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A MODEL OF A SUDDEN-DEATH FIELD-GOAL FOOTBALL GAME AS A SEQUENTIAL DUEL*

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A highly simplified 'football' game is constructed where individuals can run or kick for a field goal. It represents an attempt to extend the domain of the modeling of duels to a broader category than those previously studied. There are some worthwhile similarities (and important differences) in military, sports and economic tactics which are reasonably well modeled as zero sum games.

The theory of duels for military application has been reasonably well developed but there are at least two other activities beyond war in which the zero-sum game duel is worth considering. They are in sports and in the tactics of advertising campaigns or other economic activities where the budgets of A and B are set and the fight is a non-price duel over a market of fixed size. In this paper we concentrate on the extension of dueling to sports.

Rather than claim immediate relevance to football as it is played, a highly simplified version of 'sudden-death' scoring is introduced. The basic game is as follows. With equal probability the teams toss to see who gets the ball. The team with the ball can either run or try to kick a field goal. The first team to score wins the game.

Key words: Game theory; dueling.

1. Games and duels

In a conventional duel the duelists approach each other with their revolvers, or the two tanks close range. A key element is that both sides have revolvers or other weapons for the offensive. A football game differs inasmuch as there is only one ball. One team or the other always has the initiative. There is only one pistol, a team can close on the target or fire. It may lose the pistol or fire and miss. In either case the other team may get the pistol.

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In tactical fights one side may capture resources of the other and use them. The possession of a key position may give the initiative to one side or the other. Thus although the context of the analysis here is primarily in terms of an extremely simplified version football the type of model appears to be related to duels with potential application to tactical combat. In particular the concept of initiative, often regarded as critical in tactics, emerges from the model.

If a team kicks from point x and fails to score it loses the ball and the other team gets the ball at x.

2. The model

Let the length of the field be 2 and the center be at 0. The goal line for Player A is at 1 and for Player B is at -1.

Let $A(x, y)$ be the probability that A will win given that he has the ball at position x, attempts to run a distance $y-x$ and then kicks from y. We define $B(x, y)$ similarly.

Let $\alpha(t)$ = the probability density of Player A losing the ball after running t.
$\beta(t)$ = the probability density of Player B losing the ball after running t.

In this simplified game, unlike the actual game with the need to run 10 yards in four downs, the optimal running policy is to run a large number of small distances.

Suppose α were the probability of losing the ball in one yard. Keeping the ball is given by $1-\alpha$. If instead the individual tried to gain the yard in k downs his probability of doing so would be: $(1-\alpha/k)^k$. In the limit we obtain $e^{-\alpha}$, and hence the probability of keeping the ball for a distance t is $e^{-\alpha t}$ where $\alpha = \alpha(0)$ the density at the initial point.

Let $a(x)$, $(b(x))$ be the probability that Player A (Player B) misses a kick from position x.

Let $A(x)$ $(B(x))$ be the probability that Player A (B) will win given that they both utilize optimal strategies at point x.

$$A(x) = \max_y A(x, y).$$

We may express $A(x, y)$ as follows:

$$A(x, y) = \mathrm{e}^{\alpha x}\left[\mathrm{e}^{-\alpha y}(1 - aB) + \alpha\int_x^y \mathrm{e}^{\alpha z}(1 - B)\,\mathrm{d}z\right]. \qquad (2.1)$$

The expression in (2.1) contains three parts. They are the chance that Player A succeeds in running y yards, kicks and scores; the chance that he runs the y yards, kicks and misses and the chance that he loses the ball at point z.

Taking the total differential of (2.1) we obtain:

$$\frac{\delta A}{\delta y} = aBe^{\alpha(x-y)}\left(\frac{\alpha a - a' - \alpha}{a} - \frac{B'}{B}\right). \tag{2.2}$$

Let y^* be the optimal distance for Player 1 to run, then y^* must satisfy:

$$\frac{\alpha a - a' - \alpha}{a} = \frac{B'}{B}, \tag{2.3}$$

$$A(x) = e^{\alpha x}\left[e^{-\alpha y}(1 - aB) + \alpha \int_x^y e^{-\alpha z}(1 - B)\,dz\right]. \tag{2.4}$$

Similarly if y^+ is the optimal distance for Player 2 to run then y^+ must satisfy:

$$\frac{\beta - b' - \beta b}{b} = \frac{A'}{A}, \tag{2.5}$$

$$B(x) = e^{-\beta x}\left[e^{\beta y}(1 - Ab) + \beta \int_y^x e^{-\beta z}(1 - A)\,dz\right]. \tag{2.6}$$

2.1. The running only game

Rather than attempt to solve equations (2.3)–(2.6) directly, a simple observation enables us to break the problem up into two parts. We observe that if kicking accuracy improves continuously as a team nears the opponent's goal line then there will be an optimum point at which to kick. Thus the field can be divided into three zones, one at each end where someone kicks and the central area where both sides run.

This clear division into zones would be somewhat complicated in the real game by the differences in points for a touchdown and a field goal as well as problems concerning the amount of time left to play.

We now turn to the simple running game where the winner is the team that first runs the full length of the field.

Equation (2.4) simplifies to:

$$A(x) = \int_x^1 \alpha e^{-\alpha(z-x)}(1 - B)\,dz + e^{-\alpha(1-x)}; \tag{2.7}$$

and similarly from (2.6) we obtain

$$B(x) = \int_{-1}^x \beta e^{-\beta(x-z)}(1 - A)\,dz + e^{-\beta(1+x)}; \tag{2.8}$$

where the first term on the right shows the odds of losing the ball at point z and the second term indicates the odds of winning by running to the opponent's goal line.

Equations (2.7) and (2.8) can be rewritten as follows:

$$e^{-\alpha x}A(x)=e^{-\alpha}+\alpha\int_x^1 e^{-\alpha z}(1-B(z))\,dz, \tag{2.9}$$

$$e^{\beta x}A(x)=e^{-\beta}+\beta\int_{-1}^x e^{\beta z}(1-A(z))\,dz. \tag{2.10}$$

Taking derivatives with respect to x and writing $A(x)$ and $dA(x)/dx$ as A and A' respectively we obtain

$$-\alpha e^{-\alpha x}A+e^{-\alpha x}A'=-\alpha e^{-\alpha x}(1-B), \tag{2.11}$$

$$\beta e^{\beta x}B+e^{\beta x}B'=\beta e^{\beta x}(1-A), \tag{2.12}$$

hence

$$-\alpha A+A'=-\alpha(1-B), \tag{2.13}$$

and

$$\beta B+B'=\beta(1-A), \tag{2.14}$$

hence

$$A'=\alpha(A+B-1) \tag{2.15}$$

and

$$B'=-\beta(A+B-1) \tag{2.16}$$

or

$$(A+B-1)'=(\alpha-\beta)(A+B+1),$$

hence

$$A+B-1=ce^{(\alpha-\beta)x}, \tag{2.17}$$

where c is a constant of integration.

Also from (2.15) and (2.16) we obtain

$$\beta A'+\alpha B'=0, \tag{2.18}$$

hence we can obtain A in terms of B as

$$A=\frac{d-\alpha B}{\beta}, \tag{2.19}$$

where d is a constant of integration.

Let us assume $\alpha>\beta$ (we handle $\alpha=\beta$ later). Substituting in (2.17) we obtain

$$\frac{d-\alpha B}{\beta}+B-1=ce^{(\alpha-\beta)x} \tag{2.20}$$

or

$$B(\beta-\alpha)-\beta+d=c\beta e^{(\alpha-\beta)x} \tag{2.21}$$

hence

$$B=\frac{(\beta-d)-c\beta e^{(\alpha-\beta)x}}{(\beta-\alpha)}=\frac{d-\beta}{\alpha-\beta}+\frac{c\beta e^{(\alpha-\beta)x}}{\alpha-\beta}. \tag{2.22}$$

Substituting for B in (2.19) we have

$$\beta A = d - \alpha B = d - \frac{\alpha}{\alpha-\beta}[(d-\beta) + c\beta e^{(\alpha-\beta)x}] \tag{2.23}$$

which gives

$$A = \frac{1}{\alpha-\beta}[(\alpha-d) - c\alpha e^{(\alpha-\beta)x}]. \tag{2.24}$$

Set $\gamma = \alpha - \beta$.

From (2.24) as $A(1) = 1$

$$1 = \gamma^{-1}[(\alpha-d) - c\alpha e^{\gamma}], \text{ and from (2.22) as } B(-1) = 1, \tag{2.25}$$

$$1 = \gamma^{-1}[(d-\beta) + c\beta e^{-\gamma}], \tag{2.26}$$

adding (2.25) and (2.26)

$$2 = \gamma^{-1}[\gamma - c(\alpha e^{\gamma} - \beta e^{-\gamma})] \tag{2.27}$$

hence:

$$c = \frac{-\gamma}{\alpha e^{\gamma} - \beta e^{-\gamma}}. \tag{2.28}$$

Substituting in (2.24) for c we have:

$$A = \frac{1}{\gamma}\left[(\alpha-d) + \frac{\alpha\gamma}{\alpha e^{\gamma} - \beta e^{-\gamma}} e^{\gamma x}\right], \tag{2.29}$$

which is of the form

$$A = \frac{d^* + \alpha e^{\gamma x}}{\alpha e^{\gamma} - \beta e^{-\gamma}},$$

where d^* is some expression; but as $A(1) = 1$

$$d^* = -\beta e^{-\gamma}, \tag{2.30}$$

hence

$$A(x) = \frac{\alpha e^{\gamma x} - \beta e^{-\gamma}}{\alpha e^{\gamma} - \beta e^{-\gamma}} \tag{2.31}$$

and with some manipulation:

$$B(x) = \frac{\alpha e^{\gamma} - \beta e^{\gamma x}}{\alpha e^{\gamma} - \beta e^{-\gamma}}. \tag{2.32}$$

A sensitivity analysis of (2.31) shows the appropriate properties. It can be seen that as $\alpha \to \infty$ then $A(x) \to 0$ for $0 \le x < 1$, i.e., Player A will lose for every point except $x = 1$.

Suppose $\alpha = \beta$ then (2.31) is indeterminate. Using l'Hôpital's rule and letting α approach β we get

$$\frac{e^{\gamma x}+\alpha x e^{\gamma x}+e^{-\gamma}}{e^{\gamma x}+\alpha e^{\gamma}+\beta e^{-\gamma}} \to \frac{1+\alpha x+\alpha}{1+\alpha+\alpha}. \tag{2.33}$$

Hence for $\alpha=\beta$

$$A(x)=\frac{1+\alpha x+\alpha}{1+2\alpha} \text{ and } B(x)=A(-x)=\frac{1-\alpha x+\alpha}{1+2\alpha}. \tag{2.34}$$

2.2. Running on an asymmetric field

Before we are ready to attempt to solve the model with both running and kicking we need to consider running on an asymmetric field where the payoffs for reaching the end of the field for the players are $A(s)=p$ instead of $A(1)=1$ and $B(t)=q$ instead of $B(-1)=1$ as before.

Equations (2.22) and (2.24) are still valid. Instead of (2.25) and (2.26) we have

$$p=\frac{\alpha-d}{\gamma}-\frac{c\alpha e^{\gamma s}}{\gamma}, \tag{2.35}$$

$$q=\frac{d-\beta}{\gamma}-\frac{c\beta e^{\gamma t}}{\gamma}, \tag{2.36}$$

adding we obtain

$$p+q=1-\frac{c}{\gamma}(\alpha e^{\gamma s}-\beta e^{\gamma t}), \tag{2.37}$$

hence

$$c=\frac{\gamma(1-p-q)}{\alpha e^{\gamma s}-\beta e^{\gamma t}}. \tag{2.38}$$

Substituting (2.38) in (2.24) after some manipulation we have:

$$A(x)=\frac{p(\alpha e^{\gamma x}-\beta e^{-\gamma t})+\alpha(q-1)(e^{\gamma x}-e^{\gamma s})}{\alpha e^{\gamma s}-\beta e^{\gamma t}}. \tag{2.39}$$

Similarly

$$B(x)=\frac{q(\alpha e^{\gamma x}-\beta e^{-\gamma t})+\alpha(p-1)(e^{\gamma x}-e^{\gamma s})}{\alpha e^{\gamma s}-\beta e^{\gamma t}}. \tag{2.40}$$

Setting $p=q=1=s=t$ we see that (2.39) and (2.40) simplify to (2.31) and (2.32).

We are now in a position to make use of (2.39) and (2.40) in order to examine the game with both running and kicking.

2.3. The game with kicking

We extend the above analysis to the symmetric game with kicking. The analysis we present can be extended to the nonsymmetric cases where $\alpha\neq\beta$ and $a(x)\neq b(-x)$.

The calculations are straightforward but unattractive and we do not do them here.

Given $\alpha = \beta$ and $a(x) = b(-x)$ then we may expect that there will be two locations $-s$ and s at which each side will kick. At each point $-s$ and s the probability of success by the side kicking is given by $1 - a(s)$.

In the symmetric game with kicking we may derive simpler expressions for $A(x)$ and $B(x)$ from (2.39) and (2.40) using l'Hôpital's rule as $\alpha \to \beta$ and $\gamma \to 0$. Taking the derivatives of the numerator and denominator of (2.39) with respect to α we obtain:

$$A(x) = \lim_{\alpha \to \beta} \frac{p(e^{\gamma x} + \alpha x e^{\gamma x} - \beta t e^{\gamma t}) + \alpha(p-1)(x e^{\gamma x} - s e^{\gamma s})}{e^{\gamma s} + \alpha s e^{\gamma s} - \beta t e^{\gamma t}}$$

$$= \frac{p(1 + \alpha x + \alpha s) + \alpha(p-1)(x-s)}{1 + \alpha s + \alpha s},$$

noting that $t = -s$.

Hence:

$$A(x) = \frac{p(1 + 2\alpha x) - \alpha x + \alpha s}{1 + 2\alpha s} \tag{2.41}$$

and similarly

$$B(x) = \frac{p(1 - 2\alpha x) + \alpha x + \alpha s}{1 + 2\alpha s}. \tag{2.42}$$

We may check that for $p = 1 = s$ (2.41) yields (2.34).

At the point s we can express the value p by the equation

$$p = 1 - a(s) + a(s)(1 - A(-s)) = 1 - a(s)A(-s), \tag{2.43}$$

where $1 - a(s)$ is the probability of kicking successfully.

In order to minimize notation let $a = a(s)$ and $A = A(-s)$. From (2.41) and (2.43) we obtain:

$$(1 + 2\alpha s)A(x) = (1 - aA)(1 + 2\alpha x) - \alpha x + \alpha s. \tag{2.44}$$

For $x = -s$ we obtain

$$(1 + 2\alpha s)A = (1 - aA)(1 - 2\alpha s) + 2\alpha s, \tag{2.45}$$

$$A(1 + 2\alpha s + a(1 - 2\alpha s)) = 1. \tag{2.46}$$

Let $k = 1 + 2\alpha s + a(1 - 2\alpha s)$ then from (2.46) and (2.44)

$$k(1 + 2\alpha s)A(x) = (k - a)(1 + 2\alpha x) + k(\alpha s - \alpha x).$$

We can determine s from (2.3), the condition is that

$$\frac{B'(s)}{B(s)} = \frac{\alpha a - a' - \alpha}{a} \quad \text{noting } s = y^*. \tag{2.48}$$

On the other hand, by (2.47)

$$B(x)=A(-x)$$

$$=\frac{1}{k(1+2\alpha s)}[(k-a)(1-2\alpha x)+k(\alpha s+\alpha x)]$$

$$=\frac{1}{k(1+2\alpha s)}[k(1+\alpha s-\alpha x)-a(1-2\alpha x)].$$

So

$$B(s)=\frac{1}{k(1+2\alpha s)}[k-a(1-2\alpha x)]=\frac{1}{k(1+2\alpha s)}(1+2\alpha s)=\frac{1}{k}$$

and

$$B'(s)=\frac{1}{k(1+2\alpha s)}[-\alpha k+2a\alpha]=\frac{-\alpha(1-a)(1+2\alpha s)}{k(1+2\alpha s)},$$

therefore

$$\frac{B'(s)}{B(s)}=-\alpha(1-a), \tag{2.49}$$

hence s satisfies

$$\alpha(a-1)=\frac{\alpha(a-1)-a'}{a} \tag{2.50}$$

or

$$\alpha=-a'/(a-1)^2,$$

giving

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{(1-a)}\right]=-\alpha. \tag{2.51}$$

Thus s is the point at which the derivative of the reciprocal of the probability of a successful kick plus the chance of losing the ball at that point equals zero. We suspect that this has a physical meaning which we have not yet understood.

Suppose, for example $a(x)$ was of the form

$$a(x)=\frac{1-x}{r} \quad \text{for } 0\leq x\leq 1-r \tag{2.52}$$

$$=0 \qquad \text{for } x\geq 1-r.$$

This states that if a player is $1-r$ units or further away from the goal line, a kick always fails.

$$\frac{1}{(1-a)} \text{ becomes } \frac{r}{(r-1+x)},$$

hence (2.51) yields:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{r}{r-1+x}\right]=-\alpha, \tag{2.53}$$

which yields:

$$-r/(r-1+x)^2=-\alpha$$

or

$$x=1-r+(r/\alpha)^{1/2}. \tag{2.54}$$

If the probability of being able to kick a field goal were zero only from midfield then (2.54) would simplify to:

$$x=(1/\alpha)^{1/2} \tag{2.55}$$

and more generally the condition that $a(x)=1-x^n$ for $0\leq x\leq 1$ gives

$$x=(n/\alpha)^{1/(n+1)}. \tag{2.56}$$

An examination of (2.56) shows that for any α, as n is increased the closer to midfield is the point of optimal kicking.

A natural question to ask is if running takes time, what is the expected length of the game until a score? The simplest conditions would be a running time proportional to distance and no time for kicking.

3. Discussion

The models presented here represent only a starting point in the mathematical inquiry into some aspects of football. There are a series of questions which are suggested. We begin with the questions that our simple models answer.

A1. In the running-only game the value of the toss to Player A is given by:

$$A(0)-\hat{A}(0)=\tfrac{1}{2}A(0)+\tfrac{1}{2}(1-B(0))=\tfrac{1}{2}\left[\frac{1+\alpha+\beta-(\alpha e^{\gamma}+\beta e^{-\gamma})}{\alpha e^{\gamma}-\beta e^{-\gamma}}\right].$$

With $\alpha=\beta$ we obtain

$$A(0)-\hat{A}(0)=\frac{1+\alpha}{1+2\alpha}-\frac{1}{2}=\frac{1}{2(1+2\alpha)}.$$

Setting $\alpha=0$ the sensitivity analysis indicates that the whole game depends upon the toss, as the team with the ball wins.

A2. There is an optimal point for each team to kick which depends only upon each team's own kicking and running abilities. The points for the symmetric game are given by the condition

$$\frac{d}{dx}\left[\frac{1}{1-a}\right]=-\alpha.$$

It must be noted that the distributions $\alpha(t)$ and $\beta(t)$ really are not independent, but are of the form of: 'All other things being equal the probability that team A will lose the ball to team B if it tries to run distance t is given by $\alpha(t)$.

A3. Using the results of 2.2 we can calculate the value of the toss as a function of both running and kicking ability.

Some questions which require further complication of the model, but appear to be within the range of analysis are as follows.

Suppose the game lasts for K minutes and after each score play starts again at center field. Furthermore assume that it requires $(k_1 t)$ units of time to run distance t and k_2 units of time to kick.

Q1. How does the optimal strategy depend upon the time left to play?

As a good first approximation we have modeled the contest as a zero-sum game. In such a model, winning, maximizing score and maximizing the difference in scores appear to be the same.

Q2. At what level of model complication is this equivalence no longer true? Can we describe the sensitivity of the optimal strategies to these different goals?

Q3. Can we characterize the optimal running policy in the simple running game with four downs to gain 10 yards?

The next layer of complication calls for the introduction of kicking and passing to gain yards. There appears to be a large number of models which would serve to make more precise the nature of the strategic choice and the odds of success in football and other two team sports.

The theory of games should be broadly applicable to competitive sports much in the same way as it is relevant to military tactical doctrine and weapons evaluation. In each instance items such as morale, terrain and special detail all count. But the presence of these specific factors are complementary with the game theoretic analysis and not substitutes for the strategic insights which may be obtained from the stripped down abstract models.

4. Extensions

A considerable step towards realism would be to add the possibility of the forward pass to the model. At the expense of introducing an extra decision variable for each player and a fair amount of computation this can be handled. But a somewhat more difficult conceptual and empirical problem appears both in military and sports applications. This is the 'breaking point' phenomenon. The probability densities for running with the ball or for completing a pass are not exogenous (except as a very rough first approximation). They depend not only on the tactical behavior of the other team but also on the morale of the team. In military history it is well known that casualties occur heavily at the end of a battle after a loss in morale and breakdown of effective resistance of one side. In sports the breaking point phenomenon does not appear to be as well recognized, possibly it is less pronounced.

References

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