{G_2(a_1)}$ *and* $\widetilde{A_1}$
E_6	$\underline{D_4(a_1)}$ *and* $A_3 + A_1$	E_8	$\underline{E_8(a_1)}$, $\underline{E_8(a_2)}$ *and* $\underline{E_8(a_3)}$
E_7	$\underline{E_7(a_1)}$ *and* $\underline{E_7(a_2)}$	E_8	$\underline{E_8(a_4)}$, $\underline{E_8(b_4)}$ *and* $\underline{E_8(a_5)}$
E_7	$\underline{E_7(a_3)}$ *and* D_6	E_8	$\underline{E_7(a_1)}$, $\underline{E_8(b_5)}$ *and* $E_7(a_2)$
E_7	$\underline{E_6(a_1)}$ *and* $\underline{E_7(a_4)}$	E_8	$\underline{E_8(a_6)}$ *and* $\underline{D_7(a_1)}$
F_4	$\underline{F_4(a_1)}$ *and* $\underline{F_4(a_2)}$	E_8	$\underline{E_6(a_1)}$ *and* $\underline{E_7(a_4)}$
F_4	$\underline{F_4(a_3)}$ *and* $C_3(a_1)$	E_8	$\underline{E_8(a_7)}$, $E_7(a_5)$, $E_6(a_3) + A_1$, $D_6(a_2)$

Proof. In Bala-Carter notation, the PL-orbits are labeled by the corresponding Levi subalgebra $\mathfrak{l}$, while the other orbits have names of the form $\mathfrak{l}(*)$. Thus for any orbit $\mathcal{O}$ we can easily compute $PL(\mathcal{O})$ by looking at lower orbits whose Bala-Carter label has no parenthesis. With this in mind, the table above follows from the Bala-Carter classification tables [Car85, §§13.4] □

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