TWISTED HOMOLOGY FOR THE MIRABOLIC NILRADICAL

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Abstract. The notion of derivatives for smooth representations of $GL(n, \mathbb{Q}_p)$ was defined in [BZ77]. In the archimedean case, an analog of the highest derivative was defined for irreducible unitary representations in [Sah89] and called the "adduced" representation. In [AGS] derivatives of all orders were defined for smooth admissible Fréchet representations (of moderate growth).

A key ingredient of this definition is the functor of twisted coinvariants with respect to the nilradical of the mirabolic subgroup. In this paper we prove exactness of this functor and compute it on a certain class of representations. This implies exactness of the highest derivative functor, and allows to compute highest derivatives of all monomial representations.

In [AGS] these results are applied to finish the computation of adduced representations for all irreducible unitary representations and to prove uniqueness of degenerate Whittaker models for unitary representations.

1. Introduction

The notion of derivative was first defined in [BZ77] for smooth representations of $G_n = GL(n)$ over non-archimedean fields and became a crucial tool in the study of this category. The definition of derivative is based on the "mirabolic" subgroup P_n of G_n consisting of matrices with last row $(0, \dots, 0, 1)$. The unipotent radical of this subgroup is an $(n-1)$-dimensional linear space that we denote V_n, and the reductive quotient is G_{n-1}. The group G_{n-1} has 2 orbits on V_n and hence also on V_n^*: the zero and the non-zero orbit. The stabilizer in G_{n-1} of a non-trivial character ψ of V_n is isomorphic to P_{n-1}.

The construction of derivative is based on two functors: Φ and Ψ. This paper deals only with the functor Φ of (normalized) twisted zero homology (in other words co-equivariants) with respect to (V_n, ψ), which goes from the category of smooth representations of P_n to the category of smooth representations of P_{n-1}.

To be more precise, fix an archimedean local field F and denote $G_n := \mathrm{GL}(n, F)$. Let $\mathfrak{p}_n, \mathfrak{v}_n, ..$ denote the Lie algebras of the corresponding Lie groups. Let ψ_n be the standard non-degenerate character of V_n, given by $\psi_n(x_1, \dots, x_{n-1}) := \exp(\sqrt{-1}\pi \operatorname{Re} x_{n-1})$. We will also denote by ψ_n the corresponding character of the Lie algebra $\mathfrak{v}_n$. For all n and for all representations π of $\mathfrak{p}_n$, we define

$$\Phi(\pi) := |\det|^{-1/2} \otimes \pi_{(\mathfrak{v}_n, \psi_n)} := |\det|^{-1/2} \otimes \pi / \mathrm{Span}\{\alpha v - \psi_n(\alpha) v \,:\, v \in \pi,\, \alpha \in \mathfrak{v}_n\}$$

Now, define $E^k(\pi) := \Phi^{k-1}(\pi)$. This is one of the three notions of derivatives defined in [AGS]. The other two are obtained from this one by applying coinvariants (respectively generalized coinvariants) with respect to $\mathfrak{v}_{n-k+1}$. However, in this paper we will only prove results concerning E^k.

In the non-archimedean case there is an equivalence between the category of smooth representations of P_n and the category of G_{n-1}-equivariant sheaves on V_n^*. This equivalence is based on the Fourier transform. Under this equivalence, Φ becomes the fiber at the point ψ. It can also be viewed as the composition of two functors: restriction to the open orbit and an equivalence of categories between equivariant sheaves on the orbit and representations of the stabilizer of a point. We used this as an intuition for the archimedean case. In particular, we expected Φ to be exact, which indeed is one of the main results of this paper, that we will now formulate.

Let $\mathcal{M}(G_n)$ denote the category of smooth admissible Fréchet representations of moderate growth.

Theorem A. *For any* $0 < k \leq n$

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(1) *$E^k : \mathcal{M}(G_n) \to \mathcal{M}(\mathfrak{p}_n)$ is an exact functor.*
(2) *For any $\pi \in \mathcal{M}(G_n)$, the natural (quotient) topology on $E^k(\pi)$ is Hausdorff, i.e. $\mathfrak{u}^k(\pi \otimes (-\psi^k))$ is closed in π.*

For proof see §3.

Theorem B. *Let $n = n_1 + \cdots + n_k$ and let χ_i be characters of G_{n_i}. Let $\pi = \chi_1 \times \cdots \times \chi_k$ denote the corresponding monomial representation. Then*

$$E^k(\pi) = E^1(\chi_1) \times \cdots \times E^1(\chi_k) = ((\chi_1)|_{G_{n_1-1}} \times \cdots \times (\chi_k)|_{G_{n_k-1}}).$$

We prove this theorem in §4.

In the case of $k = n$ this becomes the Whittaker functor, which is known to be exact by [CHM00]. In this case Theorem B implies that the space of Whittaker functionals on a principal series representation is one-dimensional, which was proven for G_n by Casselman-Zuckerman and for a general quasi-split reductive group by Kostant.

1.1. **Structure of our proof.**

1.1.1. *Exactness and Hausdorffness.* It turns out to be convenient to prove the exactness and the Hausdorffness together. Let us describe the proof of exactness, and the proof of Hausdorffness will be incorporated into this argument.

The ideas of the proof come from two sources: the proof of the case $k = n$ given in [CHM00] and the analogy with the p-adic case, where one can use the language of equivariant l-sheaves.

In [CHM00], the first step of the proof is reduction to the statement that the principal series representations are acyclic, using the Casselman embedding theorem. This step works in our case as well. Then in [CHM00] they prove that the principal series representations are acyclic by decomposing them with respect to the decomposition of the flag variety into U_n^n orbits. In their case, there is a finite number of U_n^n orbits and each orbit corresponds to an acyclic representation. In our case, U_n^k has infinitely many orbits on the flag variety and some of them give rise to non-acyclic representations.

Eventually we show that these problematic orbits do not contribute to the higher $(\mathfrak{u}_n^k, \psi_n^k)$-cohomologies of principal series since they are immersed in smooth families of orbits most of which give rise to acyclic representations. It is difficult to see that directly, therefore we turn to the strategy inspired by equivariant sheaves.

If we had a nice category of P_n-representations, which like in the p-adic case was equivalent to a nice category of G_{n-1}-equivariant sheaves on V_n^*, then the functor Φ would be a composition of a restriction to an open set and an equivalence of categories, and thus it should be exact. Unfortunately it seems too hard at this point to define such categories. Rather than doing this we define a certain class of P_n-representations. We define it using specific examples and constructions and not by demanding certain properties. We do not view this class as a "nice category of representation of P_n", in particular we do not analyze it using equivariant sheaves. However it is sufficient for the proof of exactness. Namely we prove that this class includes the principal series representations of G_n. We prove that it is closed under Φ. Finally, we prove that its objects are acyclic with respect to Φ.

In order to do this we develop some tools for computing the homology of representations constructed in a geometric way.

1.1.2. *Derivative of monomial representations.* In the non-archimedean case, [BZ77] provides a description of any derivative of a Bernstein-Zelevinsky product $\rho_1 \times \rho_2$ in terms of derivatives of ρ_1 and ρ_2. This description immediately implies the natural analog of Theorem B. This description is proved using geometric analysis of the Bernstein-Zelevinsky product as the space of sections of the corresponding sheaf on the corresponding Grassmanian.

This method is hard to translate to the archimedean case for the following reasons.

- The analogous statement for the derivative as we defined it here does not hold. If we would not replace the functor of coinvariants by the functor of generalized coinvariants then the analogous statement might hold. However, we would lose the exactness, which played a crucial role in the proof in [BZ77].

- We do not have an appropriate language of infinite-dimensional sheaves.

For these reasons, we prove a product formula only for the depth derivatives, and only for products of characters. The drawback of that approach is that we have to consider product of more than two characters.

We prove the product formula using the geometric description of monomial representations. More specifically, we decompose the corresponding flag variety into P_n-orbits. This decomposition gives a filtration of the monomial representation. We compute the functor Φ on each of the associated graded pieces in geometric terms, and use the exactness of Φ to proceed by induction.

1.2. **Tools developed in this paper.**

1.2.1. *The relative Shapiro lemma.* In the proof of the exactness of the derivatives we are interested in the study of homology of representations of the following type. Let a real algebraic group G act on a real algebraic manifold X, and let $\mathcal{E}$ be a G-equivariant bundle over X. Consider the space $\mathcal{S}(X,\mathcal{E})$ of Schwartz sections of this bundle (see §2.4). This is a representation of G. For technical reasons we prefer to study Lie algebra homology rather than Lie group homology. Thus we will impose some homotopic assumptions on G and X so that there will be no difference. In the case when X is homogeneous, i.e. $X = G/H$, a version of the Shapiro Lemma states, under suitable homotopic and unimodularity assumptions on G and H, that $\mathrm{H}_*(\mathfrak{g},\mathcal{S}(X,\mathcal{E})) = \mathrm{H}_*(\mathfrak{h},\mathcal{E}_{[1]})$, where $\mathcal{E}_{[1]}$ denotes the fiber of $\mathcal{E}$ at the class of 1 in G/H (cf. [AG10]).

In Appendix A we prove a generalization of this statement to the relative case. Namely, let G act on $X \times Y$ and $\mathcal{E}$ be a G-equivariant bundle over $X \times Y$. Suppose that $X = G/H$. Then, under the same assumptions on G and H, we have

$$\mathrm{H}_*(\mathfrak{g},\mathcal{S}(X \times Y,\mathcal{E})) = \mathrm{H}_*(\mathfrak{h},\mathcal{S}([1] \times Y,\mathcal{E}_{[1]\times Y})).$$

For the zero homology this statement is in fact Frobenius descent (cf. [AG09, Appendix B.3]).

1.2.2. *Homology of families of characters.* In the proof of the exactness of the derivatives, we use the relative Shapiro lemma to reduce to the study of $\mathrm{H}_*(\mathfrak{g},\mathcal{S}(X,\mathcal{E}))$, where the action of G on X is trivial. Moreover, in our case $\mathcal{E}$ is a line bundle and $\mathfrak{g}$ is abelian. In §6 we prove that $\mathcal{S}(X,\mathcal{E})$ is acyclic and compute $\mathrm{H}_0(\mathfrak{g},\mathcal{S}(X,\mathcal{E}))$ under certain conditions on the action of G on $\mathcal{E}$.

1.2.3. *Tempered equivariant bundles.* Since we discuss non-algebraic (e.g. unitary) characters, it might become necessary to consider non-algebraic geometric objects. We tried to avoid that, but had to consider vector bundles of algebraic nature with a non-algebraic action of a group. A similar difficulty arose in [AG], and the notion of tempered equivariant bundle was defined in order to deal with it. In this paper we develop more tools to work with such bundles.

1.3. **Structure of the paper.**

The proof of exactness and Hausdorffness of Φ^k and the computation of the depth derivative of the monomial representations are based on the theory of Schwartz functions on Nash manifolds. In §2.4 we review this theory, including the notion of equivariant tempered bundle.

In §3 we prove that Φ^k is exact functor on the category of smooth admissible representations and that $\Phi^k(\pi)$ is Hausdorff for any reducible smooth admissible representation π. In §§3.1 we introduce the class of good representations of $\mathfrak{p}_n$. The construction of this class is based on representations with simple geometric descriptions that we call geometric representations. We prove that this class includes the principal series representations of G_n, is closed under Φ and that its objects are acyclic with respect to Φ. In order to prove the last two statements we use Lemma 3.1.1 (the key lemma), which is proved in §6.

In §4 we compute the depth derivative of a monomial representation. This computation is based on the computation of certain geometric representations. We formulate the results of this computation in §§4.4, and prove them in §§5.8.

In §5 we give the postponed proofs of the technical lemmas that were used in sections 3 and 4. For this we need some preliminaries on homological algebra, topological linear spaces and Schwartz functions on Nash manifolds, which we give in subsections 5.1, 5.3, and 5.5.

In §6 we prove the key lemma (Lemma 3.1.1).

In Appendix A we prove a version of the Shapiro lemma which is crucial for the proof of Lemma 3.1.1.

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2. Preliminaries

2.1. **Notation and conventions.**

- We will denote real algebraic groups by capital Latin letters and their complexified Lie algebras by small Gothic letters.
- Let $\mathfrak{g}$ be a complex Lie algebra. We denote by $\mathcal{M}(\mathfrak{g})$ the category of (arbitrary) $\mathfrak{g}$-modules. Let ψ be a character of $\mathfrak{g}$. For a module $M \in \mathcal{M}(\mathfrak{g})$ denote by $M^{\mathfrak{g}}$ the space of $\mathfrak{g}$-invariants, by $M^{\mathfrak{g},\psi}$ the space of $(\mathfrak{g},\psi)$-equivariants, by $M_{\mathfrak{g}}$ the space of coinvariants, i.e. $M_{\mathfrak{g}} := M/\mathfrak{g}M$ and by $M_{\mathfrak{g},\psi}$ the space of $(\mathfrak{g},\psi)$-coequivariants, i.e. $M_{\mathfrak{g},\psi} := (M \otimes (-\psi))_{\mathfrak{g}}$.
- $G_n := GL(n,F)$, we consider the standard embeddings $G_n \subset G_m$ for any $m > n$, we denote the union by G_∞ and all the groups we will consider will be embedded to G_∞ in a standard way.
- By a composition of n we mean a tuple $(n_1,\dots,n_k)$ of natural numbers such that $n_1+\dots+n_k = n$. By a partition we mean a composition which is non-increasing, i.e. $n_1 \geq \dots \geq n_k$.
- For a composition $\lambda = (n_1,\dots,n_k)$ of n we denote by P_λ the corresponding block-upper triangular parabolic subgroup of G_n. For example, $P_{(1,\dots,1)} = B_n$ denotes the standard Borel subgroup, $P_{(n)} = G_n$ and $P_{(n-1,1)}$ denotes the standard maximal parabolic subgroup that includes P_n.
- We denote by $P_n \subset G_n$ the mirabolic subgroup (consisting of matrices with last row $(0,\dots,0,1)$).
- Let $V_n \subset P_n$ be the last column. Note that $V_n \cong F^{n-1}$ and $P_n = G_{n-1} \ltimes V_n$. Let $U_n^k := V_{n-k+1}V_{n-k+2}\cdots V_n$ and $S_n^k := G_{n-k}U_n^k$. Note that U_n^k is the unipotent radical of S_n^k. Let $N_n := U_n^n$.
- Fix a non-trivial unitary additive character θ of F, given by $\theta(x) = \exp(\sqrt{-1}\pi \operatorname{Re} x)$.
- Let $\bar{\psi}_n^k : U_n^k \to F$ be the standard non-degenerate homomorphism, given by $\bar{\psi}_n^k(u) = \sum_{j=n-k}^{n-1} u_{j,j+1}$ and let $\psi_n^k := \theta \circ \bar{\psi}_n^k$.

We will usually omit the n from the notations U_n^k and S_n^k, and both indexes from ψ_n^k.

Definition 2.1.1. *Define the functor $\Phi : \mathcal{M}(\mathfrak{p}_n) \to \mathcal{M}(\mathfrak{p}_{n-1})$ by $\Phi(\pi) := \pi_{\mathfrak{v}_n,\psi} \otimes |\det|^{-1/2}$.*

For a $\mathfrak{p}_n$-module π we define derivative by $E^k(\pi) := \Phi^{k-1}(\pi) := \pi_{\mathfrak{u}^{k-1},\psi^{k-1}} \otimes |\det|^{-(k-1)/2}$. Clearly it has a structure of a $\mathfrak{p}_{n-k+1}$ - representation. For convenience we will also use untwisted versions of the above functors, defined by $\widetilde{\Phi}(\pi) := \Phi(\pi) \otimes |\det|^{1/2}$, and $\widetilde{E}^k(\pi) := E^k(\pi) \otimes |\det|^{(k-1)/2}$.

2.2. **Smooth representations.**

Theorem 2.2.1 (Casselman-Wallach)**.**

(i) The category $\mathcal{M}(G)$ is abelian.
(ii) Any morphism in $\mathcal{M}_\infty(G)$ has closed image.

Theorem 2.2.2 (Casselman subrepresentation theorem, see [CM82], Proposition 8.23)**.** *Let π be a finitely generated admissible $(\mathfrak{g},K)$-module and P be a minimal parabolic subgroup of G. Then there exists a finite-dimensional representation σ of P such that π may be imbedded into $Ind_P^G(\sigma)$.*

2.3. **Parabolic induction and Bernstein-Zelevinsky product.** Let G be a real reductive group, P be a parabolic subgroup, M be the Levi quotient of P and $pr : P \to M$ denote the natural map.

Notation 2.3.1.

- *For a Lie group H we denote by Δ_H the modulus character of H, i.e. the absolute value of the determinant of the infinitesimal adjoint action.*
- *For $\pi \in \mathcal{M}(M)$ we denote by $I_M^G(\pi)$ the normalized parabolic induction of π, i.e. the space of smooth functions $f : G \to \pi$ such that $f(pg) = \Delta_H(p)^{1/2}\Delta_G(p)^{-1/2}\pi(pr(p))f(g)$, with the action of G given by $(I_M^G(\pi)(g)f)(x) := f(xg)$.*

2.3.1. *Bernstein-Zelevinsky product.* We now introduce the Bernstein-Zelevinsky product notation for parabolic induction.

Definition 2.3.2. *If $\alpha = (n_1, \dots, n_k)$ is a composition of n and $\pi_i \in \mathcal{M}(G_{n_i})$ then $\pi_1 \otimes \cdots \otimes \pi_k$ is a representation of $L_\alpha \approx G_{\alpha_1} \times \cdots \times G_{\alpha_k}$. We define*

$$\pi_1 \times \cdots \times \pi_k = I_{P_\alpha}^{G_n}(\pi_1 \otimes \cdots \otimes \pi_k)$$

$\pi_1 \times \cdots \times \pi_k$ will be referred to below as the Bernstein-Zelevinsky product, or the BZ-product, or sometimes just the product of $\pi_1, \dots, \pi_k$. It is well known (see e.g. [Wall92, §§12.1]) that the product is commutative in the Grothendieck group.

2.4. **Schwartz functions on Nash manifolds.** In the proofs of Theorems A and B we will use the language of Schwartz functions on Nash manifolds, as developed in [AG08].

Nash manifolds are smooth semi-algebraic manifolds. In sections 3 and 4 only algebraic manifolds are considered and thus the reader can safely replace the word "Nash" by "real algebraic". However, in section 5 we will use Nash manifolds which are not algebraic.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. We also need the notion of tempered functions, i.e. smooth functions that grow not faster than a polynomial, and so do all their derivatives. For precise definitions of those notions we refer the reader to [AG08]. In this section we summarize some elements of the theory of Schwartz functions. We will give more details in §§5.5.

Notation 2.4.1. *Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X. For any Nash vector bundle $\mathcal{E}$ over X we denote by $\mathcal{S}(X, \mathcal{E})$ the space of Schwartz sections of $\mathcal{E}$.*

Proposition 2.4.2 ([AG08], Theorem 5.4.3)**.** *Let $U \subset X$ be a (semi-algebraic) open subset, then*

$$\mathcal{S}(U, \mathcal{E}) \cong \{\phi \in \mathcal{S}(X, \mathcal{E}) | \quad \phi \text{ is 0 on } X \setminus U \text{ with all derivatives}\}.$$

Notation 2.4.3. *Let Z be a locally closed semi-algebraic subset of a Nash manifold X. Let $\mathcal{E}$ be a Nash bundle over X. Denote*

$$\mathcal{S}_X(Z, \mathcal{E}) := \mathcal{S}(X - (\overline{Z} - Z), \mathcal{E})/\mathcal{S}(X - \overline{Z}, \mathcal{E}).$$

Here we identify $\mathcal{S}(X - \overline{Z}, \mathcal{E})$ with a closed subspace of $\mathcal{S}(X - (\overline{Z} - Z), \mathcal{E})$ using the description of Schwartz functions on an open set (Proposition 2.4.2).

Corollary 2.4.4. *Let $X := \bigcup_{i=1}^k X_i$ be a Nash stratification of a Nash manifold X, i.e. $\bigcup_{i=j}^k X_i$ is an open Nash subset of X for any j. Let $\mathcal{E}$ be a Nash bundle over X.*

Then $\mathcal{S}(X, \mathcal{E})$ has a natural filtration of length k such that $Gr^i(\mathcal{S}(X, \mathcal{E})) = \mathcal{S}_X(X_i, \mathcal{E})$.

Moreover, if Y is a Nash manifold and $X \subset Y$ is a (locally closed) Nash submanifold then $\mathcal{S}_Y(X, \mathcal{E})$ has a natural filtration of length k such that $Gr^i(\mathcal{S}_Y(X, \mathcal{E})) = \mathcal{S}_Y(X_i, \mathcal{E})$.

Notation 2.4.5. *Let X be a Nash manifold.*
(i) We denote by D_X the Nash bundle of densities on X. It is the natural bundle whose smooth sections are smooth measures. For the precise definition see e.g. [AG10].
(ii) Let $\phi : X \to Y$ be a map of (Nash) manifolds. Denote $D_Y^X := D_\phi := \phi^*(D_Y^*) \otimes D_X$.
(iii) Let $\mathcal{E} \to Y$ be a (Nash) bundle. Denote $\phi^?(\mathcal{E}) = \phi^*(\mathcal{E}) \otimes D_Y^X$.

Remark 2.4.6. *If ϕ is a submersion then for all $y \in Y$ we have $D_Y^X|_{\phi^{-1}(y)} \cong D_{\phi^{-1}(y)}$.*

2.5. **Tempered functions.** Analyzing smooth representations from a geometric point of view we encounter two related technical difficulties:

- Most characters of $F^\times$ are not Nash
- ψ is not Nash.

This makes the group action on the geometric objects that we consider to be not Nash. This could cause us to consider geometric objects outside the realm of Nash manifolds. In order to prevent that we introduce some technical notions, including the notion of G-tempered bundle which is, roughly speaking, a Nash bundle with a tempered G-action. The price of that is the need to make sure that all our constructions produce only Nash geometric objects.

Notation 2.5.1. *Let X be a Nash manifold. Denote by $\mathcal{T}(X)$ the space of tempered functions on X. For any Nash vector bundle $\mathcal{E}$ over X we denote by $\mathcal{T}(X,\mathcal{E})$ the space of tempered functions of $\mathcal{E}$.*

Note that any unitary character of a real algebraic group G is a tempered function on G.

Definition 2.5.2. *Let X be a Nash manifold and let $\mathcal{E}$ and $\mathcal{E}'$ be Nash bundles over it. A morphism of bundles $\phi:\mathcal{E}\to\mathcal{E}'$ is called tempered if it corresponds to a tempered section of the bundle* $\mathrm{Hom}(\mathcal{E},\mathcal{E}')$.

Note that if $\phi:\mathcal{E}\to\mathcal{E}'$ is a tempered morphism of bundles, then the induced map $C^\infty(X,\mathcal{E})\to C^\infty(X,\mathcal{E}')$ maps $\mathcal{S}(X,\mathcal{E})$ to $\mathcal{S}(X,\mathcal{E}')$ and $\mathcal{T}(X,\mathcal{E})$ to $\mathcal{S}(X,\mathcal{E}')$.

Definition 2.5.3. *Let a Nash group G act on a Nash manifold X. A* **G-tempered bundle** *$\mathcal{E}$ over X is a Nash bundle $\mathcal{E}$ with an equivariant structure $\phi:a^*(\mathcal{E})\to p^*(\mathcal{E})$ (here $a:G\times X\to X$ is the action map and $p:G\times X\to X$ is the projection) that is a tempered morphism of bundles such that for any element L in the Lie algebra of G and any open (semi-algebraic) subset $U\subset X$ the derived map $a(L):C^\infty(U,\mathcal{E})\to C^\infty(U,\mathcal{E})$ preserves the sub-space of Nash sections of $\mathcal{E}$ on U.*

Definition 2.5.4.

- *We call a character χ of a Nash group G* multiplicative *if $\chi=\mu\circ\overline{\chi}$, where $\overline{\chi}:G\to F^\times$ is a homomorphism of Nash groups and $\mu:F^\times\to\mathbb{C}^\times$ is a character. Note that all multiplicative characters are tempered.*
- *A multiplicative representation of G is a product of (finite-dimensional) Nash representation of G with a multiplicative character.*
- *Let a Nash group G act on a Nash manifold X. A G-tempered bundle $\mathcal{E}$ over X is called* **G-multiplicative bundle** *if for any point $x\in X$, the fiber $\mathcal{E}_x$ is a multiplicative representation of the stabilizer G_x.*

3. Proof of exactness and Hausdorffness (Theorem A)

We will prove Theorem A for the "untwisted" functor $\widetilde{E}^k$. Clearly, these versions are equivalent.

The proof is based on the following theorem.

Theorem 3.0.1. *There exists a sequence of collections $\{C_n\}_{n=1}^\infty$ of topological Hausdorff $\mathfrak{p}_n$-representations such that*

1. *Any representation parabolically induced from a finite-dimensional representation of the torus lies in C_n.*
2. *For any $\pi\in C_n$, $\pi\otimes(-\psi)$ is $\mathfrak{v}_n$-acyclic (as a linear representation).*
3. *For any $\pi\in C_n$, we have $\pi_{\mathfrak{v}_n,\psi}\in C_{n-1}$.*

We will prove this theorem in §§3.1.

Corollary 3.0.2. *Any $\mathfrak{p}_n$-representation π of the class C_n is acyclic with respect to $\widetilde{E}^k$ and $\widetilde{E}^k(\pi)$ is Hausdorff, for any $0<k\le n$.*

This corollary follows from Theorem 3.0.1, using the following lemma that we will prove in §§5.2.

Lemma 3.0.3. *Let π be a representation of $\mathfrak{p}_n$. Suppose that π is $\widetilde{\Phi}$-acyclic and $\widetilde{\Phi}(\pi)$ is $\widetilde{E}^k$-acyclic. Then π is $\widetilde{E}^{k+1}$-acyclic.*

Corollary 3.0.4. *Any representation π of G_n parabolically induced from a finite-dimensional representation of the torus is acyclic with respect to $\widetilde{E}^k$ and $\widetilde{E}^k(\pi)$ is Hausdorff, for any $0 < k \leq n$.*

Now we are ready to deduce Theorem A. The deduction follows the lines of the proof of [CHM00, Lemma 8.4].

Proof of Theorem A. (1) First of all, note that it is enough to show that for any $\pi \in \mathcal{M}(G_n)$, the $\mathfrak{u}^{n,k}$-representation $\pi \otimes (-\psi_n^k)$ is acyclic.

We will prove by downward induction on l that $H^l(\mathfrak{u}_n^k, \pi \otimes (-\psi_n^k)) = 0$ for any $\pi \in \mathcal{M}(G_n)$. For $l > \dim \mathfrak{u}_n^k$ it follows from the Koszul complex (see §§§5.1.1). Let us assume that the statement holds for $l+1$ and prove it for l. By the Casselman subrepresentation theorem (Theorem 2.2.2), one can embed π into a representation I parabolically induced from a finite dimensional representation of the torus. Consider the short exact sequence

$$0 \to \pi \to I \to I/\pi \to 0.$$

By Theorem 2.2.1, $I/\pi \in \mathcal{M}(G_n)$ and thus, by the induction hypothesis, $H^{l+1}(\mathfrak{u}_n^k, (I/\pi) \otimes (-\psi_n^k)) = 0$. By Corollary 3.0.4, $H^i(\mathfrak{u}_n^k, I \otimes (-\psi_n^k)) = 0$ for all $i > 0$. Thus, $H^l(\mathfrak{u}_n^k, \pi \otimes (-\psi_n^k)) = H^{l+1}(\mathfrak{u}_n^k, (I/\pi) \otimes (-\psi_n^k)) = 0$.

(2) By the Casselman subrepresentation theorem (Theorem 2.2.2), one can embed π into a representation I parabolically induced from a finite dimensional representation of the torus. By (1), we have an embedding $\widetilde{E}^k(\pi) \hookrightarrow \widetilde{E}^k_\infty(I)$. By Corollary 3.0.4, $\widetilde{E}^k(I)$ is Hausdorff and hence $\widetilde{E}^k(\pi)$ is Hausdorff.

□

3.1. **Good $\mathfrak{p}_n$-representations and the key lemma.** The following lemma will play a key role in this and the next section.

Lemma 3.1.1 (The key lemma)**.** *Let T be a Nash linear group, i.e. a Nash subgroup of $GL_k(\mathbb{R})$ for some k, and let $R := P_n \times T$ and $R' := G_{n-1} \times T$. Let $Q < R$ be a Nash subgroup and let $X := R/Q$ and $X' := V_n \backslash X$. Note that $X' = R'/Q'$, where $Q' = Q/(Q \cap V_n)$. Let $X_0 = \{x \in X : \psi|_{(V_n)_x} = 1\}$. Let X_0' be the image of X_0 in X'. Let $\mathcal{E}'$ be an R'-tempered bundle on X' and $\mathcal{E} := p_{X'}^?(\mathcal{E}')$ be its pullback to X. Then*

(i) $\mathrm{H}_i(\mathfrak{v}_n, \mathcal{S}(X, \mathcal{E}) \otimes (-\psi)) = 0$ *for any* $i > 0$.
(ii) X_0' *is smooth*
(iii) $\mathrm{H}_0(\mathfrak{v}_n, \mathcal{S}(X, \mathcal{E}) \otimes (-\psi)) \cong \mathcal{S}(X_0', \mathcal{E}|_{X_0'})$ *as representations of* $\mathfrak{p}_{n-1}$.

We will prove this lemma in §6.

Definition 3.1.2. *Let $B_n < G_n$ denote the standard (upper-triangular) Borel subgroup. In the situation of the above lemma, assuming that $(B_n \cap P_n) \times T$ has finite number of orbits on X, and that $\mathcal{E}'$ is R'-multiplicative we call the representation $\mathcal{S}(X, \mathcal{E})$ a geometric representation of $\mathfrak{p}_n$.*

Definition 3.1.3. *We define a good extension of nuclear Fréchet spaces to be the following data:*

(1) a nuclear Fréchet space W
(2) a countable descending filtration $F^i(W)$ by closed subspaces
(3) a sequence of nuclear Fréchet spaces W_i
(4) isomorphisms $\phi_i : F^{i+1}(W)/F^i(W) \to W_i$

such that the natural map $W \to \varprojlim W/F^i(W)$ is an isomorphism of (non-topological) linear spaces.

In this situation we will also say that W is a good extension of W_i. If a Lie algebra $\mathfrak{g}$ acts on W and on W_i, preserving $F^i(W)$ and commuting with ϕ we will say that this good extension is $\mathfrak{g}$-invariant.

Definition 3.1.4. *We define the collection of good representations of $\mathfrak{p}_n$ to be the smallest collection that includes the geometric representations and is closed with respect to $\mathfrak{p}_n$-invariant good extensions.*

We will prove the following concretization of Theorem 3.0.1:

Theorem 3.1.5.

(1) The representations of G_n parabolically induced from a finite dimensional representation of the torus are good representations of $\mathfrak{p}_n$.

(2) For any good representation π of $\mathfrak{p}_n$, $\pi \otimes (-\psi)$ is $\mathfrak{v}_n$-acyclic (as a linear representation).
(3) For any good representation π of $\mathfrak{p}_n$, the representation $\pi_{\mathfrak{v}_n,\psi}$ is good representation of $\mathfrak{p}_{n-1}$.

For the proof we will need some lemmas.

Lemma 3.1.6. *Let T be a Nash linear group, and let $R := P_n \times T$ and $R' := G_{n-1} \times T$. Let Y be an R-Nash manifold. Let $X \subset Y$ be an R-invariant (locally closed) Nash submanifold such that $(B_n \cap P_n) \times T$ has finite number of orbits on X. Let $\mathcal{E}$ be an R-multiplicative bundle on Y. Then the representation $\mathcal{S}_Y(X, \mathcal{E})$ is good.*

We will prove this lemma in §§5.6.

Lemma 3.1.7. *Let G be a Nash group and H be a closed Nash subgroup. Let χ be a multiplicative character of H. Let K be a closed Nash subgroup of G such that $KH = G$ and $\chi|_{K\cap H}$ is a Nash character. Let $\mathcal{E} = G \times_H \chi$ be the smooth bundle on G/H obtained by inducing the character χ. Then $\mathcal{E}$ admits a structure of a G-multiplicative bundle.*

We will prove this lemma in §§5.7.

Corollary 3.1.8. *Let χ be a character of the torus $T_n < B_n < G_n$, continued trivially to B_n. Let $\pi = Ind_{B_n}^{G_n}(\chi)$. Then there exists a G_n-multiplicative bundle $\mathcal{E}$ on G_n/B_n such that $\pi = \mathcal{S}(G_n/B_n, \mathcal{E})$.*

Lemma 3.1.9. *Let $\mathfrak{g}$ be a Lie algebra. Let (π, F, π_i) be a $\mathfrak{g}$-invariant good extension. Suppose that π_i are $\mathfrak{g}$-acyclic and $\mathrm{H}_0(\mathfrak{g}, \pi_i)$ is Hausdorff for any i. Then π is acyclic and $\mathrm{H}_0(\mathfrak{g}, \pi)$ together with the induced filtration $F^i(\mathrm{H}_0(\mathfrak{g}, \pi)) = F^i(\pi)/(\mathfrak{g}\pi \cap F^i(\pi))$ and the natural morphisms $F^i(\mathrm{H}_0(\mathfrak{g}, \pi))/F^{i+1}(\mathrm{H}_0(\mathfrak{g}, \pi)) \to \mathrm{H}_0(\mathfrak{g}, \pi_i)$ form a good extension. In particular, $\mathrm{H}_0(\mathfrak{g}, \pi)$ is Hausdorff.*

We will prove this lemma in §§5.4.

Proof of Theorem 3.1.5.

(1) Let σ be a finite dimensional representation of the torus and let $\pi \in \mathcal{M}(G_n)$ be its parabolic induction. Without loss of generality, by the definition of goodness, we can assume that σ is a character. In this case π is good by Corollary 3.1.8 and Lemma 3.1.6.
(2) The key lemma (Lemma 3.1.1) implies that $\pi \otimes (-\psi)$ is acyclic for any geometric representation π. Hence Lemma 3.1.9 implies by induction that $\pi \otimes (-\psi)$ is acyclic for any good representation π.
(3) If $\pi = \mathcal{S}(X, \mathcal{E})$ is a geometric representation then by the key lemma (Lemma 3.1.1), $\pi_{\mathfrak{v}_n,\psi} \cong \mathcal{S}(X'_0, \mathcal{E}'|_{X'_0})$, where $\mathcal{E}'$ and X'_0 are as in the key lemma. Let $\widetilde{T}$ denote the product of T with the center of G_{n-1}. Clearly, $\widetilde{T}$ acts on X'_0 and $B_{n-1} \times T = (B_{n-1} \cap P_{n-1}) \times \widetilde{T}$ has a finite number of orbits on X'_0. Thus, by Lemma 3.1.6, $\pi_{\mathfrak{v}_n,\psi}$ is good.
Now let π be a good representation. We can assume by induction that π is a good extension of π_i, where π_i are good and $(\pi_i)_{\mathfrak{v}_n,\psi}$ are good. Then Lemma 3.1.9 implies that $\pi_{\mathfrak{v}_n,\psi}$ is a good extension of $(\pi_i)_{\mathfrak{v}_n,\psi}$ and hence is good.

□

For the proof of Theorem B we will need the following corollary of Lemma 3.1.9 and Theorem 3.1.5.

Corollary 3.1.10. *Let (π, F, π_i) be a $\mathfrak{p}_n$-invariant good extension of good representations of $\mathfrak{p}_n$. Then $E^k(\pi)$ together with the induced filtration defined by the images of $E^k(F^i(\pi))$ and the natural morphisms $F^i(E^k(\pi))/F^{i+1}(E^k(\pi)) \to E^k(\pi_i)$ form a good extension.*

4. Highest derivative of monomial representations (Proof of Theorem B)

We first sketch the proof for the case of product of two characters. We believe that using an appropriate notion of Fréchet bundles it will be possible to upgrade this proof to the case highest derivative of product of two arbitrary representations, which by induction will give a more straightforward proof of Theorem B.

Since we do not currently have a proof for a product of two representations, in §§4.3 - 4.5 we prove Theorem B directly for a product of k characters.

4.1. **Sketch of proof for the case $k = 2$.** First, note that there exists a G_n-multiplicative line bundle $\mathcal{E}$ over $G_n/P_{(n_1,n_2)}$ such that $\chi_1 \times \chi_2 = \mathcal{S}(G_n/P_{(n_1,n_2)}, \mathcal{E})$ and the action of $P_{(n_1,n_2)}$ on the fiber is given by trivial extension of $\chi_1 \otimes \chi_2$, twisted by a power of the determinant.

Let $w_n \in G_n$ denote the longest Weyl group element. Let $x_1, x_2 \in X$ be the classes of 1 and w_n in correspondence and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be their P_n-orbits. Note that $X = \mathcal{O}_1 \cup \mathcal{O}_2$. Thus, by Corollary 2.4.4, we have a short exact sequence

$$0 \to \mathcal{S}(\mathcal{O}_2, \mathcal{E}) \to \mathcal{S}(X, \mathcal{E}) \to \mathcal{S}_X(\mathcal{O}_1, \mathcal{E}) \to 0.$$

By Lemma 3.1.6, $\mathcal{S}(O_2, \mathcal{E})$ and $\mathcal{S}_X(O_1, \mathcal{E})$ are good. By Theorem 3.1.5, $\mathcal{S}(\mathcal{O}_2, \mathcal{E})$ and $\mathcal{S}_X(\mathcal{O}_1, \mathcal{E})$ are acyclic with respect to Φ and thus we have a short exact sequence

$$0 \to \Phi(\mathcal{S}(\mathcal{O}_2, \mathcal{E})) \to \Phi(\mathcal{S}(X, \mathcal{E})) \to \Phi(\mathcal{S}_X(\mathcal{O}_1, \mathcal{E})) \to 0.$$

The proposition follows from the following 2 statements:

(1) $\Phi(\mathcal{S}_X(\mathcal{O}_1, \mathcal{E})) = 0$
(2) $\Phi(\mathcal{S}(\mathcal{O}_2, \mathcal{E})) = (\pi_1)|_{G_{n_1-1}} \times (\pi_2)|_{G_{n_2-1}}$.

Let us first show (1). Using the results of §3 and a version of the Borel lemma (Lemma 5.5.17), we reduce to the statement $\Phi(\mathcal{S}(\mathcal{O}_1, \mathcal{E}')) = 0$ for any P_n-multiplicative equivariant bundle $\mathcal{E}'$. Let $p : \mathcal{O}_1 \to V_n\backslash\mathcal{O}_1$ be the natural projection. Using the nilpotency of the action of V_n on $\mathcal{E}'$ we reduce to the case $\mathcal{E}' = p^?(\mathcal{E}'')$. This case follows by the key lemma (Lemma 3.1.1) from the inclusion $P_{x_1} = P_n \cap P_{(n_1,n_2)} \supset V_n$.

The proof of (2) is a computation based on the key lemma (Lemma 3.1.1).

4.2. **Structure of the proof.** Let $\lambda = (n_1, \dots, n_k)$ be a composition of n. The monomial representation is isomorphic to $\mathcal{S}(G_n/P_\lambda, \mathcal{E})$ for some G_n-multiplicative line bundle $\mathcal{E}$ over $X := G_n/P_\lambda$. Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ be the P_n-orbits on G_n/P_λ, where $\mathcal{O}_1$ is the open orbit. We reduce the problem to the following two tasks:

- to compute $E^k(\mathcal{S}(\mathcal{O}_1, \mathcal{E}))$
- to show that $E^k(\mathcal{S}_X(\mathcal{O}_i, \mathcal{E}'))$ for any P_n-multiplicative equivariant bundle $\mathcal{E}'$ on X any $2 \le i \le k$.

We do both tasks by induction using the key lemma (Lemma 3.1.1) and the fact that the P_{n-1}-orbits on the geometric quotient $V_n\backslash\mathcal{O}_i$ looks exactly like $\mathcal{O}_j$ when n is replaced by $n-1$.

In §§4.3 we study the geometry of P_n-orbits on X. In §§4.4 we reformulate the above tasks in explicit lemmas, that will be proven in §§5.8. In §§4.5 we prove Theorem B using those lemmas.

4.3. **Geometry of P_n-orbits on flag varieties.** Let $\lambda = (n_1, \dots, n_k)$ be a composition of n. In this subsection we describe the orbits of P_n on G_n/P_λ. Note that the scalar matrices act trivially on G/P_λ and thus P_n orbits coincide with $P_{n-1,1}$-orbits. By Bruhat theory, $P_{(n-1,1)}\backslash G_n/P_\lambda$ can be identified with the set of double cosets of the symmetric group permuting the set $\{1, \dots, n\}$ by subgroups corresponding to $\mathcal{P}_{(n-1,1)}$ and P_λ respectively. The first subgroup is the stabilizer of the point n, and the second is the subgroup of all permutations that preserve the segment $[n_{i-1}+1, n_i]$ for each i. Thus, the double quotient consists of k elements. Now we want to find a representative in G_n for each double coset. We can choose all of them to be powers of the standard cyclic permutation matrix. This discussion is formalized in the following notation and lemma.

Notation 4.3.1.

- *Let $c \in G_n$ denote the standard cyclic permutation matrix, i.e. $ce_i = e_{i+1}$ for $i < n$ and $ce_n = e_1$, where e_i are basis vectors.*
- $m^i_\lambda := \sum_{l=1}^{i} n_l$. *In particular, $m^k_\lambda = n$.*
- *For $1 \le i \le k$, let $w^i_\lambda := c^{n-m^{k-i+1}_\lambda} \in G_n$*
- $P^i_\lambda := w^i_\lambda P_\lambda (w^i_\lambda)^{-1}$.
- *Let x^i_λ be the class of w^i_λ in G_n/P_λ and $\mathcal{O}^i_\lambda := P_n x^i_\lambda$.*
- $Q^i_\lambda := P^i_\lambda \cap P_n$.

See Appendix B for examples of the objects described in this and the following notations.

Lemma 4.3.2.
(i) The stabilizer of x^i_λ is Q^i_λ.
(ii) $\mathcal{O}^i_\lambda$ are all distinct.
(iii) $G_n/P_\lambda = \bigcup_{i=1}^k \mathcal{O}^i_\lambda$
(iv) For any $l \leq k$, $\bigcup_{i=1}^l \mathcal{O}^i_\lambda$ is closed in G_n/P_λ .

We will also be interested in the representatives of P_n-orbits on G_n/P^i_λ.

Notation 4.3.3. *Let $\lambda = (n_1, \dots, n_k)$ be a composition of n. For $1 \leq i, j \leq k$,*
- *Let $w^{ij}_\lambda := (w^i_\lambda)^{-1} w^j_\lambda \in G_n$*
- *Let x^{ij}_λ be the class of w^{ij}_λ in G_n/P^i_λ and $\mathcal{O}^{ij}_\lambda := P_n x^{ij}_\lambda$.*

Corollary 4.3.4. *Fix $i \leq k$. Then*
(i) The stabilizer of x^{ij}_λ is Q^j_λ.
(ii) $\mathcal{O}^{ij}_\lambda$ are all distinct.
(iii) $G_n/P^i_\lambda = \bigcup_{j=1}^k \mathcal{O}^{ij}_\lambda$
(iv) For any $l \leq k$, $\bigcup_{j=1}^l \mathcal{O}^{ij}_\lambda$ is closed in G_n/P^i_λ.

$\{x^{ij}_\lambda\}_{j=1}^k$ is a full system of representatives for P_n-orbits in G_n/P^i_λ, and the stabilizer of x^{ij}_λ is Q^j_λ.

Notation 4.3.5. *For any $1 \leq i \leq k$, denote $\lambda^-_i := (n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_k)$.*

The following lemma is a straightforward computation (see Appendix B for a particular example.).

Lemma 4.3.6. *Let $\lambda = (n_1, \dots, n_k)$ be a composition of n. Let $X := P/Q^i_\lambda$ and $Z := (V_n)\backslash X$ be the geometric quotient. Note that $Z = G_{n-1}/P^i_{\lambda'}$, where $\lambda' = \lambda^-_{k-i+1}$.*

Let $L = Q^i_\lambda \cap V_n$ and $\widetilde{Z}_0 := \{g \in G_{n-1} : \psi(gL) = 1\}$. Note that $\widetilde{Z}_0$ is right-invariant with respect to $P^i_{\lambda'}$. Let $Z_0 := \widetilde{Z}_0/P^i_{\lambda'} \subset Z$.

Then $x^{ij}_{\lambda'} \in Z_0$ if and only if $1 \leq j < i$.

4.4. **Derivatives of quasi-regular representations on P_n-orbits on flag varieties.** The proof of Theorem B is based on two lemmas that we formulate in this subsection and prove in §§5.8.

Lemma 4.4.1. *Let $\lambda = (n_1, \dots, n_k)$ be a composition of n. Let Y be a Nash P_n-manifold and $\mathcal{E}$ be a P_n-multiplicative bundle on Y. Let $x_0 \in Y$ be a point with stabilizer Q^i_λ and let $X := P_n x_0$. Let $\pi := \mathcal{S}_Y(X, \mathcal{E})$. Then $E^{i+1}(\pi) = 0$.*

Notation 4.4.2.
- *Denote by $\mathfrak{X}$ the set of all characters of $F^\times$.*
- *Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathfrak{X}^k$. Denote by $\overline{\xi}_{\lambda,\alpha}$ the character of L_λ defined by*

$$\overline{\xi}_{\lambda,\alpha}(g_1, \dots, g_k) = \prod_{i=1}^k (\alpha_i(\det(g_i)).$$

and by $\xi_{\lambda,\alpha}$ the character of L_λ defined by

$$\xi_{\lambda,\alpha}(g_1, \dots, g_k) = \overline{\xi}_{\lambda,\alpha} \Delta^{1/2}_{P_\lambda}|_{L_\lambda} = \prod_{i=1}^k (\alpha_i(\det(g_i))|\det(g_i)|^{(n-m^i_\lambda - m^{i-1}_\lambda)/2}).$$

- *Let $\chi_{\lambda,\alpha}$ and $\overline{\chi}_{\lambda,\alpha}$ be the extensions of $\xi_{\lambda,\alpha}$ and $\overline{\xi}_{\lambda,\alpha}$ to P_λ.*
- *Denote by $\chi^i_{\lambda,\alpha}$ and $\overline{\chi}^i_{\lambda,\alpha}$ the characters of P^i_λ defined by $\chi^i_{\lambda,\alpha}(g) = \chi_{\lambda,\alpha}((w^i_\lambda)^{-1} g w^i_\lambda)$ and $\overline{\chi}^i_{\lambda,\alpha}(g) = \overline{\chi}_{\lambda,\alpha}((w^i_\lambda)^{-1} g w^i_\lambda)$.*

Lemma 4.4.3. *Let $\lambda = (n_1, \dots, n_k)$ be a composition of n. Let $X := P_n/Q^i_\lambda$. Let $\mathcal{E}$ be a P_n-multiplicative line bundle on X. Let $\pi := \mathcal{S}(X, \mathcal{E})$. Let $x_0 \in X$ be the class of the trivial element of P_n. Consider $\mathcal{E}|_{x_0}$ as a character of Q^i_λ. Suppose that this character is the restriction of $\chi^i_{\lambda,\alpha}$ to Q^i_λ, for some $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathfrak{X}^k$. Then*

$$E^i(\pi)|_{G_{n-i}} = \chi_{(n_1),(\alpha_1|\det|^{-1/2})} \times \dots \times \chi_{(n_{k-i}),(\alpha_{k-i}|\det|^{-1/2})} \times \chi_{(n_{k-i+1}-1),(\alpha_{k-i+1})} \times \dots \times \chi_{(n_k-1),(\alpha_k)}$$

4.5. **Proof of Theorem B.**

Lemma 4.5.1. *Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathfrak{X}^k$. There exists a G_n-multiplicative line bundle $\mathcal{E}$ over G/P_λ such that $\chi_{(n_1),(\alpha_1)} \times \dots \times \chi_{(n_k),(\alpha_k)} = \mathcal{S}(G/P_\lambda, \mathcal{E})$ and the action of P_λ on the fiber $\mathcal{E}_{x^1_\lambda}$ is given by the character $\chi_{\lambda,\alpha}$.*

This lemma follows from Lemma 3.1.7.

Proof of Theorem B. Let $X := G_n/P_\lambda$ and let $\mathcal{E}$ be the G_n-multiplicative equivariant bundle given by Lemma 4.5.1. We have to show that $E^k(\mathcal{S}(X,\mathcal{E})) = \chi_{(n_1-1),(\alpha_1)} \times \dots \times \chi_{(n_k-1),(\alpha_k)}$. Recall that by Lemma 4.3.2, $G_n/P_\lambda = \bigcup_{i=1}^k \mathcal{O}^i_\lambda$. Thus, by Corollary 2.4.4 there is a finite filtration on $\mathcal{S}(X,\mathcal{E})$ such that $Gr^i(\mathcal{S}(X,\mathcal{E})) = \mathcal{S}_X(\mathcal{O}^i_\lambda, \mathcal{E})$. By Lemma 3.1.6, each $\mathcal{S}_X(\mathcal{O}^i_\lambda,\mathcal{E})$ is a good representation of $\mathfrak{p}_n$ and thus, by Theorem 3.1.5, $\mathcal{S}_X(\mathcal{O}^i_\lambda,\mathcal{E})$ is E^k-acyclic. Thus, $E^k(\mathcal{S}(X,\mathcal{E}))$ has a filtration such that $Gr^i(E^k(\mathcal{S}(X,\mathcal{E}))) = E^k(\mathcal{S}_X(\mathcal{O}^i_\lambda,\mathcal{E}))$. By Lemma 4.4.1, $E^k(\mathcal{S}_X(\mathcal{O}^i_\lambda,\mathcal{E})) = 0$ for all $i < k$. By Lemma 4.4.3,

$$E^k(\mathcal{S}(\mathcal{O}^i_\lambda,\mathcal{E}))|_{G_{n-k}} = \chi_{(n_1-1),(\alpha_1)} \times \dots \times \chi_{(n_k-1),(\alpha_k)}$$

Thus,

$$E^k(\mathcal{S}(X,\mathcal{E}))|_{G_{n-k}} = E^k(\mathcal{S}(\mathcal{O}^k_\lambda,\mathcal{E}))|_{G_{n-k}} = \chi_{(n_1-1),(\alpha_1)} \times \dots \times \chi_{(n_k-1),(\alpha_k)}$$

□

Remark 4.5.2. *Note that this proof gives a description of $E^l(\chi_1 \times \dots \times \chi_k)$ for an arbitrary l. More precisely, we get a filtration on this space and a description of each quotient. However, this is an infinite filtration, indexed by a rather complicated ordered set.*

In fact, we have a recipe to compute $\Phi(\mathcal{S}(P/Q^i_\lambda,\mathcal{E}))$ for any multiplicative bundle $\mathcal{E}$. First, we use Lemma 5.6.1 to reduce to the case when $\mathcal{E}$ is the pullback of a bundle $\mathcal{E}'$ on the geometric quotient $P_n/Q^i_\lambda V_n$. Then the key lemma (Lemma 3.1.1) says $\Phi(\mathcal{S}(P/Q^i_\lambda,\mathcal{E})) = \mathcal{S}(X'_0,\mathcal{E}')$, where $X'_0 \subset P_n/Q^i_\lambda V_n$. The set X'_0 is described in Lemma 4.3.6. This gives us a filtration on $\mathcal{S}(X'_0,\mathcal{E}')$ such that the associated graded pieces are of the type $\mathcal{S}(P_{n-1}/Q^j_{\lambda^-_i}, \mathcal{E}_{jk})$ where $1 \le j < i$ and $k \ge 0$.

Since the orbits of P_n on X/P_λ are P_n/Q^i_λ, we can use this recipe in order to describe the value of the functor Φ on products of finite-dimensional representations and then, proceeding inductively, we can describe the functor E^i on such representations.

5. Proofs of some technical lemmas

5.1. **Preliminaries on homological algebra.**

Definition 5.1.1. *Let $\mathcal{C}$ be an abelian category. A family of objects $\mathcal{A} \subset Ob(\mathcal{C})$ is called a **generating** family if for any object $X \in Ob(\mathcal{C})$ there exists an object $Y \in \mathcal{A}$ and an epimorphism $Y \twoheadrightarrow X$.*

Definition 5.1.2 ([GM88], III.6.3)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a right-exact additive functor. A family of objects $\mathcal{A} \subset Ob(\mathcal{C})$ is called **$\mathcal{F}$-adapted** if it is generating, closed under direct sums and for any acyclic complex $\dots \to A_3 \to A_2 \to A_1 \to 0$ with $A_i \in \mathcal{A}$, the complex $\dots \to \mathcal{F}(A_3) \to \mathcal{F}(A_2) \to \mathcal{F}(A_1) \to 0$ is also acyclic.*

For example, a generating, closed under direct sums system consisting of projective objects is $\mathcal{F}$-adapted for any right-exact functor $\mathcal{F}$. For a Lie algebra, the system of free $\mathfrak{g}$-modules (i.e. direct sums of copies of $U(\mathfrak{g})$) is an example of such system.

The following results are well-known.

Theorem 5.1.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a right-exact additive functor. Suppose that there exists an $\mathcal{F}$-adapted family $\mathcal{A} \subset Ob(\mathcal{C})$. Then $\mathcal{F}$ has derived functors.*

Remark 5.1.4. *Note that if the functor $\mathcal{F}$ has derived functors, and $\mathcal{A}$ is a generating class of acyclic objects then $\mathcal{A}$ is $\mathcal{F}$-adapted.*

Lemma 5.1.5. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories. Let $\mathcal{F}:\mathcal{A}\to\mathcal{B}$ and $\mathcal{G}:\mathcal{B}\to\mathcal{C}$ be right-exact additive functors. Suppose that both $\mathcal{F}$ and $\mathcal{G}$ have derived functors.*

(i) Suppose that $\mathcal{F}$ is exact. Suppose also that there exists a class $\mathcal{E}\subset Ob(\mathcal{A})$ which is $\mathcal{G}\circ\mathcal{F}$-adapted and such that $\mathcal{F}(X)$ is $\mathcal{G}$-acyclic for any $X\in\mathcal{E}$. Then the functors $L^i(\mathcal{G}\circ\mathcal{F})$ and $L^i\mathcal{G}\circ\mathcal{F}$ are isomorphic.

(ii) Suppose that there exists a class $\mathcal{E}\subset Ob(\mathcal{A})$ which is $\mathcal{G}\circ\mathcal{F}$-adapted and $\mathcal{F}$-adapted and such that $\mathcal{F}(X)$ is $\mathcal{G}$-acyclic for any $X\in\mathcal{E}$. Let $Y\in\mathcal{A}$ be an $\mathcal{F}$-acyclic object. Then $L^i(\mathcal{G}\circ\mathcal{F})(Y)$ is (naturally) isomorphic to $L^i\mathcal{G}(\mathcal{F}(Y))$.

(iii) Suppose that $\mathcal{G}$ is exact. Suppose that there exists a class $\mathcal{E}\subset Ob(\mathcal{A})$ which is $\mathcal{G}\circ\mathcal{F}$-adapted and $\mathcal{F}$-adapted. Then the functors $L^i(\mathcal{G}\circ\mathcal{F})$ and $\mathcal{G}\circ L^i\mathcal{F}$ are isomorphic.

5.1.1. *Lie algebra homology.* Let $\mathfrak{g}$ be a Lie algebra and $\pi\in\mathcal{M}(\mathfrak{g})$. The homology of $\mathfrak{g}$ with coefficients in π are defined by $\mathrm{H}_i(\mathfrak{g},\pi):=L^i\mathcal{C}(\pi)$, where $\mathcal{C}:\mathcal{M}(\mathfrak{g})\to Vect$ is the functor of coinvariants and $L^i\mathcal{C}$ denotes the derived functor. The homology $\mathrm{H}_i(\mathfrak{g},\pi)$ is equal to the i-th homology of the Koszul complex.

$$0\leftarrow\pi\leftarrow\mathfrak{g}\otimes\pi\leftarrow\cdots\leftarrow\Lambda^n(\mathfrak{g})\otimes\pi\leftarrow 0. \tag{1}$$

The following lemmas are standard and can be easily deduced from Lemma 5.1.5

Lemma 5.1.6. *Let $\mathfrak{h}\subset\mathfrak{g}_1\subset\mathfrak{g}$ be Lie algebras. Let ψ be a character of $\mathfrak{h}$. Suppose that $\mathfrak{g}_1$ stabilizes $(\mathfrak{h},\psi)$. Let $\mathcal{F}:\mathcal{M}(\mathfrak{g})\to\mathcal{M}(\mathfrak{g}_1)$ be the functor defined by $\mathcal{F}(\pi)=\pi_{\mathfrak{h},\psi}$. Then $L^i(\mathcal{F})(\pi)=\mathrm{H}_i(\mathfrak{h},\pi\otimes(-\psi))$.*

Lemma 5.1.7. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ be its Lie ideal. Let π be a representation of $\mathfrak{g}$. Suppose that π is $\mathfrak{i}$-acyclic and $\mathrm{H}_0(\mathfrak{i},\pi)$ is $\mathfrak{g}/\mathfrak{i}$-acyclic. Then π is $\mathfrak{g}$-acyclic.*

The last lemma is in fact a special case of the Hochschild-Serre spectral sequence.

5.2. **Acyclicity with respect to composition of derivatives(Proof of Lemma 3.0.3).** By Lemma 5.1.5(ii), it is enough to show that there exists a class of representations of $\mathfrak{p}_n$ that is $\widetilde{\Phi}$-adapted, $\widetilde{E}^{k+1}$-adapted, and such that for every object τ in the class, $\widetilde{\Phi}(\tau)$ is $\widetilde{E}^k$-acyclic. We claim that the class of free $U(\mathfrak{p}_n)$-modules satisfies this conditions.

For this it is enough to show that $U(\mathfrak{p}_n)$ is $\widetilde{E}^{k+1}$ and $\widetilde{\Phi}$ acyclic and $\widetilde{\Phi}(U(\mathfrak{p}_n))$ is $\widetilde{E}^k$ acyclic. By Lemma 5.1.6 it is equivalent to showing that

$$\mathrm{H}_i(\mathfrak{u}_n^{k+1},U(\mathfrak{p}_n)\otimes(-\psi_n^{k+1}))=\mathrm{H}_i(\mathfrak{v}_n,U(\mathfrak{p}_n)\otimes(-\psi_n))=\mathrm{H}_i(\mathfrak{u}_{n-1}^k,U(\mathfrak{p}_n)_{\mathfrak{v}_n,\psi_n}\otimes(-\psi_{n-1}^k))=0,$$

for any $i>0$. This follows from the fact that $U(\mathfrak{p}_n)\otimes(-\psi_n^{k+1})$ is free $\mathfrak{u}_n^{k+1}$ module, $U(\mathfrak{p}_n)\otimes(-\psi_n)$ is free $\mathfrak{v}_n$- module and $U(\mathfrak{p}_n)_{\mathfrak{v}_n,\psi_n}\otimes(-\psi_{n-1}^k)$ is free $\mathfrak{u}_{n-1}^k$-module. This is immediate by the PBW theorem.

5.3. **Preliminaries on topological linear spaces.** We will need some classical facts from the theory of nuclear Fréchet spaces. A good exposition on nuclear Fréchet spaces can be found in [CHM00, Appendix A].

Definition 5.3.1. *We call a complex of topological vector spaces **admissible** if all its differentials have closed images.*

Proposition 5.3.2 (see e.g. [CHM00], Appendix A)**.**

(1) Let V be a nuclear Fréchet space and W be a closed subspace. Then both W and V/W are nuclear Fréchet spaces.

(2) Let
$$\mathcal{C}:0\to C_1\to\cdots\to C_n\to 0$$
be an admissible complex of nuclear Fréchet spaces. Then $H^i(\mathcal{C})$ are nuclear Fréchet spaces.

(3) Let
$$\mathcal{C}:0\to C_1\to\cdots\to C_n\to 0$$
be an admissible complex of nuclear Fréchet spaces. Then the complex $\mathcal{C}^*$ is also admissible and $H_i(\mathcal{C}^*)\cong H^i(\mathcal{C})^*$.

Corollary 5.3.3. *Let $\mathfrak{g}$ be a (finite dimensional) Lie algebra and π be its nuclear Fréchet representation (i.e. a representation in a nuclear Fréchet space such that the action map $\mathfrak{g}\otimes\pi\to\pi$ is continuous). Suppose that $\mathrm{H}_i(\mathfrak{g},\pi)$ are Hausdorff for all $i\geq 0$. Then $\mathrm{H}_i(\mathfrak{g},\pi)$ are nuclear Fréchet and $H^i(\mathfrak{g},\pi^*)=\mathrm{H}_i(\mathfrak{g},\pi)^*$.*

Proposition 5.3.4 (see e.g. [CHM00], Appendix A)**.**
Let $0\to V\to W\to U\to 0$ be an exact sequence of nuclear Fréchet spaces. Suppose that the embedding $V\to W$ is closed. Let L be a nuclear Fréchet space. Then the sequence $0\to V\widehat{\otimes}L\to W\widehat{\otimes}L\to U\widehat{\otimes}L\to 0$ is exact and the embedding $V\widehat{\otimes}L\to W\widehat{\otimes}L$ is closed.

5.4. **Co-invariants of good extensions (Proof of Lemma 3.1.9).** Let us first prove the following special case.

Lemma 5.4.1. *Let $0\to L\to M\to N\to 0$ be a short exact sequence of nuclear Fréchet representations of a Lie algebra $\mathfrak{g}$. Suppose that L and N are acyclic and that $\mathrm{H}_0(\mathfrak{g},L)$ and $\mathrm{H}_0(\mathfrak{g},N)$ are Hausdorff.*

Then M is acyclic, $\mathrm{H}_0(\mathfrak{g},M)$ is Hausdorff, and $0\to\mathrm{H}_0(\mathfrak{g},L)\to\mathrm{H}_0(\mathfrak{g},M)\to\mathrm{H}_0(\mathfrak{g},N)\to 0$ is a short exact sequence.

Proof. The long exact sequence implies that M is acyclic and $0\to\mathrm{H}_0(\mathfrak{g},L)\to\mathrm{H}_0(\mathfrak{g},M)\to\mathrm{H}_0(\mathfrak{g},N)\to 0$ is a short exact sequence.

By Proposition 5.3.2(3) $0\to N^*\to M^*\to L^*$ is exact, by Corollary 5.3.3 L^* is $\mathfrak{g}$-coacyclic (i.e. $\mathrm{H}^i(\mathfrak{g},L^*)=0$ for $i>0$) and hence $0\to(N^*)^{\mathfrak{g}}\to(M^*)^{\mathfrak{g}}\to(L^*)^{\mathfrak{g}}\to 0$ is exact. We have to show that $\mathfrak{g}M$ is closed in M. For that it is enough to show that for any $x\in M$, if for any $f\in(M^*)^{\mathfrak{g}}$ we have $f(x)=0$ then $x\in\mathfrak{g}M$. Let y be the image of x in N. We know that any $f\in(N^*)^{\mathfrak{g}}$ vanishes on y, and hence $y\in\mathfrak{g}N$. Hence $\exists x'\in\mathfrak{g}M$ such that $x-x'\in L$. Since the map $(M^*)^{\mathfrak{g}}\to(L^*)^{\mathfrak{g}}$ is onto, every element in $(L^*)^{\mathfrak{g}}$ vanishes on $x-x'$. Thus $x-x'\in\mathfrak{g}L$ and hence $x\in\mathfrak{g}M$. □

We will also need the following lemma.

Lemma 5.4.2. *Let $C_1\to\cdots\to C_k$ be a complex of linear spaces. Suppose that each C_j is equipped with a descending filtration $F^i(C_j)$ such that $F^0(C_j)=C_j$, $\bigcap F^i(C_j)=0$ and $\lim_{\leftarrow}C_j/F^i(C_j)\cong C_j$, and $d_j(F^i(C_j))\subset F^i(C_{j+1})$. Suppose that $F^i(C_1)/F^{i+1}(C_1)\to\cdots\to F^i(C_k)/F^{i+1}(C_k)$ is exact. Then $C_1\to\cdots\to C_k$ is also exact.*

Proof. Let $m\in\operatorname{Ker}d_j\subset C_j$. Let $m_j:=[m]\in C_j/F^i(C_j)$. We want to construct $n\in C_{j-1}$ such that $d_{j-1}(n)=m$. Since $C_{j-1}\cong\lim_{\leftarrow}C_{j-1}/F^i(C_{j-1})$ for that it is enough to construct a compatible system of representatives $n_i\in C_{j-1}/F^i(C_{j-1})$. We build this system by induction. Suppose we constructed n_i and left n_i' be an arbitrary lift of n_i to $C_{j-1}/F^{i+1}(C_{j-1})$. Consider $\varepsilon:=m_{i+1}-d(n_i')$. Then $e\in F^i(C_j)/F^{i+1}(C_j)$. Moreover, $d(\varepsilon)=0$. Hence there exists $\delta\in F^i(C_{j-1})/F^{i+1}(C_{j-1})$ such that $d(\delta)=\varepsilon$. Define $n_{i+1}:=n_i'+\delta$. □

Proof of Lemma 3.1.9. Lemma 5.4.1 implies by induction that $\pi/F^i(\pi)$ is acyclic and $A_i:=\mathrm{H}_0(\mathfrak{g},\pi/F^i(\pi))$ is Hausdorff, and hence is a nuclear Fréchet space. Lemma 5.4.1 also implies that $A_i\to A_{i+1}$ is onto. Define $A_\infty:=\lim_{\leftarrow}A_i$ (as a linear, non-topological, representation). Consider the Koszul complex of π extended by A_∞:

$$0\leftarrow A_\infty\leftarrow\pi\leftarrow\mathfrak{g}\otimes\pi\leftarrow\cdots\leftarrow\Lambda^n(\mathfrak{g})\otimes\pi\leftarrow 0. \tag{2}$$

Lemma 5.4.2 implies that the sequence (2) is exact. Hence π is acyclic and the natural map $\mathrm{H}_0(\mathfrak{g},\pi)\to A_\infty$ is an isomorphism (of vector spaces).

Let $p_i:\pi\to\pi/F^i(\pi)$ denote the natural projection. The exactness of (2) implies

$$\mathfrak{g}\pi=\bigcap_i p_i^{-1}(\mathfrak{g}(\pi/F^i(\pi)))$$

which in turn implies that $\mathfrak{g}\pi$ is closed and thus $\mathrm{H}_0(\mathfrak{g},\pi)$ is a nuclear Fréchet space.

Consider the short exact sequence $0\to F^i(\pi)\to\pi\to\pi/(F^i(\pi))\to 0$. We showed that it consists of acyclic objects and that $\mathrm{H}_0(\mathfrak{g},F^i(\pi))$ and $\mathrm{H}_0(\mathfrak{g},\pi/(F^i(\pi)))$ are Hausdorff. By Lemma 5.4.1 this implies

that $0 \to \mathrm{H}_0(\mathfrak{g}, F^i(\pi)) \to \mathrm{H}_0(\mathfrak{g}, \pi) \to \mathrm{H}_0(\mathfrak{g}, \pi/(F^i(\pi))) \to 0$ is a short exact sequence of nuclear Fréchet spaces and hence the image of $\mathrm{H}_0(\mathfrak{g}, F^i(\pi)) \hookrightarrow \mathrm{H}_0(\mathfrak{g}, \pi)$ is closed.

Note that $F^i(\mathrm{H}_0(\mathfrak{g}, \pi)) = \mathrm{H}_0(\mathfrak{g}, F^i(\pi))$. To sum up, $\mathrm{H}_0(\mathfrak{g}, \pi)$ is a nuclear Fréchet space, $F^i(\mathrm{H}_0(\mathfrak{g}, \pi))$ are closed, the natural morphisms $F^i(\mathrm{H}_0(\mathfrak{g}, \pi)) \to \mathrm{H}_0(\mathfrak{g}, \pi_i)$ are isomorphisms and $\mathrm{H}_0(\mathfrak{g}, \pi) \cong A_\infty \cong \varprojlim \mathrm{H}_0(\mathfrak{g}, \pi)/(F^i(\mathrm{H}_0(\mathfrak{g}, \pi)))$. □

5.5. **More on Schwartz functions on Nash manifolds.** In this subsection we will recall some properties of Nash manifolds and Schwartz functions over them. We work in the notation of [AG08], where one can read about Nash manifolds and Schwartz distributions over them. More detailed references on Nash manifolds are [BCR98] and [Shi87].

Nash manifolds are equipped with the **restricted topology**, in which open sets are open semi-algebraic sets. This is not a topology in the usual sense of the word as infinite unions of open sets are not necessarily open sets in the restricted topology. However, finite unions of open sets are open and therefore in the restricted topology we consider only finite covers. In particular, if $\mathcal{E}$ over X is a Nash vector bundle it means that there exists a finite open cover U_i of X such that $\mathcal{E}|_{U_i}$ is trivial.

Fix a Nash manifold X and a Nash bundle $\mathcal{E}$ over X.

An important property of Nash manifolds is

Theorem 5.5.1 (Local triviality of Nash manifolds; [Shi87], Theorem I.5.12)**.** *Any Nash manifold can be covered by a finite number of open submanifolds Nash diffeomorphic to $\mathbb{R}^n$.*

Together with [AG08, Corollary 3.6.3] this implies

Theorem 5.5.2. *Let $Y \subset X$ be a closed Nash submanifold. Then there exist open sunsets $\{U_i\}_{i=1}^n \subset X$ and Nash diffeomorphisms $\phi_i : \mathbb{R}^n \xrightarrow{\sim} U_i$ such that $Y \subset \cup_{i=1}^n U_i$ and $\phi_i^{-1}(U_i \cap Y)$ is a linear subspace of $\mathbb{R}^n$.*

Theorem 5.5.3 ([AG10], Theorem 2.4.16)**.** *Let $s : X \to Y$ be a surjective submersive Nash map. Then locally it has a Nash section, i.e. there exists a finite open cover $Y = \bigcup_{i=1}^{k} U_i$ such that s has a Nash section on each U_i.*

Corollary 5.5.4. *An étale map $\phi : X \to Y$ of Nash manifolds is locally an isomorphism. That means that there exists a finite cover $X = \bigcup U_i$ such that $\phi|_{U_i}$ is an isomorphism onto its open image.*

Corollary 5.5.5 ([AG09], Theorem B.2.3)**.** *Let $p : X \to Y$ be a Nash submersion of Nash manifolds. Then there exist a finite open (semi-algebraic) cover $X = \bigcup U_i$ and isomorphisms $\phi_i : U_i \cong W_i$ and $\psi_i : p(U_i) \cong \mathcal{O}_i$ where $W_i \subset \mathbb{R}^{d_i}$ and $\mathcal{O}_i \subset \mathbb{R}^{k_i}$ are open (semi-algebraic) subsets, $k_i \leq d_i$ and $p|_{U_i}$ correspond to the standard projections.*

Notation 5.5.6. *Let $U \subset X$ be an open (Nash) subset. We denote by $\mathrm{Nash}(U, \mathcal{E})$ the space of Nash sections of $\mathcal{E}$ on U.*

Definition 5.5.7. *We call a Nash action of a Nash group G on X **strictly simple** if it is simple (i.e. all stabilizers are trivial) and there exists a geometric (separated) Nash quotient $G\backslash X$ (see* [AG10, Definition 4.0.4]*).*

Proposition 5.5.8 ([AG10], Proposition 4.0.14)**.** *Let $H < G < G_n$ be Nash groups. Then the action of H on G is strictly simple.*

Corollary 5.5.9. *Let $H < G < G_n$ be Nash groups. Let X be a transitive Nash G-manifold. Then there exists a geometric quotient $H\backslash X$.*

We will use the following theorem that describes the basic properties of Schwartz functions on Nash manifolds.

Theorem 5.5.10.

(i) $\mathcal{S}(\mathbb{R}^n)$ = Classical Schwartz functions on $\mathbb{R}^n$.

(ii) The space $\mathcal{S}(X, \mathcal{E})$ is a nuclear Fréchet space.

(iii) *Let $Z \subset X$ be a closed Nash submanifold. Then the restriction maps $\mathcal{S}(X,\mathcal{E})$ onto $\mathcal{S}(Z,\mathcal{E}|_Z)$.*

(iv) *Let $X = \bigcup U_i$ be a finite open cover of X. Then a global section f of $\mathcal{E}$ on X is a Schwartz section if and only if it can be written as $f = \sum_{i=1}^n f_i$ where $f_i \in \mathcal{S}(U_i,\mathcal{E})$ (extended by zero to X).*

Moreover, there exists a tempered partition of unity $1 = \sum_{i=1}^n \lambda_i$ such that for any Schwartz section $f \in \mathcal{S}(X,\mathcal{E})$ the section $\lambda_i f$ is a Schwartz section of $\mathcal{E}$ on U_i (extended by zero to X).

Note that this property implies that Schwartz sections form a cosheaf in the restricted topology.

(v) *Tempered sections of a Nash bundle $\mathcal{E}$ over a Nash manifold X form a sheaf in the restricted topology. Namely, let U_i be a (finite) open Nash cover of X and let $s \in \Gamma(X,\mathcal{E})$. Then s is a tempered section if and only if $s|_{U_i}$ is a tempered section for any i.*

(vi) *For any Nash manifold Y,*

$$\mathcal{S}(X \times Y) = \mathcal{S}(X)\widehat{\otimes}\mathcal{S}(Y).$$

Part (i) is [AG08, Theorem 4.1.3], part (ii) is [AG10, Corollary 2.6.2], for (iii) see [AG08, §§1.5], for (iv) and (v) see [AG08, §§5], for (vi) see e.g. [AG10, Corollary 2.6.3].

Lemma 5.5.11 ([AOS12], Appendix A)**.** *Suppose*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is an exact sequence of Nash bundles on X. Then

$$0 \to \mathcal{S}(X,\mathcal{E}_1) \to \mathcal{S}(X,\mathcal{E}_2) \to \mathcal{S}(X,\mathcal{E}_3) \to 0.$$

is an exact sequence of Fréchet spaces.

For the proof we will need the following lemma.

Lemma 5.5.12. *Suppose*

$$0 \to \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_2 \xrightarrow{d_2} \mathcal{E}_3 \to 0$$

is a finite exact sequence of Nash bundles on X. Then there exist an open Nash cover $X = \bigcup_1^n U_j$ s.t. the sequence

$$0 \to \mathcal{E}_1|_{U_j} \xrightarrow{d_1} \mathcal{E}_2|_{U_j} \xrightarrow{d_2} \mathcal{E}_3|_{U_j} \to 0$$

is isomorphic to the sequence

$$0 \to U_j \times V_1 \to U_j \times (V_1 \oplus V_2) \to U_j \times V_2 \to 0 \tag{3}$$

Where V_i are finite dimensional vector spaces, $U_i \times V$ denotes the constant bundle with fiber V and the maps in the sequence come from the the standard embedding and projection.

This lemma is essential proven in [BCR98], but for completeness let us prove it here.

Proof. By the definition of a Nash bundle we can find a finite Nash open cover $X = \bigcup_1^n V_j$ s.t. the bundles $\mathcal{E}_i|_{V_i}$ are constant. Thus we may assume that $\mathcal{E}_i$ are constant bundles with fibers W_i. Choose a basis for W_2 and let I be the collection of coordinate subspaces. For any $V \in I$ denote $U_V = \{x \in X | ((d_2)_x)|_V \text{ is an isomorphism}\}$. Clearly U_V form a Nash cover of X. Thus we may assume that $X = U_V$ for some V. in this case d_1 gives an isomorphism between the constant bundle $X \times (W_1 \oplus V)$ and the constant bundle $X \times W_2$. Also d_2 gives an isomorphism between $X \times V$ and $X \times W_3$. Those isomorphisms gives identification of the sequence

$$0 \to X \times W_1 \to X \times W_2 \to X \times W_3 \to 0$$

with

$$0 \to X \times W_1 \to X \times (W_1 \oplus V) \to X \times V \to 0.$$

□

Proof of Lemma 5.5.11.

Step 1 The case when the sequence is as in (3).

It follows immediately from the definition of Schwartz section of a Nash bundle.

Step 2 The general case
Let $X=\bigcup U_j$ be a finite open Nash cover s.t. the sequence

$$0\to\mathcal{E}_1|_{U_j}\overset{d_1}{\to}\mathcal{E}_2|_{U_j}\overset{d_2}{\to}\mathcal{E}_3|_{U_j}\to 0$$

is isomorphic to a sequence

$$0\to U_j\times V_1\to U_j\times(V_1\oplus V_2)\to U_j\times V_2\to 0$$

as in Lemma 5.5.12. By Theorem 5.5.10(iv) we can choose a partition of unity $1_X=\sum e_j$ where $e_j\in C^\infty(X)$ with $\mathrm{Supp}(e_j)\subset U_j$ and such that for any i,j and any $\phi\in\mathcal{S}(X,\mathcal{E}_i)$ we have $e_j\phi\in\mathcal{S}(U_j,E_i)$. Let $\phi\in Ker(d_i)\subset\mathcal{S}(X,\mathcal{E}_i)$ then we have

$$\phi=\sum e_j\phi\in\sum\mathrm{Ker}(d_i|_{\mathcal{S}(U_j,\mathcal{E}_i|_{U_j})})=\sum\mathrm{Im}(d_{i-1}|_{\mathcal{S}(U_j,\mathcal{E}_{i-1}|_{U_j})})\subset\mathrm{Im}(d_{i-1}).$$

□

The following Lemma follows immediately from Corollary 5.5.5 and Theorem 5.5.10(v).

Lemma 5.5.13. *Let $p:Y\to X$ be a Nash surjective submersion of Nash manifolds. Let $s\in\Gamma(X,\mathcal{E})$ be a set-theoretical global section of $\mathcal{E}$. Then s is tempered if and only if p^*s is a tempered section of $p^*\mathcal{E}$.*

Lemma 5.5.14 ([AG09],Theorem B.2.4)**.** *Let $\phi:Y\to X$ be a Nash submersion of Nash manifolds. Then*
(i) there exists a unique continuous linear map $\phi_*:\mathcal{S}(Y,\phi^?(\mathcal{E}))\to\mathcal{S}(X,\mathcal{E})$ such that for any $f\in\mathcal{S}(X,\mathcal{E}^*\otimes D_X)$ and $\mu\in\mathcal{S}(Y,\phi^?(\mathcal{E}))$ we have

$$\int_{x\in X}\langle f(x),\phi_*\mu(x)\rangle=\int_{y\in Y}\langle\phi^*f(y),\mu(y)\rangle.$$

In particular, we mean that both integrals converge. Here, $\phi^?(\mathcal{E})=\phi^*(\mathcal{E})\otimes D_Y^X$, as in Notation 2.4.5.
(ii) If ϕ is surjective then ϕ_* is surjective.

Lemma 5.5.15. *[[AG] ,Lemma B.1.4] Let G be a connected Nash group and $\mathfrak{g}$ be the Lie algebra of G. Let $p:G\times X\to X$ be the projection. Let G act on $\mathcal{S}(G\times X,p^?(\mathcal{E}))$ by acting on the G coordinate. Consider the pushforward $p_*:\mathcal{S}(G\times X,p^?(\mathcal{E}))\to\mathcal{S}(X,\mathcal{E})$. Then $\mathfrak{g}\mathcal{S}(G\times X,p^?(\mathcal{E}))=\mathrm{Ker}(p_*)$.*

Lemma 5.5.16. *Let $f:X\to\mathbb{R}$ be a Nash function such that 0 is a regular value of f. Then $f\mathcal{S}(X)=\{\phi\in\mathcal{S}(X)\,:\,\phi(f^{-1}(0))=0\}$.*

Proof. By Theorem 5.5.2 and partition of unity (Theorem 5.5.10(iv)), we can assume that $X=\mathbb{R}^n$ and $f^{-1}(0)=\mathbb{R}^{n-1}\subset\mathbb{R}^n$. Let $x:\mathbb{R}^n\to\mathbb{R}$ be the last coordinate function. Since 0 is a regular value of f, f/x is a smooth invertible function. Since f is a Nash function, f/x is also a Nash function. Thus, we can assume $f=x$. We have to show that the following sequence is exact.

$$0\to\mathcal{S}(\mathbb{R}^n)\overset{x}{\to}\mathcal{S}(\mathbb{R}^n)\overset{Res}{\to}\mathcal{S}(\mathbb{R}^{n-1})\to 0,$$

where $Res:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^{n-1})$ is the restriction map.

By the Schwartz Kernel Theorem (Proposition vi) this sequence is isomorphic to

$$0\to\mathcal{S}(\mathbb{R}^{n-1})\widehat{\otimes}\mathcal{S}(\mathbb{R})\overset{Id\widehat{\otimes}x}{\to}\mathcal{S}(\mathbb{R}^{n-1})\widehat{\otimes}\mathcal{S}(\mathbb{R})\overset{Id\widehat{\otimes}Res}{\to}\mathcal{S}(\mathbb{R}^{n-1})\to 0,$$

where $Res:\mathcal{S}(\mathbb{R})\to\mathbb{C}$ is evaluation at 0.

By Proposition 5.3.4 it is enough to prove the exactness in the case $n=1$, which is obvious. □

We will use the following version of Borel's lemma.

Lemma 5.5.17. *Let $Z\subset X$ be Nash submanifold.*
Then $\mathcal{S}_X(Z)$ has a canonical countable decreasing filtration by closed subspaces $(\mathcal{S}_X(Z))^i$ satisfying
(1) $\bigcap(\mathcal{S}_X(Z))^i=0$
(2) $gr_i(\mathcal{S}_X(Z,\mathcal{E}))\cong\mathcal{S}(Z,\mathrm{Sym}^i(CN_Z^X)\otimes\mathcal{E})$

(3) the natural map

$$\mathcal{S}_X(Z,\mathcal{E}) \to \lim_{\leftarrow}(\mathcal{S}_X(Z,\mathcal{E}))/\mathcal{S}_X(Z,\mathcal{E}))^i)$$

is an isomorpihsm.

For proof see [AG, Lemmas B.0.8 and B.0.9].

5.6. **Good representations of geometric origin (Proof of Lemma 3.1.6).** For the proof we will fix T, R and R'. We will need the following statements.

Lemma 5.6.1. *Let X be an R-transitive Nash manifold. Let $\mathcal{E}$ be an R-tempered bundle on X. Suppose that for each $x \in X$, the V_n-stabilizer of x acts trivially on the fiber $\mathcal{E}_x$. Note that by Corollary 5.5.9 there exists geometric quotient $V_n\backslash X$. Let $p : X \to V_n\backslash X$ denote the projection. Then*

(1) there exists an R'-tempered bundle $\mathcal{E}'$ on the geometric quotient $V_n\backslash X$ and a (tempered) isomorphism of R-tempered bundles $\mathcal{E} \xrightarrow{\sim} p^*(\mathcal{E}')$.
(2) there exists an R'-tempered bundle $\mathcal{E}''$ on the geometric quotient $V_n\backslash X$ and a (tempered) isomorphism of R-tempered bundles $\mathcal{E} \xrightarrow{\sim} p^?(\mathcal{E}'')$.

Proof. (1) Let $x \in X$ and let R_x be the stabilizer of x. Identify $V_n\backslash X$ with $Y := R'x$. Let $\mathcal{E}' := \mathcal{E}|_Y$.

We will construct a (tempered) isomorphism of R-tempered bundles $\phi : \mathcal{E} \xrightarrow{\sim} p^*(\mathcal{E}')$. In order to do that we have to construct, for any $x \in X$, an element $\phi_x \in \mathrm{Hom}(\mathcal{E}_x, p^*(\mathcal{E}')_x)$. Note that $\mathrm{Hom}(\mathcal{E}_x, p^*(\mathcal{E}')_x) = \mathrm{Hom}(\mathcal{E}_x, \mathcal{E}_{p(x)})$. We know that there exists $g \in V_n$ such that $gx = p(x)$. This g gives an element in $\phi_x \in \mathrm{Hom}(\mathcal{E}_x, p^*(\mathcal{E}')_x)$. The element g is not unique, but the assumptions of the lemma imply that ϕ_x does not depend on the choice of g. Now we have to show that the constructed $\phi \in \Gamma(X, \mathrm{Hom}(\mathcal{E}, p^*(\mathcal{E}')))$ is a tempered section. Consider the action map $a : V_n \times Y \to X$. Clearly, it is a surjective submersion. It is easy to see that $a^*(\phi) \in \Gamma(V_n \times Y, \mathrm{Hom}(a^*\mathcal{E}, a^*p^*(\mathcal{E}')))$ is a tempered section. Thus, Lemma 5.5.13 implies that $\phi \in \Gamma(X, \mathrm{Hom}(\mathcal{E}, p^*(\mathcal{E}')))$ is a tempered section.

(2) By part (1) there exists a bundle $(D_Y^X)'$ on $V_n\backslash X$ such that $p^*((D_Y^X)') \simeq D_Y^X$. We define $\mathcal{E}'' := \mathcal{E}' \otimes ((D_Y^X)')^*$. □

Remark 5.6.2. *An analog of this lemma holds for Nash equivariant bundles, and for multiplicative equivariant bundles, with an analogous proof. In particular, the isomorphism $p^*((D_Y^X)') \simeq D_Y^X$ is an isomorphism of Nash bundles.*

Corollary 5.6.3. *Let X be a transitive R-Nash manifold. Let $\mathcal{E}$ be a R-multiplicative bundle on X (see Definition 2.5.4). Then there exists a filtration F^i of $\mathcal{E}$ by R-multiplicative bundles such that for every i there exists an R'-multiplicative bundle $\mathcal{E}'_i$ on the geometric quotient $V_n\backslash X$ such that $F^i(\mathcal{E})/F^{i-1}(\mathcal{E}) = p^?(\mathcal{E}'_i)$.*

To deduce this corollary we will use the following obvious lemma.

Lemma 5.6.4. *Let π be a multiplicative (finite dimensional) representation of a subgroup L of V_n. Then the Lie algebra of L acts nilpotently on π.*

Proof of Corollary 5.6.3. Let $F^i(\mathcal{E}) \subset \mathcal{E}$ be defined by $F^i(\mathcal{E}) := \{(x,e) \in \mathcal{E} | ((\mathfrak{v}_n)_x)^{\otimes i} e = 0\}$. By the last lemma (Lemma 5.6.4) the filtration $F^i(\mathcal{E})$ is exhaustive. Since $\mathcal{E}$ is a tempered bundle, the action of the Lie algebra $\mathfrak{r}$ is Nash and hence $F^i(\mathcal{E})$ are Nash. Clearly the action $(\mathfrak{v}_n)_x$ on $F^i(\mathcal{E})/F^{i-1}(\mathcal{E})$ is trivial. Thus, by Lemma 5.6.1, we are done. □

Proof of Lemma 3.1.6. The case $Y = X$ and X is R-transitive follows from Corollary 5.6.3 and Lemma 5.5.11. By the Borel lemma (Lemma 5.5.17), this implies the case when X is R-transitive and Y is arbitrary. This in turn implies the general case by Corollary 2.4.4. □

5.7. **Induced representations as sections of multiplicative bundles (Proof of Lemma 3.1.7).** Identify the variety $X := G/H$ with $K/(K \cap H)$. Note that under this identification $\mathcal{E} \cong K \times_{K\cap H} \chi$. This is a Nash bundle.

Let us show that the action of G on $\mathcal{E}$ is multiplicative. The only non-trivial part is that any $\alpha \in \mathfrak{g}_n$ preserve the space of Nash sections of $\mathcal{E}$ on any open (Nash) set. For this we note that

$$\mathrm{Nash}(U, \mathcal{E}) \cong \{f \in C^\infty(p^{-1}(U)) | f|_K \text{ is Nash and } f(gh) = \chi(h)f(g) \text{ for any } h \in H\},$$

where $p : G \to X$ is the quotient map. Under this identification, an element $\alpha \in \mathfrak{g}$ acts on $\mathrm{Nash}(U, \mathcal{E})$ via a Nash vector field β on $p^{-1}(U)$. This field can be interpreted as a Nash map $\beta : p^{-1}(U) \to \mathfrak{g}$. Here we identify the Lie algebra $\mathfrak{g}$ with the tangent space at a point $g \in G$ using the differential of the left translation by g. Restrict β to $K \cap p^{-1}(U)$ and decompose it to a sum of $\beta_1 : K \cap p^{-1}(U) \to \mathfrak{k}$ and $\beta_2 : K \cap p^{-1}(U) \to \mathfrak{h}$, where $\mathfrak{h}$ and $\mathfrak{k}$ denote the Lie algebras of H and K respectively. Let $f \in C^\infty(p^{-1}(U))$ such that $f|_K$ is Nash and $f(gh) = \chi(h)f(g)$ for any $h \in H$. Then $(\beta f)|_K = \beta_1 f|_K + d\chi \circ \beta_2 \cdot f$ which is clearly a Nash function. □

5.8. **Derivatives of quasi-regular representations on P_n-orbits on flag varieties (Proofs of Lemmas 4.4.1 and 4.4.3).**

Proof of Lemma 4.4.1. The proof is by induction on n. In step 1 we reduce to the case when $Y = X = P_n/Q^i_\lambda$. Let Z and Z_0 to be as in Lemma 4.3.6 and $p : X \to Z$ denote the natural projection. In step 2 we reduce to the case $\mathcal{E} = p^?(\mathcal{E}')$, where $\mathcal{E}'$ is a G_{n-1}-multiplicative bundle on Z. In step 3 we prove the lemma for this case.

Step 1 Reduction to the case $Y = X = P_n/Q^i_\lambda$
By Lemma 5.5.17, $\mathcal{S}_Y(X, \mathcal{E})$ is a good extension of $\pi_i := \mathcal{S}(X, \mathcal{E}|_X \otimes Sym^i(CN^Y_X)$. By Lemma 3.1.6, π_i are good. Thus, by Corollary 3.1.10, $E^{i+1}(\pi)$ is a good extension of $E^{i+1}(\pi_i)$. Thus it is enough to show that $E^{i+1}(\pi_i) = 0$.

Step 2 Reduction to the case $\mathcal{E} = p^?(\mathcal{E}')$, where $\mathcal{E}'$ is a G_{n-1}-multiplicative bundle on Z.
By Corollary 5.6.3, we have a finite filtration on $\mathcal{E}$ and G_{n-1}-multiplicative bundles $\mathcal{E}'_i$ on Z such that $p^?(\mathcal{E}_i) \simeq Gr^i(\mathcal{E})$. By Lemma 5.5.11, this gives a filtration on $\mathcal{S}(X, \mathcal{E})$ such that $Gr^i(\mathcal{S}(X, \mathcal{E})) = \mathcal{S}(X, p^?(\mathcal{E}_i))$. As before, this means that it is enough to show that $E^{i+1}(\mathcal{S}(X, p^?(\mathcal{E}_i))) = 0$.

Step 3 Proof for the case $\mathcal{E} = p^?(\mathcal{E}')$.
By the key lemma (Lemma 3.1.1), $\widetilde{\Phi}(\pi) = \mathcal{S}(Z_0, \mathcal{E}')$. Denote $O_j := P_{n-1}x^{ij}_\lambda$ for $1 \le j < i$. By Corollary 4.3.4 and Lemma 4.3.6, $Z_0 = \bigcup_{j=1}^{i-1} \mathcal{O}^{ij}_\lambda$. By Corollary 2.4.4, there is a filtration F^j on $\mathcal{S}(Z_0, \mathcal{E}')$ such that $Gr^j(\mathcal{S}(Z_0, \mathcal{E}')) = \mathcal{S}_{Z_0}(\mathcal{O}^{ij}_\lambda, \mathcal{E}')$. Thus it is enough to show that $E^i(\mathcal{S}_{Z_0}(\mathcal{O}^{ij}_\lambda, \mathcal{E}')) = 0$ for any j. By Corollary 4.3.4, $\mathcal{O}^{ij}_\lambda \cong P_{n-1}/Q^j_{\lambda'}$. Thus, by the induction hypothesis, $E^i(\mathcal{S}_{Z_0}(\mathcal{O}^{ij}_\lambda, \mathcal{E}')) = 0$.

□

For the proof of Lemma 4.4.3, consider the embedding of G_{n-1} to G_n obtained by conjugating the standard embedding by the permutation matrix corresponding to the permutation $(m^i_\lambda, m^i_\lambda + 1, \dots, n)$. Under this embedding, $P_{\lambda^-_i}$ embeds into P_λ. Denote $\beta^i_k := (|\det|^{-1/2}, \dots, |\det|^{-1/2}, 1, |\det|^{1/2}, \dots, |\det|^{1/2}) \in \mathfrak{X}^k$, where 1 stands on place number i. We will need the following straightforward computation.

Lemma 5.8.1. *Let $\alpha \in \mathfrak{X}^k$. Then*

$$\chi_{\lambda,\alpha}|_{P_{\lambda^-_i}} = \chi_{\lambda^-_i, \alpha\cdot\beta^i_k},$$

where $\alpha \cdot \beta^i_k \in \mathfrak{X}^k$ denotes the coordinate-wise product.

Proof of Lemma 4.4.3. The proof is by induction. Let us first prove the base $i = 1$. Note that $X = P_n/Q^1_\lambda \cong G_{n-1}/P_{\lambda^-_k}$. Thus $\pi|_{G_{n-1}} = \mathcal{S}(G_{n-1}/P_{\lambda^-_k}, \mathcal{E})$. Consider the fiber of $\mathcal{E}$ at the class of 1 as a character of $P_{\lambda^-_k}$. We get $\mathcal{E}_{x_0} = \chi_{\lambda,\alpha}|_{P_{\lambda^-_k}} = \chi_{\lambda^-_k, \alpha\cdot\beta^k_k}$. Thus

$$\pi|_{G_{n-1}} = \chi_{(n_1),(\alpha_1|\det|^{-1/2})} \times \cdots \times \chi_{(n_{k-1}),(\alpha_{k-1}|\det|^{-1/2})} \times \chi_{(n_k-1),(\alpha_k)}.$$

For the induction step, we assume $i>0$ and let Z, Z_0 and λ' be as in Lemma 4.3.6 and $p: X\to Z$ denote the natural projection. By Lemma 5.6.1 (and Remark 5.6.2) there exists a G_{n-1}-multiplicative equivariant bundle $\mathcal{E}'$ on Z such that $\mathcal{E}=p^?(\mathcal{E}')$. By the key lemma (Lemma 3.1.1), $\Phi(\pi)=\mathcal{S}(Z_0,\mathcal{E}')\otimes|\det|^{-1/2}$. By Corollary 4.3.4 and Lemma 4.3.6, $Z_0=\bigcup_{j=1}^{i-1}\mathcal{O}^{ij}_{\lambda'}$. By Corollary 2.4.4, there is a filtration F^j on $\mathcal{S}(Z_0,\mathcal{E}')$ such that $Gr^j(\mathcal{S}(Z_0,\mathcal{E}'))=\mathcal{S}_{Z_0}(\mathcal{O}^{ij}_{\lambda'},\mathcal{E}')$. By Corollary 4.3.4, $\mathcal{O}^{ij}_{\lambda'}\cong P_{n-1}/Q^j_{\lambda'}$. Thus, by Lemma 4.4.1, $\Phi^{i-2}(\mathcal{S}_{Z_0}(\mathcal{O}^{ij}_{\lambda'},\mathcal{E}'))=0$ for $j<i-1$. Thus

$$E^i(\pi)=\Phi^{i-1}(\pi)=\Phi^{i-2}(\Phi(\pi))=\Phi^{i-2}(\mathcal{S}_{Z_0}(\mathcal{O}^{i,i-1}_{\lambda'},\mathcal{E}')\otimes|\det|^{-1/2})=$$
$$\Phi^{i-2}(\mathcal{S}(\mathcal{O}^{i,i-1}_{\lambda'},\mathcal{E}'\otimes|\det|^{-1/2}))=E^{i-1}(\mathcal{S}(\mathcal{O}^{i,i-1}_{\lambda'},\mathcal{E}'\otimes|\det|^{-1/2})).$$

In the last expression we consider the character $|\det|^{-1/2}$ of P_{n-1} as a constant P_{n-1}-equivariant bundle on Z_0.

Consider $\nu_1:=\mathcal{E}'|_{x^{i,i-1}_{\lambda}}$ and $\nu_2:=\mathcal{E}|_{x^{i,i-1}_{\lambda}}$ as characters of $(P_{n-1})_{x^{i,i-1}_{\lambda'}}=Q^{i-1}_{\lambda'}$. Note that

$$\mathcal{E}'|_{x^{i,i-1}_{\lambda'}}=\mathcal{E}|_{x^{i,i-1}_{\lambda'}}\otimes D_X^*|_{x^{i,i-1}_{\lambda'}}\otimes D_Z|_{x^{i,i-1}_{\lambda'}}$$

and thus $\nu_1=\nu_2\otimes\overline{\chi}^{i-1}_{\lambda',\beta}|_{Q^{i-1}_{\lambda'}}$, where $\beta=(|\det|,\dots,|\det|,1,\dots,1)$, where the last appearance of $|\det|$ is in the place $k-i+1$. Thus

$$(\mathcal{E}'\otimes|\det|^{-1/2})|_{x^{i,i-1}_{\lambda}}=\nu_1\otimes|\det|^{-1/2}=\nu_2\cdot\overline{\chi}^{i-1}_{\lambda',\beta}|_{Q^{i-1}_{\lambda'}}\cdot|\det|^{-1/2}=$$
$$(\chi^i_{\lambda,\alpha}|_{Q^{i-1}_{\lambda'}}\cdot\overline{\chi}^{i-1}_{\lambda',\beta}|_{Q^{i-1}_{\lambda'}}\cdot|\det|^{-1/2})|=((\chi^i_{\lambda,\alpha}|_{P^{i-1}_{\lambda'}})\cdot\overline{\chi}^{i-1}_{\lambda',\beta}\cdot|\det|^{-1/2})|_{Q^{i-1}_{\lambda'}}=$$
$$(\chi^{i-1}_{\lambda',\alpha\cdot\beta_k^{k-i+1}}\cdot\overline{\chi}^{i-1}_{\lambda',\beta}\cdot|\det|^{-1/2})|_{Q^{i-1}_{\lambda'}}=\chi^{i-1}_{\lambda',\alpha'}|_{Q^{i-1}_{\lambda'}},$$

where $\alpha'=(\alpha_1,\dots,\alpha_{k-i},\alpha_{k-i+1}|\det|^{1/2},\alpha_{k-i+2},\dots,\alpha_k)$.

Thus, by the induction hypothesis,

$$E^i(\pi)|_{G_{n-i}}=E^{i-1}(\mathcal{S}(\mathcal{O}^{i-1,j}_{\lambda},\mathcal{E}'\otimes|\det|^{-1/2}))=$$
$$\chi_{(n_1),(\alpha_1|\det|^{-1/2})}\times\dots\times\chi_{(n_{k-i}),(\alpha_{k-i}|\det|^{-1/2})}\times\chi_{(n_{k-i+1}-1),(\alpha_{k-i+1})}\times\dots\times\chi_{(n_k-1),(\alpha_k)}$$

□

6. Homology of geometric representations and the proof of the key lemma (Lemma 3.1.1)

6.1. **Sketch of the proof.** We have to compute $\mathfrak{v}_n$-homology of the representation $\mathcal{S}(X,\mathcal{E})\otimes(-\psi)$. We first note that this task is local on $X'=V_n\backslash X$, i.e. if we cover X' by subsets U_i it is enough to compute the homology of $\mathcal{S}(p^{-1}(U_i),\mathcal{E})\otimes(-\psi)$. Now we can cover X' by refined enough cover s.t. for each U_i , the space $p^{-1}(U_i)$ will look like a product $U_i\times W$ where W is a V_n-orbit and the bundle $\mathcal{E}|_{U_i}$ is trivial. Thus we reduced to computation of homology of $\mathcal{S}(U_i\times W)\otimes(-\psi)$.

Note that the action of V_n on $U_i\times W$ is not the usual product action but rather a twisted product, i.e. the action V_n on its orbit $\{x\}\times W$ depends on the point $x\in U_i$. We can untwist this product (see Proposition 6.2.5) but this will cause a twist in the character ψ. Namely, it will replace it by a line bundle $\mathcal{E}$ where the action of V_n on $\mathcal{E}_{\{x\}\times W}$ depends on the point $x\in U_i$. Thus we reduce to computation of homology of $\mathcal{S}(U_i\times W,\mathcal{E})$.

Now we use a relative version of the Shapiro Lemma (see Theorem 6.2.1) in order to reduce to the computation of the homology of $\mathcal{S}(U_i,\mathcal{E})$ as a representation of the stabilizer $(V_n)_0$ in V_n of a point $0\in W$. Note that the action of $(V_n)_0$ on U_i is trivial and thus we can view $\mathcal{S}(U_i,\mathcal{E})$ as a family of characters, i.e. it is defined by a map ϕ from U_i to the space $(V_n)_0^*$ of characters of $(V_n)_0$. In Lemma 6.2.2 we compute homology of such families under the assumption that ϕ is submersive at the trivial character. This assumption is satisfied in our case due to the action of P_n.

Remark 6.1.1. *The proof is based on series of reductions. If we were interested only in acyclicity then one could give a relatively simple proof in which each of those reductions is given by general statements. However we are also interested in computation of* H_0 *and this makes those reductions more complicated.*

For example in the first step when we say that the computation is local one should explain what does this mean. We do it by constructing an explicit morphism $I : \mathrm{H}_0(\mathfrak{v}_n, \mathcal{S}(X, \mathcal{E}) \otimes (-\psi)) \to \mathcal{S}(X'_0, \mathcal{E}|_{X'_0})$ *and proving that this morphism is an isomorphism rather than proving that there exists some isomorphism. This forces us to make each reduction more explicit. This is sometimes unpleasant since some of the reductions e.g. the Shapiro lemma are using the general machinery of homological algebra which usually is not so explicit.*

Therefore if the reader is not interested in all the details we recommend him to skip all the parts that regard H_0 *and concentrate only on the acyclicity. The computation of* H_0 *is essentially the same but its exposition is much longer.*

6.2. **Ingredients of the proof.** We will need the following version of Shapiro lemma.

Theorem 6.2.1 (Relative Shapiro lemma)**.** *Let* G *be an affine Nash group and* X *be a transitive Nash* G*-manifold. Let* Y *be a Nash manifold. Let* $x \in X$ *and denote* $H := G_x$*. Let* $\mathcal{E} \to X \times Y$ *be a* G *equivariant Nash bundle. Suppose that* G *and* H *are homologically trivial (i.e. all their homology except* H_0 *vanish and* $\mathrm{H}_0 = \mathbb{R}$*). Let* $\mathfrak{X} : G \times Y \to \mathbb{R}$ *be a Nash map such that for any* $y \in Y$*, the map* $\mathfrak{X}|_{G \times \{y\}}$ *is a group homomorphism. Let* $\mathfrak{X}'$ *be 1-dimensional* G*-equivariant bundle on* Y*, with action of* G *given by* $g(y, v) = (y, \theta(\mathfrak{X}(g, y))v)$*. Let* $\mathcal{E}' := \mathcal{E} \otimes (\mathbb{C} \boxtimes \mathfrak{X}')$*, where* $\mathbb{C}$ *denotes the trivial bundle on* X *and* $\boxtimes$ *denotes exterior tensor product. Then*

$$\mathrm{H}_i(\mathfrak{g}, \mathcal{S}(X \times Y, \mathcal{E}')) \cong \mathrm{H}_i(\mathfrak{h}, \mathcal{S}(\{x\} \times Y, \mathcal{E}'_{\{x\} \times Y} \otimes \Delta_H \cdot (\Delta_G^{-1})|_H)),$$

where Δ_H *and* Δ_G *denote the modular characters of the groups* H *and* G*.*

The proof of this theorem is along the lines of the proof of [AG10, Theorem 4.0.9]. We give it in Appendix A.

Lemma 6.2.2. *Let* X *be a Nash manifold and* V *be a real vector space. Let* $\phi : X \to V^*$ *be a Nash map. Suppose that* $0 \in V^*$ *is a regular value of* ϕ*. It gives a map* $\chi : V \to \mathcal{T}(X)$ *given by* $\chi(v)(x) = \theta(\phi(x)(v))$ *(recall that* $\mathcal{T}(X)$ *denotes the space of tempered functions). This gives an action of* V *on* $S(X)$ *by* $\pi(v)(f) := \chi(v) \cdot f$*. Then*

(i) $\mathrm{H}_i(\mathfrak{v}, \mathcal{S}(X)) = 0$ *for* $i > 0$*.*

(ii) Let $X_0 := \phi^{-1}(0)$*. Note that it is smooth. Let* r *denote the restriction map* $r : \mathcal{S}(X) \to \mathcal{S}(X_0)$*. Then* r *gives an isomorphism* $\mathrm{H}_0(\mathfrak{v}, \mathcal{S}(X)) \xrightarrow{\sim} \mathcal{S}(X_0)$*.*

We will prove this lemma in §§6.5

Notation 6.2.3. *In the situation of the Lemma we will denote the action of* $\mathfrak{v}$ *on* $\mathcal{S}(X)$ *by* $\tau_{X,\phi}$*.*

In order to check the conditions of Lemma 6.2.2 we will need the following lemma that we will prove in §§6.6.

Lemma 6.2.4. *Let a Nash group* G *act linearly on a finite-dimensional vector space* V *over* F*, such that the action on* $V^* \setminus 0$ *is transitive. Let* Q *be a closed Nash subgroup of* G *and* L *be a vector subspace of* V *stabilized by* Q*. Let* $\varphi \in V^*$ *be a functional. Consider the map* $a : G \to L^*$ *defined by* $a(g) := g\varphi|_L$*. Let* $U \subset G/Q$ *be an open (Nash) subset and* $s : U \to G$ *be a local Nash section of the canonical projection* $p : G \to G/Q$*. Then* 0 *is a regular value of* $\mu := a \circ s$*.*

Proposition 6.2.5. *Let a Nash group* G *act transitively on a Nash manifold* Y *and let* X *be a Nash manifold.*

Let X *"act" on* G *i.e. let* G' *be a Nash group acting on* G *by automorphisms and* $a : X \to G'$ *be a Nash map. This defines a twisted action of* G *on* $X \times Y$*. More precisely* $\rho_1(g)(x, y) = (x, a(x)(g)(y))$*. Let* χ *be a fixed tempered character of* G*. Let* ρ_2 *denote the non-twisted action of* G *on* $X \times Y$*, i.e.* $\rho_2(g)(x, y) = (gx, y)$*.*

Define function $\mathfrak{X}(g, x, y) = \chi((a(x))^{-1}(g))$*. Note that it does not depend on* y*. It defines a tempered* $(G, \rho_2)-$*equvariant structure on the trivial line bundle on* $X \times Y$*. We denote the resulting bundle by* $\mathcal{E}$*. Let* π_1 *denote the representation of* $\mathfrak{g}$ *on* $\mathcal{S}(X \times Y) \otimes \chi$ *given by the action* ρ_1 *and* π_2 *denote the representation of* $\mathfrak{g}$ *on* $\mathcal{S}(X \times Y, \mathcal{E})$ *given by the action* ρ_2*.*

Then $\mathrm{H}_*(\mathfrak{g}, \pi_1) = \mathrm{H}_*(\mathfrak{g}, \pi_2)$*.*

We will prove this proposition in §§6.4.

Notation 6.2.6. *In the situation of the proposition we will denote* ρ_1 *by* $\rho_{a,X,Y}$ *and* π_1 *by* $\pi_{a,X,Y}$.

6.3. **Proof of the key lemma.** First, let us prove the following version of the lemma.

Lemma 6.3.1. *Let* X' *be a Nash manifold. Let* $0 \to L \to V \xrightarrow{p} W \to 0$ *be an exact sequence of finite dimensional vector spaces over* F. *Let* $\nu : X' \to V^*$ *be a Nash map, such that* 0 *is a regular value of the composition* $X' \to V^* \to L^*$. *Let* X'_0 *be the preimage of* 0 *under this composition. Note that it is smooth. Fix a Haar measure on* W. *Let* π_ν *be the representation of* V *on* $\mathcal{S}(W \times X')$ *given by*

$$\pi_\nu(v)(f)(z,w) := \theta(\langle \nu(z), v\rangle) f(z, w + p(v)).$$

Then

(i) π_ν *is acyclic*

(ii) Note that $\nu(X'_0) \subset W^* \subset V^*$. *Let* $\nu_0 : X'_0 \to W^*$ *be the map given by restriction of* ν. *Let* $F \in \mathcal{T}(X'_0 \times W)$ *be given by* $F(z,w) := \langle \nu_0(z), w\rangle$. *Consider the map* $I_\nu : \mathcal{S}(W \times X') \to \mathcal{S}(X'_0)$ *given by* $I_\nu(f) := (pr_{X'})_*(f|_{X'_0 \times W} \cdot F)$, *where* $pr_{X'} : W \times X' \to W$ *denotes the projection. Then* I_ν *defines an isomorphism* $\mathrm{H}_0(\mathfrak{v}, \pi_\nu) \xrightarrow{\sim} \mathcal{S}(X'_0)$.

Proof. (i) By the relative Shapiro Lemma (Lemma 6.2.1) it is enough to show that $\tau_{X',\nu}$ (see Notation 6.2.3) is acyclic. This follows from Lemma 6.2.2.

(ii) Fix a Haar measure on L. Since we fixed a Haar measure on W, this defines a Haar measure on V as well. Define $\bar{F} \in \mathcal{T}(X' \times V)$ by $\bar{F}(z,v) := \langle \nu(z), v\rangle$. Define $\bar{I} : \mathcal{S}(X' \times V) \to \mathcal{S}(X')$ by $\bar{I}(f) = (pr_{X'})_*(f) \cdot \bar{F}$. Let $r : \mathcal{S}(X') \to \mathcal{S}(X'_0)$ denote the restriction. Let $\mu : X' \to L^*$ be the composition of ν with $V^* \twoheadrightarrow L^*$. Define an action $\tilde{\pi}_\nu$ of V on $\mathcal{S}(X' \times V)$ given by

$$\pi_\nu(v)(f)(z,v') := \theta(\langle \nu(z), v\rangle) f(z, v' + v).$$

Define an action σ of L on $\mathcal{S}(X' \times V)$ by $\sigma(l)(f)(z,v') := f(z, v' + l)$.

Note that the following diagram is commutative

(1) (2) (3)

$$\begin{array}{ccccccc}
(1) & & \mathfrak{v}\otimes\mathcal{S}(X'\times V) & \xrightarrow{Id_{\mathfrak{v}}\otimes(Id_{X'}\times p)_*} & \mathfrak{v}\otimes\mathcal{S}(W\times X') & & \\
 & & \downarrow{\scriptstyle d\tilde{\pi}_\nu} & & \downarrow{\scriptstyle d\pi_\nu} & & \\
(2) & \mathfrak{l}\otimes\mathcal{S}(X'\times V) & \xrightarrow{d\sigma} & \mathcal{S}(X'\times V) & \xrightarrow{(Id_{X'}\times p)_*} & \mathcal{S}(W\times X') & \longrightarrow 0 \\
 & \downarrow{\scriptstyle Id_{\mathfrak{l}}\otimes\bar{I}} & & \downarrow{\scriptstyle \bar{I}} & & \downarrow{\scriptstyle I} & \\
(3) & \mathfrak{l}\otimes\mathcal{S}(X') & \xrightarrow{\tau_\mu} & \mathcal{S}(X') & \xrightarrow{r} & \mathcal{S}(X'_0) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

Recall that τ_μ, used in the diagram, is defined in Notation 6.2.3. It is enough to show that column (3) is exact. First, let us show that column (2) is exact. The map $\bar{I}$ is onto by Lemma 5.5.14 and the exactness in the place of $\mathcal{S}(X' \times V)$ follows from 5.5.15. The exactness of row (2) is proven in the same way. The exactness of row (3) follows from Lemma 6.2.2. The exactness of column (2) implies the exactness or column (1). Let us prove that column (3) is exact. First, note that I is onto by Lemma 5.5.14 and Theorem 5.5.10(iii).

Now, let $f \in \mathrm{Ker}(I) \subset \mathcal{S}(W \times X')$. Let $\tilde{f} \in \mathcal{S}(X' \times V)$ be its preimage under $(Id \times p)_*$. Then $\bar{I}(\tilde{f}) \in \mathrm{Ker}(r) = \mathrm{Im}(\tau_\mu)$. Let $h \in \mathfrak{l} \otimes \mathcal{S}(X')$ be a preimage of $\bar{I}(\tilde{f})$ and let $\tilde{h}$ be a preimage of h in $\mathfrak{l} \otimes \mathcal{S}(X' \times V)$. Then $d\sigma(\tilde{h}) - \tilde{f} \in \mathrm{Ker}(\bar{I}) = \mathrm{Im}\, d\tilde{\pi}_\nu$. Now,

$$f = (Id \times p)_*(\tilde{f}) = (Id \times p)_*(\tilde{f} - d\sigma(\tilde{h}) + d\sigma(\tilde{h})) = (Id \times p)_*(\tilde{f} - d\sigma(\tilde{h})) \in (Id \times p)_*(\mathrm{Im}\, d\tilde{\pi}_\nu) \subset \mathrm{Im}(d\pi_\nu)$$

□

We will prove the following Lemma which clearly implies the key lemma.

Lemma 6.3.2. *Let T be a Nash linear group and let $R := P_n \times T$ and $R' := G_{n-1} \times T$. Let $Q < R$ be a Nash subgroup and let $X \subset R/Q$ be a V_n-invariant open Nash subset and $X' := V_n \backslash X$. Note that X' is an open Nash subset of R'/Q', where $Q' = Q/(Q \cap V_n)$. Let $X_0 = \{x \in X : \psi|_{(V_n)_x} = 1.\}$ Let X_0' be the image of X_0 in X'. Let $\mathcal{E}'$ be a R'-equivariant tempered bundle on R'/Q' and $\mathcal{E} := p_{X'}^?(\mathcal{E}'|_{X'})$. Then*

(i) $\mathrm{H}_i(\mathfrak{v}_n, \mathcal{S}(X, \mathcal{E}) \otimes (-\psi)) = 0$ *for any* $i > 0$.

(ii) X_0' *is smooth*

(iii) Since $X' \subset R'/Q'$ and $X \subset R/Q$ we have a canonical section to p_Z and will consider X' as a closed submanifold of X. Consider the action map $a : X_0' \times V_n \to X_0$. Let $\Xi = 1 \boxtimes \psi \in \mathcal{T}(X_0' \times V_n)$. Note that Ξ_0 is constant along the fibers of a. Define $\Xi \in \mathcal{T}(X_0)$ such that $a^*(\Xi) = \Xi_0$. Consider the map $I : \mathcal{S}(X, p^?\mathcal{E}) \to \mathcal{S}(X_0', \mathcal{E}|_{X_0'})$ given by $I(f) := (p_{X'}|_{X_0})_*(f|_{X_0'} \cdot \Xi)$. Then I gives an isomorphism $\mathrm{H}_0(\mathfrak{v}_n, S(X, p^?\mathcal{E})) \xrightarrow{\sim} \mathcal{S}(X_0', \mathcal{E}|_{X_0'})$, as representations of $\mathfrak{p}_{n-1}$.

Remark 6.3.3. *The open subset $X \subset R/Q$ is not necessary P_{n-1}-invariant, but $\mathfrak{p}_{n-1}$ will still act on the homology.*

First let us prove the following special case.

Lemma 6.3.4. *Lemma 6.3.2 holds under the assumption that there exists a Nash section $a : X' \to R'$ of the quotient map $R' \to R'/Q'$.*

Proof. Consider R'/Q' as a subset of R/Q. Without loss of generality we assume that the class of the unity element lies in X'. We will denote it by z_0. Without loss of generality we assume $a(z_0) = 1$. Let $W := p_{X'}^{-1}(z_0)$. Let $L := (V_n)_{i(z_0)} = Q \cap V_n$. It is a Nash subgroup of V_n and hence is a real vector space. Note that $W \cong V_n/L$. Define $\phi : X' \times V_n \to X$ by $\phi(z, v) := (a(z)v)z$, where $a(z)v$ denotes the action of $a(z) \in G_{n-1}$ on $v \in V_n$. Note that ϕ factors through $X' \times W$ and let $\overline{\phi}$ be the corresponding map $X' \times W \to X$. It is easy to see that $\overline{\phi}$ is a Nash diffeomorphism and intertwines $\rho_{a,X',W}$ with the action of V_n on X (see Notation 6.2.6).

Note that the section a gives a tempered isomorphism between $\mathcal{E}'|_{X'}$ and the trivial bundle with fiber $\mathcal{E}'_{z_0}$. Thus $\overline{\phi}$ gives an V_n -isomorphism $\overline{\phi}^* : \mathcal{S}(X, \mathcal{E}) \otimes (-\psi) \xrightarrow{\sim} \pi_{a,X',W} \otimes \mathcal{E}'_{z_0}$. Define $\nu : X' \to V_n^*$ by $\nu(z)(v) := \bar{\psi}(a(z)v)$. Note that X_0' coincides with X_0' from Lemma 6.3.1. Note that the underlying vector space of $\pi_{a,X',W}$ coincides with the underlying vector space of π_ν from Lemma 6.3.1. Thus we can consider $I_\nu \otimes Id$ as a map $\pi_{a,X',W} \otimes \mathcal{E}'_{z_0} \to \mathcal{S}(X_0') \otimes \mathcal{E}'_{z_0} \cong \mathcal{S}(X_0', \mathcal{E}'|_{X_0'})$. Note that $\overline{\phi}^*$ intertwines the map I to $I_\nu \otimes Id$.

By Proposition 6.2.5, $\mathrm{H}_*(\mathfrak{v}_n, \pi_{a,X',W}) \cong \mathrm{H}_*(\mathfrak{v}_n, \pi_\nu)$ and on H_0 the isomorphism is just equality of quotient spaces. Thus it is enough to show that π_ν is acyclic and I_ν defines an isomorphism $\mathrm{H}_0(\mathfrak{v}_n, \pi_\nu) \xrightarrow{\sim} \mathcal{S}(X_0')$. To show this, by Lemma 6.3.1 it is enough to show that 0 is a regular value of the composition $\nu_L : X' \xrightarrow{\nu} V_n^* \to L^*$. This follows from Lemma 6.2.4.

□

Proof of Lemma 6.3.2. Let $q : R' \to R'/Q'$ be the quotient map. By Theorem 5.5.3, there exists a finite Nash cover $\{U_i\}$ of $X' \subset R'/Q'$ and sections $s_i : U_i \to R'$ of q. By Lemma 6.3.4, $\mathcal{S}(p_{X'}^{-1}(U_i), \mathcal{E}|_{p_{X'}^{-1}(U_i)})$ is acyclic and $I|_{\mathcal{S}(p_{X'}^{-1}(U_i), \mathcal{E}|_{p_{X'}^{-1}(U_i)})}$ gives an isomorphism

$$\mathrm{H}_0(\mathfrak{v}_n, \mathcal{S}(p_{X'}^{-1}(U_i), \mathcal{E}|_{p_{X'}^{-1}(U_i)})) \xrightarrow{\sim} \mathcal{S}(U_i \cap X_0', \mathcal{E}'|_{U_i \cap X_0'}).$$

Consider the extended Kozsul complex of $\mathcal{S}(X, \mathcal{E}) \otimes (-\psi)$:

$$0 \to \Lambda^n(\mathfrak{v}_n) \otimes \mathcal{S}(X) \otimes (-\psi) \xrightarrow{d_n} \cdots \xrightarrow{d_2} \mathfrak{v}_n \otimes \mathcal{S}(X) \otimes (-\psi) \xrightarrow{d_1} \mathcal{S}(X) \otimes (-\psi) \xrightarrow{d_0 := I} \mathcal{S}(X_0', \mathcal{E}'|_{X_0'}) \to 0 \quad (4)$$

We have to show that it is exact. The fact that I is onto follows from Theorem 5.5.10(iii) and Lemma 5.5.14. Let us show that it is exact in place $l \geq 0$, i.e. at the object $\Lambda^l(\mathfrak{v}_n) \otimes \mathcal{S}(X)$. Choose a partition of unity e_i corresponding to the cover $X' = \bigcup U_i$.

Let $\alpha \in \operatorname{Ker} d_l \subset \Lambda^l(\mathfrak{v}_n) \otimes \mathcal{S}(X) \otimes (-\psi)$. Consider

$$p^*(e_i)\alpha \in \operatorname{Ker} d_l|_{\Lambda^l(\mathfrak{v}_n)\otimes\mathcal{S}(p^{-1}(U_i)\otimes(-\psi))}.$$

By Lemma 6.3.4, we have

$$p^*(e_i)\alpha \in \operatorname{Im}(d_{l+1}|_{\Lambda^l(\mathfrak{v}_n)\otimes\mathcal{S}(p^{-1}(U_i)\otimes(-\psi))}) \text{ and thus } \alpha = \sum p^*(e_i)\alpha \in \operatorname{Im}(d_{l+1}).$$

□

6.4. **Untwisting a product (Proof of Proposition 6.2.5).**

Lemma 6.4.1. *Let $\mathfrak{g}$ be a Lie algebra and A be a commutative algebra with 1. Let M be a $(U(\mathfrak{g}) \otimes A)$-module. Then $\mathrm{H}_*(\mathfrak{g}, M) = \mathrm{H}_*(\mathfrak{g} \otimes A, M)$.*

Proof. Let $\mathcal{A}$ be the category of $(U(\mathfrak{g}) \otimes A)$-modules and $\mathcal{B}$ be the category of $\mathfrak{g}$-modules. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be the forgetful functor and $\mathcal{G} : \mathcal{B} \to Vect$ be the functor of $\mathfrak{g}$ co-invariants. Note that $\mathcal{G} \circ \mathcal{F}$ is the functor of coinvariants with respect to the Lie algebra $\mathfrak{g} \otimes A$. By Lemma 5.1.5 we have

$$L^i(\mathcal{G} \circ \mathcal{F}) = L^i(\mathcal{G}) \circ \mathcal{F}.$$

This proves the assertion. □

Proof of Proposition 6.2.5. Extend the representations π_i to representations Π_i of $\mathcal{T}(X) \otimes \mathfrak{g} \cong \mathcal{T}(X, \mathfrak{g})$.

Let $b : \mathcal{T}(X, \mathfrak{g}) \to \mathcal{T}(X, \mathfrak{g})$ be given by $b(f)(x) = a(x)(f(x))$. Note that b is invertible. Let us show that $\Pi_1 \cong \Pi_2 \circ b$.

First, note that as linear spaces both can be identified with $\mathcal{S}(X \times Y)$. Now, denote by $d\rho_i$ the corresponding maps from $\mathfrak{g}$ to the space of vector fields on $X \times Y$ and by $d\mathfrak{X} : \mathfrak{g} \times X \times Y$ the differential of $\mathfrak{X}$ in the first variable. Let $f \in \mathcal{T}(X, \mathfrak{g})$, $h \in \mathcal{S}(X \times Y)$, and $(x, y) \in X \times Y$. Then

$$\begin{aligned}(\Pi_1(f)h)(x,y) = (d\rho_1(f(x))h)(x,y) + \chi(f(x))h(x,y) = (d\rho_2(a(x)(f(x)))h)(x,y) + \chi(f(x))h(x,y) = \\ (d\rho_2(a(x)(f(x)))h)(x,y) + \chi(a(x)^{-1}(a(x)(f(x)))) \cdot h(x,y) = \\ (d\rho_2(a(x)(f(x)))h)(x,y) + d\mathfrak{X}(a(x)(f(x)), x, y) \cdot h(x,y) = \\ (d\rho_2((b(f)(x))h)(x,y) + d\mathfrak{X}(b(f)(x), x, y) \cdot h(x,y) = (\Pi_2(b(f))(h))(x,y)\end{aligned}$$

Now

$$\mathrm{H}_*(\mathfrak{g}, \pi_2) = \mathrm{H}_*(\mathcal{T}(X) \otimes \mathfrak{g}, \Pi_2) \cong \mathrm{H}_*(\mathcal{T}(X) \otimes \mathfrak{g}, \Pi_1) = \mathrm{H}_*(\mathfrak{g}, \pi_1).$$

□

6.5. **Homology of families of characters (Proof of Lemma 6.2.2).** We prove the lemma by induction on $\dim V$.

Base: $\dim V = 1$. In this case we have to show that the following extended Koszul complex is exact:

$$0 \leftarrow \mathcal{S}(X_0) \xleftarrow{r} \mathcal{S}(X) \leftarrow V \otimes \mathcal{S}(X) \leftarrow 0.$$

Let $v \in V$ be a generator. Then the complex is isomorphic to

$$0 \leftarrow \mathcal{S}(X_0) \xleftarrow{r} \mathcal{S}(X) \xleftarrow{m_h} \mathcal{S}(X) \leftarrow 0,$$

where $m_h(f) = hf$, and $h(x) = d\theta(\langle v, \phi(x)\rangle)$. Clearly, m_h is injective. The map r is onto by Theorem 5.5.10(iii) Note that ih is a real-valued Nash function and 0 is its regular value. The exactness in the middle follows now from Lemma 5.5.16.

Induction step. Let us first prove (i). Let $L < V$ be a one-dimensional subspace. Let ϕ_L denote the composition $X \to V^* \to L^*$. Note that 0 is a regular value of ϕ_L and let $X_{L,0} := \phi_L^{-1}(0)$. By the induction base, $\mathcal{S}(X)$ is $\mathfrak{l}$-acyclic and $r_L : \mathrm{H}_0(\mathfrak{l}, \mathcal{S}(X)) \xrightarrow{\sim} \mathcal{S}(X_{L,0})$. Note that this is an isomorphism of representations of V/L. Note also that $\phi(X_{L,0}) \subset (V/L)^* = L^\perp \subset V^*$ and let ϕ' denote $\phi|_{X_{L,0}} : X_{L,0} \to (V/L)^*$. Note that 0 is a regular value of ϕ'. Finally, note that the action of V/L on $\mathcal{S}(X_{L,0})$ is $\tau_{X_L,\phi'}$. Thus, by the induction hypothesis, $\mathrm{H}_0(\mathfrak{l}, \mathcal{S}(X))$ is V/L-acyclic and Lemma 5.1.7 implies that $\mathcal{S}(X)$ is V-acyclic.

To prove (ii) note first that r is onto by Theorem 5.5.10(iii). Now we have to show that $\mathfrak{v}\mathcal{S}(X) = \operatorname{Ker} r$. The inclusion $\subset$ is obvious. Let $f \in \operatorname{Ker} r$. Consider $f|_{X_{L,0}}$. As before, the induction hypothesis implies $f|_{X_{L,0}} \in (\mathfrak{v}/\mathfrak{l})\mathcal{S}(X_{L,0})$. Let $h \in (\mathfrak{v}/\mathfrak{l})\otimes\mathcal{S}(X_{L,0}) = \tau_{X_{L,0},\phi'}((\mathfrak{v}/\mathfrak{l})\otimes\mathcal{S}(X_{L,0}))$ such that $f|_{X_{L,0}} = \tau_{X_{L,0},\phi'}(h)$. By Theorem 5.5.10(iii),we may extend h to $\widetilde{h} \in \mathfrak{v}\otimes\mathcal{S}(X)$. Now, $(\tau_{X,\phi}(\widetilde{h}) - f)_{X_L} = 0$ and hence by the induction base $\tau_{X,\phi}(\widetilde{h}) - f \in \mathfrak{l}\mathcal{S}(X)$. This implies that $f \in \mathfrak{v}\mathcal{S}(X)$. □

6.6. **Proof of Lemma 6.2.4.** For the proof we will need the following straightforward lemma from linear algebra.

Lemma 6.6.1. *Let W_1, W_2 and W be linear spaces. Let $T : W_1 \twoheadrightarrow W_2$ and $A : W_1 \twoheadrightarrow W$ be epimorphisms. Suppose that $\operatorname{Ker} T \supset \operatorname{Ker} A$. Then $A \circ S$ is onto, for any section S of T.*

Proof of Lemma 6.2.4. Let $x \in U$ such that $\mu(x) = 0$. Without loss of generality, using the left action of G we can suppose $s(x) = 1$. Then $\phi|_L = 0$ and we have to show $d_1 a \circ d_x s$ is onto L^*. The map $d_1 a$ is onto by transitivity of the action of G on $V^* \setminus 0$, and $\operatorname{Ker} d_1 p \subset \operatorname{Ker} d_1 a$ since Q stabilizes L. Thus, by the previous lemma, $d_1 a \circ d_x s$ is onto. □

Appendix A. Proof of the relative Shapiro lemma

We prove here Theorem 6.2.1 which is a version of the Shapiro lemma. A similar version was proven in [AG10] and our proof follows the lines of the proof there.

Proposition A.0.1 ([AG10], Proposition 4.0.6)**.** *Let G be a Nash group and X be a Nash G-manifold. Suppose that the action is strictly simple (see Definition 5.5.7). Then the projection $\pi : X \to G \setminus X$ is a Nash locally trivial fibration.*

Corollary A.0.2. *Let G be a Nash group and X be a Nash G-manifold with strictly simple action. Let $F \to X$ be a Nash G-equivariant locally-trivial fibration. Then the action of G on the total space F is strictly simple.*

Proof. By the previous proposition we may assume without loss of generality that the map $X \to G \setminus X$ is a trivial fibration. Hence we can identify $X \cong (G \setminus X) \times G$. Now it is easy to see that $G \setminus F \cong F|_{(G\setminus X)\times\{1\}}$. □

Corollary A.0.3. *Let G be a Nash group and X be a Nash G-manifold. Suppose that the action is strictly simple. Let $\mathcal{E}$ be a Nash G-equivariant bundle on X. Then there exists a Nash bundle $\mathcal{E}'$ on $G\setminus X$ such that $\mathcal{E} = \pi^*\mathcal{E}'$, where $\pi : X \to G \setminus X$ is the standard projection.*

Proof. Consider the action of G on the total space of $\mathcal{E}$. Denote $\mathcal{E}' := G \setminus \mathcal{E}$ and consider it as a Nash bundle over $G \setminus X$. It is easy to see that $\mathcal{E} = \pi^*\mathcal{E}'$. □

Definition A.0.4. *Let X be a Nash manifold and F be a Nash locally trivial fibration over X. Denote by $H^i_c(F \to X)$ the natural Nash bundle on X such that for any $x \in X$ we have $H^i_c(F \to X)_x = H^i_c(F_x)$. For its precise definition see* [AG10, Notation 2.4.11]*. Note that if F is a trivial fibration then $H^i_c(F \to X)$ are trivial bundles.*

Definition A.0.5. *Let $F \xrightarrow{\pi} X$ be a locally trivial fibration. Let $\mathcal{E} \to X$ be a Nash bundle. We define $T_{F\to X} \subset T_F$ by $T_{F\to X} = \operatorname{Ker}(d\pi)$. We denote*

$$\Omega^{i,\mathcal{E}}_{F\to X} := ((T_{F\to X})^*)^{\wedge i} \otimes \pi^*\mathcal{E}$$

*Now we define **the relative de-Rham complexes** $DR^{\mathcal{E}}_{\mathcal{S}}(F \to X)$ and $DR^{\mathcal{E}}_{C^\infty}(F \to X)$ by*

$$DR^{\mathcal{E}}_{\mathcal{S}}(F \to X)^i := \mathcal{S}(F, \Omega^{i,\mathcal{E}}_{F\to X}) \text{ and } DR^{\mathcal{E}}_{C^\infty}(F \to X)^i := C^\infty(F, \Omega^{i,\mathcal{E}}_{F\to X}).$$

The differentials in those complexes are defined as the differential in the classical de-Rham complex. If $\mathcal{E}$ is trivial we will omit it.

Theorem A.0.6 (see [AG10], Theorem 3.2.3)**.** *Let $\pi : F \to X$ be an affine Nash locally trivial fibration. Then*

$$H^k(DR^{\mathcal{E}}_{\mathcal{S}}(F \to X)) \cong \mathcal{S}(X, H^k_c(F \to X) \otimes \mathcal{E}).$$

This theorem gives us the following recipe for computing Lie algebra homology.

Theorem A.0.7. *Let G be an affine Nash group. Let K be a Nash G-manifold and L be a Nash manifold. Let $X := K \times L$. Let $\mathcal{E} \to X$ be a Nash G-equivariant bundle. Let $\mathfrak{X} : G \times L \to \mathbb{R}$ be a Nash map such that for any $l \in L$, the map $\mathfrak{X}|_{G\times\{l\}}$ is a group homomorphism. Let $\mathfrak{X}'$ be the 1-dimensional G-equivariant bundle on L, with action of G given by $g(l, v) = (l, \psi(\mathfrak{X}(g))v)$. Let $\mathcal{E}' := \mathcal{E} \otimes (\mathbb{C} \boxtimes \mathfrak{X}')$. Note that $\mathcal{E}'$ is isomorphic to $\mathcal{E}$ as a Nash bundle, though the G-equivariant structure on $\mathcal{E}'$ is not necessarily Nash.*

Let N be a strictly simple Nash G-manifold. Suppose that N and G are homologically trivial and affine. Denote $F = X \times N$. Note that the bundle $\mathcal{E}' \boxtimes \Omega^i_N$ has G-equivariant structure given by diagonal action. Hence the relative de-Rham complex $DR^{\mathcal{E}'}_{C^\infty}(F \to X)$ is a complex of representations of G. Consider the relative de-Rham complex $DR^{\mathcal{E}}_{\mathcal{S}}(F \to X)$ without the action of G as a subcomplex of $DR^{\mathcal{E}'}_{C^\infty}(F \to X)$. Note that it is G-invariant. We denote by $DR^{\mathcal{E}'}_{\mathcal{S}}(F \to X)$ the complex $DR^{\mathcal{E}}_{\mathcal{S}}(F \to X)$ with the action of G induced from $DR^{\mathcal{E}'}_{C^\infty}(F \to X)$.

Then

$$\mathrm{H}_i(\mathfrak{g}, \mathcal{S}(X, \mathcal{E}')) = H^{n-i}((DR^{\mathcal{E}'}_{\mathcal{S}}(F \to X))_{\mathfrak{g}}),$$

where n is the dimension of X.

To prove this theorem we will need the following statements.

Lemma A.0.8. *Let G, L, N, $\mathfrak{X}$, $\mathfrak{X}'$ be as in Theorem A.0.7. Let X be a Nash manifold. Let $\mathcal{E}$ be a Nash bundle over $X \times L \times N$. Let $\mathcal{E}' = \mathcal{E} \otimes (\mathbb{C} \boxtimes \mathfrak{X}' \boxtimes \mathbb{C})$. Then $\mathcal{S}(X \times L \times N, \mathcal{E}) \simeq \mathcal{S}(X \times L \times N, \mathcal{E}')$ as representations of G.*

Proof. By Theorem 5.5.10(iv) and Proposition A.0.1 we can assume that $N = G \times N'$, for some Nash manifold N'. Hence we can assume $N = G$. Note that $\mathcal{S}(X \times L \times G, \mathcal{E})$ and $\mathcal{S}(X \times L \times G, \mathcal{E}')$ are identical as linear spaces. Now, the required isomorphism between them is given by $\alpha \mapsto (1 \boxtimes (\psi \circ \mathfrak{X}))\alpha$. □

Lemma A.0.9. *Let G be an affine Nash group. Let F be a strictly simple Nash G-manifold. Denote $X := G \setminus F$. Let $\mathcal{E} \to X$ be a Nash bundle. Then the relative de-Rham complex $DR^{\mathcal{E}}_{\mathcal{S}}(F \to X)^i$ is isomorphic to the complex $\widetilde{C}(\mathfrak{g}, \mathcal{S}(F, \pi^*\mathcal{E}))^{\dim \mathfrak{g}-i}$, where $\pi : F \to X$ is the standard projection, and $\widetilde{C}(\mathfrak{g}, W)$ denotes the Koszul complex of a representation W.*

The proof of this lemma is the same as the proof of [AG10, Lemma 4.0.10].

Corollary A.0.10. *Let an affine Nash group G act strictly simply on a Nash manifold X. Let $\mathcal{E}$ be a Nash G-equivariant bundle on X. Suppose that G is homologically trivial. Then $\mathcal{S}(X, \mathcal{E})$ is an acyclic representation of $\mathfrak{g}$.*

Proof. Follows from Lemma A.0.9, Theorem A.0.6 and Corollary A.0.3. □

Proof of Theorem A.0.7. From Theorem A.0.6 we know that the complex $DR^{\mathcal{E}}_{\mathcal{S}}(F \to X)$, as a complex of vector spaces, is a resolution of the vector space $\mathcal{S}(X, \mathcal{E})$. Hence the complex $DR^{\mathcal{E}'}_{\mathcal{S}}(F \to X)$ is a resolution of $\mathcal{S}(X, \mathcal{E}')$ as a representation of $\mathfrak{g}$.

So it is enough to prove that the representations $\mathcal{S}(F, \mathcal{E}' \boxtimes \Omega^i_N)$ are $\mathfrak{g}$- acyclic. By Lemma A.0.8,

$$\mathcal{S}(F, \mathcal{E} \boxtimes \Omega^i_N) \cong \mathcal{S}(F, \mathcal{E}' \boxtimes \Omega^i_N)$$

as $\mathfrak{g}$-representations (though those isomorphisms do not commute with differentials). Hence it is enough to show that $\mathcal{S}(F, \mathcal{E} \boxtimes \Omega^i_N)$ is acyclic. This follows from Corollary A.0.10. □

Lemma A.0.11 ([AG], Corollary B.1.8)**.** *Let G be a connected affine Nash group, X be a Nash manifold and $\mathcal{E}$ be a Nash bundle over X. Let $\mathfrak{g}$ be the Lie algebra of G. Let $p : G \times X \to X$ be the projection. Let G act on $\mathcal{S}(G \times X, p^?(\mathcal{E}))$ by acting on the G coordinate. Then the map $p_* : \mathcal{S}(G \times X, p^?(\mathcal{E})) \to \mathcal{S}(X, \mathcal{E})$ induces an isomorphism $\mathcal{S}(G \times X, p^?(\mathcal{E}))_{\mathfrak{g}} \cong \mathcal{S}(X, \mathcal{E})$.*

Corollary A.0.12. *Let G be a connected affine Nash group. Let F be a strictly simple Nash G-manifold. Denote $X := G \setminus F$. Let $\mathcal{E} \to F$ be a G-equivariant Nash bundle. Let π denote the projection map $\pi : F \to X$. Let B be a Nash bundle on X such that $\mathcal{E} = \pi^*B$ (B exists by Corollary A.0.3).*

Then the map $\pi_* : \mathcal{S}(F, \mathcal{E} \otimes D^F_X) \to \mathcal{S}(X, B)$ induces an isomorphism $\mathcal{S}(F, \mathcal{E} \otimes D^F_X)_{\mathfrak{g}} \cong \mathcal{S}(X, B)$.

Corollary A.0.13. *Let X be a Nash manifold and Y be a closed Nash submanifold. Let an affine Nash group G act on X strictly simply and H be a subgroup of G that acts on Y strictly simply. Suppose that the natural map $H \setminus Y \to G \setminus X$ is a Nash diffeomorphism. Suppose that G and H are homologically trivial. Let $\mathcal{E}$ be a Nash G-equivariant bundle on X. Then*

$$\mathcal{S}(X, \mathcal{E} \otimes D_X)_{\mathfrak{g}} \cong \mathcal{S}(Y, \mathcal{E}|_Y \otimes D_Y)_{\mathfrak{h}}.$$

Moreover, let $a : G \times Y \to X$ denote the action map and $p_2 : G \times Y \to Y$ denote the projection on the second coordinate. Then
(i) The map a_* gives an isomorphism

$$a_* : \mathcal{S}(G \times Y, (\mathbb{C} \boxtimes \mathcal{E}|_Y) \otimes D_{G\times Y} \otimes a^*(D_X^{-1}))_{\mathfrak{g}\times\mathfrak{h}} \xrightarrow{\sim} \mathcal{S}(X, \mathcal{E})_{\mathfrak{g}}$$

(ii) The map $(p_2)_*$ gives an isomorphism

$$(p_2)_* : \mathcal{S}(G \times Y, (\mathbb{C} \boxtimes \mathcal{E}|_Y) \otimes D_{G\times Y} \otimes a^*(D_X^{-1}))_{\mathfrak{g}\times\mathfrak{h}} \xrightarrow{\sim} \mathcal{S}(Y, \mathcal{E}|_Y \otimes D_Y \otimes ((D_X)|_Y)^{-1})_{\mathfrak{h}}$$

Corollary A.0.14. *Let G be an affine Nash group and X be a transitive Nash G-manifold. Let N be a Nash manifold. Let $x \in X$ and denote $H := G_x$. Suppose that G and H are homologically trivial. Let $X = X \times N \times G$ and $Y = \{x\} \times N \times G$. Let $\mathcal{E}$ be a bundle over X. Let $\mathfrak{X} : G \times N \to \mathbb{R}$ be a Nash map such that for any $n \in N$, the map $\mathfrak{X}|_{G\times\{n\}}$ is a group homomorphism. Let $\mathfrak{X}'$ be the 1-dimensional G-equivariant bundle on N, with the action of G given by $g(n, v) = (n, \psi(\mathfrak{X}(g))v)$. Let $\mathcal{E}' := \mathcal{E} \otimes (\mathbb{C} \boxtimes \mathfrak{X}' \boxtimes \mathbb{C})$. Then*
(i) The map a_* gives an isomorphism

$$a_* : \mathcal{S}(G \times Y, (\mathbb{C} \boxtimes \mathcal{E}'|_Y) \otimes D_{G\times Y} \otimes a^*(D_X^{-1}))_{\mathfrak{g}\times\mathfrak{h}} \cong \mathcal{S}(X, \mathcal{E}')_{\mathfrak{g}}$$

(ii) The map $(p_2)_*$ gives an isomorphism

$$(p_2)_* : \mathcal{S}(G \times Y, (\mathbb{C} \boxtimes \mathcal{E}'|_Y) \otimes D_{G\times Y} \otimes a^*(D_X^{-1}))_{\mathfrak{g}\times\mathfrak{h}} \cong \mathcal{S}(Y, \mathcal{E}'|_Y \otimes D_Y \otimes ((D_X)|_Y)^{-1})_{\mathfrak{h}}$$

Now we are ready to prove the relative Shapiro Lemma.

Proof of Theorem 6.2.1. From Theorem A.0.7 we see that

$$\mathrm{H}_i(\mathfrak{g}, \mathcal{S}(X \times N, \mathcal{E}')) \cong H^{\dim \mathfrak{g}-i}((DR_{\mathcal{S}}^{\mathcal{E}'}(X \times N \times G \to X \times N))_{\mathfrak{g}}).$$

Now, by Corollary A.0.14,

$$\mathcal{S}(X \times N \times G, \Omega^{i,\mathcal{E}'}_{X\times N\times G\to X\times N})_{\mathfrak{g}} \cong \mathcal{S}(\{x\} \times N \times G, \Omega^{i,\mathcal{E}'|_{\{x\}\times N}\otimes D_{\{x\}\times N}\otimes D^{-1}_{X\times N}|_{\{x\}\times N}}_{\{x\}\times N\times G\to\{x\}\times N})_{\mathfrak{h}}.$$

and this isomorphism commutes with de-Rham differential. Therefore

$$(DR_{\mathcal{S}}^{\mathcal{E}'}(X \times N \times G \to X \times N))_{\mathfrak{g}} \cong (DR_{\mathcal{S}}^{\mathcal{E}'|_{\{x\}\times N}\otimes D_{\{x\}\times N}\otimes D^{-1}_{X\times N}|_{\{x\}\times N}}(\{x\} \times N \times G \to \{x\} \times N))_{\mathfrak{h}}$$

and hence

$$H^{\dim \mathfrak{g}-i}((DR_{\mathcal{S}}^{\mathcal{E}'}(X\times N\times G \to X\times N))_{\mathfrak{g}}) \cong H^{\dim \mathfrak{g}-i}((DR_{\mathcal{S}}^{\mathcal{E}'|_{\{x\}\times N}\otimes(D_X^{-1}|_{\{x\}}\boxtimes\mathbb{C})}(\{x\}\times N\times G \to \{x\}\times N))_{\mathfrak{h}})$$

and again by Theorem A.0.7,

$$\begin{aligned} H^{\dim \mathfrak{g}-i}((DR_{\mathcal{S}}^{\mathcal{E}'|_{\{x\}\times N}\otimes(D_X^{-1}|_{\{x\}}\boxtimes\mathbb{C})}(\{x\} \times N \times G \to \{x\} \times N))_{\mathfrak{h}}) \cong \\ \cong \mathrm{H}_i(\mathfrak{h}, \mathcal{S}(\{x\} \times N, \mathcal{E}'|_{\{x\}\times N} \otimes (D_X^{-1}|_{\{x\}} \boxtimes \mathbb{C}))) \cong \\ \cong \mathrm{H}_i(\mathfrak{h}, \mathcal{S}(\{x\} \times N, \mathcal{E}'|_{\{x\}\times N} \otimes \Delta_H \cdot (\Delta_G^{-1})|_H)). \end{aligned}$$

□

Appendix B. Examples for the notation of §§4.3

In order to help the reader to read Notations 4.3.1 and 4.4.2, Lemma 4.3.2 and Corollary 4.3.4, we describe explicitly the objects discussed in them on one example. Let $n = 6$ and $\lambda = (1, 3, 2)$. Then $m_\lambda^1 = 1$, $m_\lambda^2 = 4$, $m_\lambda^3 = 6$. Also, c is the permutation (234561), $w_1 = c^0 = Id_6$, $w_2 = c^2 = (345612)$, $w_3 = c^5 = (612345)$. We have

$\overline{P_n w_1 P_\lambda} = P_n w_1 P_\lambda, \overline{P_n w_2 P_\lambda} = P_n w_2 P_\lambda \cup P_n w_1 P_\lambda$, $\overline{P_n w_3 P_\lambda} = P_n w_1 P_\lambda \cup P_n w_2 P_\lambda \cup P_n w_3 P_\lambda = G_6$, $P_\lambda^1 = P_\lambda$

Typical matrices in P_λ, $\overline{P_n w_1 P_\lambda}$, $\overline{P_n w_2 P_\lambda}$, P_λ^2 are of the forms

$$\begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & 0 & * & * & * \\ * & * & 0 & * & * & * \\ * & * & 0 & * & * & * \end{pmatrix}$$

Typical matrices in P_λ^3, Q_λ^1, Q_λ^2, Q_λ^3 are of the forms

$$\begin{pmatrix} * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ * & * & * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & 0 & * & * & * \\ * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In order to help the reader to read Lemma 4.3.6, we describe explicitly the objects discussed there for $i = 2$. Let $\lambda' = \lambda^-_{3-2+1} = \lambda_2^- = (1, 2, 2)$. Then $w_{\lambda'}^{21}$ is the permutation $(45123), w_{\lambda'}^{22} = Id, w_{\lambda'}^{23} = (34512)$

References

[AG08] A. Aizenbud, D. Gourevitch: *Schwartz functions on Nash Manifolds,* International Mathematics Research Notices, Vol. 2008, n.5, Article ID rnm155, 37 pages. DOI: 10.1093/imrn/rnm155. See also arXiv:0704.2891 [math.AG].

[AG10] A. Aizenbud, D. Gourevitch: *De-Rham theorem and Shapiro lemma for Schwartz functions on Nash manifolds.* Israel Journal of Mathematics, Volume 177, Number 1, 155-188 (2010). See also arXiv:0802.3305v2 [math.AG].

[AG09] A. Aizenbud, D. Gourevitch:*Generalized Harish-Chandra descent, Gelfand pairs and an archimedean analog of Jacquet-Rallis' Theorem,* Duke Mathematical Journal, Volume 149, Number 3,509-567 (2009). See also arXiv: 0812.5063[math.RT].

[AG] A. Aizenbud, D. Gourevitch: *Smooth Transfer of Kloostermann Integrals*, arXiv:1001.2490[math.RT], to appear in American Journal of Mathematics.

[AOS12] A. Aizenbud, O. Offen, E. Sayag: *Disjoint pairs for $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$*, C. R. Math. Acad. Sci. Paris **350**, no. 1-2, pp 911 (2012).

[AGS] A. Aizenbud, D. Gourevitch, S. Sahi: *Derivatives for smooth representations of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$*, arXiv:1109.4374v2.

[BCR98] J. Bochnak, M. Coste, M-F. Roy: *Real Algebraic Geometry* Berlin: Springer, 1998.

[BZ77] I.N. Bernstein, A.V. Zelevinsky: *Induced representations of reductive p-adic groups. I.* Ann. Sci. Ec. Norm. Super, 4^eserie **10**, 441-472 (1977).

[Cas89] W. Casselman: *Canonical extensions of Harish-Chandra modules to representations of G,* Can. J. Math., Vol. XLI, No. 3, pp. 385-438 (1989).

[CHM00] W. Casselman, H. Hecht, D. Miličić: *Bruhat filtrations and Whittaker vectors for real groups.* The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), 151-190, Proc. Sympos. Pure Math., **68**, Amer. Math. Soc., Providence, RI, (2000)

[CM82] W. Casselman, D. Miličić: *Asymptotic behavior of matrix coefficients of admissible representations.* Duke Math. J. Volume **49**, Number 4, 869-930 (1982).

[GM88] S. Gelfand, Y. Manin: *Metody gomologicheskoi algebry. Tom 1. [Methods in homological algebra. Vol. 1] Vvedenie v teoriyu kogomologii i proizvodnye kategorii. [Introduction to the theory of cohomology, and derived categories]*, with an English summary. "Nauka", Moscow, 1988. 416 pp.

[Kos78] B. Kostant: *On Whittaker vectors and representation theory.*, Invent. Math. 48, 101-184 (1978).

[Sah89] S. Sahi: *On Kirillov's conjecture for Archimedean fields,* Compositio Mathematica 72, 67-86 (1989).

[Shi87] M. Shiota: *Nash Manifolds*, Lecture Notes in Mathematics **1269** (1987).

[Wall88] N. Wallach: *Real Reductive groups I*, Pure and Applied Math. **132**, Academic Press, Boston, MA (1988).

[Wall92] N. Wallach: *Real Reductive groups II*, Pure and Applied Math. **132**, Academic Press, Boston, MA (1992).

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