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Analysis in matrix space and Speh's representations

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Introduction

The irreducible unitary representations of $GL(n, \mathbb{R})$ have been classified by D. Vogan in [V2]. In his work, four families of representations play a special role. They are (i) the one dimensional unitary representations and (ii) their complementary series; (iii) Speh's representations and (iv) their complementary series. These representations serve as "building blocks" in the sense that every irreducible unitary representation is obtained, via unitary parabolic induction, from a tensor product of these representations.

Vogan's techniques and results are completely algebraic in nature; indeed they represent a considerable triumph for the algebraic method. What Vogan actually classifies are the irreducible, admissible $(\mathfrak{g}, K)$-modules which possess an invariant, positive-definite Hermitian form. However, by a well known theorem of Harish-Chandra [H], every such module is the space of K-finite vectors of a unique irreducible unitary representation.

For many purposes, the $(\mathfrak{g}, K)$-module is an adequate substitute for the unitary representation itself. However, for certain other problems, especially those of an analytic nature, it is important to have an explicit construction of the unitary representation together with its Hilbert space. For representations of type (i), this is a trivial problem. For those of type (ii), there is an analytic construction of the type carried out in [St] for the case of $GL(n, \mathbb{C})$; similar results are true for any local field. An alternative, uniform approach to this problem is described in [S2].

This brings us to Speh's representations which are the subject of this paper. They occur for each $GL(n, \mathbb{R})$ with n even; and for each such group there is, up to tensoring by a one-dimensional unitary representation, exactly one Speh representation for each positive integer m.

The existence of these representations was conjectured as early as 1956 by Gelfand and Graev in [GG]; and they are mentioned again in [B]. However, all

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attempts to construct these representations were unsuccessful until B. Speh [Sp] showed that they occur in the residual spectrum of $L^2(G/\Gamma)$ for suitable arithmetic subgroups Γ. They may also be constructed by the cohomological induction procedure (a.k.a. the derived functor construction) of Zuckerman (see [V1]), and then their unitarizability is a consequence of more general unitarizability results for this procedure. (See §4 for further discussion.)

All of the existing approaches are purely algebraic and construct only a unitarizable $(\mathfrak{g}, K)$-module.

In this paper we give a new construction and proof of unitarity for Speh's representations. Our construction differs from earlier realizations in that we give explicit Hilbert spaces on which the representations act unitarily. To describe our result it is convenient to introduce some notation.

We will write M_n for the vector space of $n \times n$ real matrices, and M_n^+ (respectively M_n^-) for the set of matrices with nonnegative (resectively nonpositive) determinant. Also we will write G_n for $GL(n, \mathbb{R})$. For any square matrix x, we will write $|x|$ for $|\det x|$, $\varepsilon(x)$ for sign $(\det x)$ and $\chi(x)$ for $\exp(i\,\mathrm{trace}(x))$.

This notation will be used without further comment in the rest of the paper.

The pairing $(x, \xi) \mapsto \chi(x\xi)$ allows us to identify M_n with its unitary dual and to define the (additive) Fourier transform $\mathscr{F}$ by

$$\mathscr{F}f(\xi) = \int_{M_n} \chi(-x\xi) f(x)\,dx \tag{0}$$

For each $m \in \mathbb{Z}_+$, we define

$$H_m \equiv H_m(n) \equiv L^2(M_n, |\xi|^{-m} d\xi) \tag{1}$$

and

$$H_m^\pm \equiv \{\varphi \in H_m | \operatorname{supp}(\varphi) \subset M_n^\pm\} \tag{2}$$

Then H_m and $H_m^\pm$ are Hilbert spaces with respect to the norm

$$\langle \varphi, \psi \rangle = \int \varphi \bar{\psi} |\xi|^{-m} d\xi\,. \tag{3}$$

Furthermore, it is easily checked that functions in H_m are tempered distributions on M_n. Consequently, we may define

$$\check{H}_m = \{\mathscr{F}^{-1}\varphi | \varphi \in H_m\} \tag{4}$$

and

$$\check{H}_m^\pm = \{\mathscr{F}^{-1}\varphi | \varphi \in H_m^\pm\} \tag{5}$$

as spaces of tempered distributions. Let us equip $\check{H}_m$ and $\check{H}_m^\pm$ with the norm

$$\langle \mathscr{F}^{-1}\varphi, \mathscr{F}^{-1}\psi \rangle = \langle \varphi, \psi \rangle\,. \tag{6}$$

Then $\check{H}_m$ (resp. $\check{H}_m^\pm$) becomes a Hilbert space isometric (via $\mathscr{F}$) to H_m (resp. $H_m^\pm$).

We show in this paper (Theorem 1) that the m-th Speh representation for G_{2n} is realized naturally on $\check{H}_m$. Furthermore, the C^∞-vectors of the representation consist of certain functions in $C^\infty(M_n)$, on which the action of G_{2n} is by fractional linear transformations.

The explicit description allows us to prove Kirillov's conjecture for these representations. This conjecture [K] says that every irreducible unitary representation of $GL(n, \mathbb{R})$ remains irreducible upon restriction to the subgroup P_n consisting of matrices whose last row is $(0, \ldots, 0, 1)$.

In fact we prove much more. Theorem 2 contains a stronger version of this conjecture and shows that these representations stay irreducible even when we restrict them to $G_n \ltimes M_n$. Theorem 3 is a "finer" version of this conjecture and in it we compute the "adduced" representation (see [S1] and [S2]) of a Speh representation. We show that it is the Speh representation of G_{2n-2} with the same parameter. Finally, in Theorem 4 we show that the restriction to $SL(2n, \mathbb{R})$ of the m-th Speh representation is the sum of two irreducible pieces realized on the spaces $\check{H}_m^+$ and $\check{H}_m^-$.

Our construction depends on the possibility of realizing Speh's representations as subrepresentations of certain degenerate principal series for G_{2n} in a manner analogous to the classical imbedding of the discrete series inside the non-unitary principal series for $GL(2, \mathbb{R})$ (see [BSS]). The key step in our approach is an inductive argument, similar to [S2], which reduces everything to the classical situation.

The organization of this paper is as follows. In Section 1 we introduce most of the notation and state the main results. In Section 2, we recall the relevant results for $GL(2, \mathbb{R})$; and in Section 3 we present the inductive argument. Finally, in Section 4, we discuss the applications to Kirillov's conjecture, describe the restriction to $SL(2n, \mathbb{R})$, and also bring out the connection between our construction and the derived functor realization.

§1. Degenerate principal series for $GL(2n, \mathbb{R})$

Let us write G_{2n} as 2×2 matrices of $n \times n$ blocks, writing a typical element g as

$$g = \begin{bmatrix} a & c \\ b & d \end{bmatrix}; \quad a, b, c, d \in M_n\,. \tag{7}$$

Also, let us write

$$\begin{gathered} Q = Q_n = \left\{ \begin{bmatrix} a & c \\ 0 & d \end{bmatrix} \right\}, \quad \bar{Q} = \bar{Q}_n = \left\{ \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} \right\}, \\ N = N_n = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \right\}, \quad R = R_n = \left\{ \begin{bmatrix} a & c \\ 0 & 1 \end{bmatrix} \right\}, \end{gathered} \tag{8}$$

for the indicated subgroups of G_{2n}.

The degenerate principal series of interest to us is induced from one dimensional characters of $\bar{Q}$ as follows:

For each $m \in \mathbb{Z}_+$, let L_m denote the character

$$L_m\left(\begin{bmatrix} a & 0 \\ b & d \end{bmatrix} \right) = \varepsilon(d)^{m+1} |a|^{-(n+m)/2} |d|^{(n+m)/2} \tag{9}$$

and write $I_m \equiv I_m(n)$ for the space of smooth vectors of the representation $\mathrm{Ind}_{\bar{Q}}^{G_{2n}}(L_m)$.

In other words,

$$I_m = \{F \in C^\infty(G_{2n}): F(\bar{q}g) = L_m(\bar{q})F(g), \quad \text{for } \bar{q} \in \bar{Q}\} . \tag{10}$$

We will write π_m for the representation of G_{2n} on I_m by right translations.

Now, by the Gelfand-Naimark decomposition, functions in I_m are determined by their restriction to $N \approx M_n$. For $F \in I_m$, let F_N denote this restriction, then for g as in (7), we have

$$(\pi_m(g)F)_N(x) = L_m\left(\begin{bmatrix} a' & 0 \\ b' & d' \end{bmatrix}\right)F_N(x') \quad \text{a.e} . \tag{11}$$

where a', d', x' are obtained by solving

$$\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a' & 0 \\ b' & d' \end{bmatrix}\begin{bmatrix} 1 & x' \\ 0 & 1 \end{bmatrix} . \tag{12}$$

This gives $a' = (a + xb)$, $x' = (a + xb)^{-1}(c + xd)$ and $\det d' = (\det g)(\det a')^{-1} = (\det g)\det(a + xb)^{-1}$.

Consequently (11) holds for all x such that $\det(a + xb) \neq 0$ and the action is

$$\begin{aligned}\pi_m\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right)F_N(x) &= \varepsilon(d')^{m+1}|a'|^{-(n+m)/2}|d'|^{(n+m)/2}F_N(x') \\ &= \varepsilon(g)^{m+1}\varepsilon(a + xb)^{m+1}|g|^{(n+m)/2}|a + xb|^{-(n+m)} \\ &\quad \times F_N((a + xb)^{-1}(c + xd))\end{aligned} \tag{13}$$

The next lemma allows us to characterize those functions on M_n which are restrictions of functions in I_m.

Lemma 1. *A smooth function f on M_n is of the form F_N for a (necessarily unique) function F in I_m if, for each g as in* (7), *the function* $x \mapsto L_m\left(\begin{bmatrix} a' & 0 \\ b' & d' \end{bmatrix}\right)f(x')$ *(defined initially for those x for which* $\det(a + xb) \neq 0$*) extends to a smooth function on M_n.*

Proof. Let $U = N \cdot \bar{Q}$. Then U is open and dense in $G = G_{2n}$, and f extends to a smooth function F_e on U defined by $F_e(x\bar{q}) = L_m(\bar{q})f(x)$. The assumption of the lemma implies that for each $g \in G$, there is a smooth function F_g defined on $U_g \equiv Ug$ which agrees with F_e on $U_g \cap U$. For g and h in G, F_g and F_h agree with F_e on $U_g \cap U_h \cap U$; but this set is dense in $U_g \cap U_h$, so F_g and F_h must agree on $U_g \cap U_h$. It follows that the functions F_g have a common (smooth) extension F to all of G. □

The Iwasawa decomposition on G implies that $G = K \cdot \bar{Q}$. Consequently a function in I_m is also determined by its values on K. An easy calculation (which we omit) shows readily that for $F \in I_m$,

$$|F_N(x)| \leqq A|1 + x^t x|^{-(n+m)/2} \tag{14}$$

where $A = \sup\{|F(k)|; k \in K\}$.

Let us remark that $|1 + x^t x| \geqq 1$ for all x in M_n. The basic estimate for this function is the following:

Lemma 2.

$$\int_{M_n} |1 + x^t x|^{-s} dx < \infty \quad \textit{if} \quad s > n - 1/2 \, .$$

Proof. This follows easily by transferring the integral to G_n and using the Cartan (KA_+K) decomposition (see [Hu] p. 63). □

Corollary. *If $F \in I_m$ for some $m(\geqq 0)$, then F_N is in $L^2(M_n)$.*

Let us write

$$S_m = S_m(n) = \{F \in I_m | \mathscr{F}(F_N) \in H_m\} \, . \tag{15}$$

We first show that S_m is not zero. To see this, let us write $\partial_{ij} = \partial/\partial x_{ij}$ and $\square = \det(\partial_{ij})$. Now specializing (13) we see that

$$\left(\pi_m\left(\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\right) F\right)_N (x) = F_N(x + c) \, . \tag{16}$$

Thus the Lie algebra $\mathfrak{n}$ of N acts by differential operators; and we can find an element X_m in its enveloping algebra $\mathscr{U}(\mathfrak{n})$ such that

$$(\pi_m(X_m)F)_N = \square^m F_N \, . \tag{17}$$

Lemma 3. *$\pi_m(X_m)$ is an injective map from I_m into S_m.*

Proof. Suppose F is a nonzero function in I_m, then $\pi_m(X_m)F$ belongs to I_m as well. Now by Lemma 2 and (17), F_N and $\square^m F_N$ are both in $L^2(M_n)$. Taking the Fourier transform we get.

$$\mathscr{F}(\square^m F_N) = i^{mn}(\det \xi)^m \mathscr{F}(F_N) \quad \text{a.e} \, . \tag{18}$$

which shows that $\square^m F_N \neq 0$.

It remains only to check that $\pi_m(X_m)F$ is in S_m. To see this, we calculate

$$\begin{aligned} \int |\mathscr{F}((\pi_m(X_m)F)_N)|^2 |\xi|^{-m} d\xi &= \int |\mathscr{F}(\square^m F_N)| \, |\mathscr{F}(F_N)| d\xi \\ &\leqq \|\mathscr{F}(\square^m F_N)\|_{L^2} \|\mathscr{F}(F_N)\|_{L^2} \\ &= \|\square^m F_N\|_{L^2} \|F_N\|_{L^2} < \infty \, . \quad \square \end{aligned}$$

Recalling the definition of $\check{H}^m$ from (4), we see that the map $F \mapsto F_N$ gives us an imbedding of S_m inside $\check{H}_m$. In what follows, we will implicitly identify S_m with its image in $\check{H}_m$, and write $\langle , \rangle$ for the inner product on $\check{H}_m$ restricted to S_m.

Lemma 4. *S_m is a Q-invariant subspace of (π_m, I_m) and Q preserves the inner product $\langle , \rangle$.*

Proof. First of all, observe that if $q = \begin{bmatrix} a & c \\ 0 & d \end{bmatrix}$, then by (13)

$$\pi_m(q) f(x) = \varepsilon(d)^{m+1} |a|^{-(n+m)/2} |d|^{(n+m)/2} f(a^{-1}(c + xd)) \, .$$

Taking the Fourier transform gives

$$\mathscr{F}((\pi_m(q))f)(\xi) = \chi(cd^{-1}\xi)\varepsilon(d)^{m+1}|a|^{(n-m)/2}|d|^{(m-n)/2}\mathscr{F}f(d^{-1}\xi a) \, . \tag{19}$$

Now the Lemma follows by an easy calculation. □

As the reader may have observed, Lemma 4 has very little to do with the fact that m is an integer! Our main result is that for $m \in \mathbb{Z}_+$ we have

Theorem 1. *S_m is G_{2n}-invariant and G_{2n} preserves the inner product $\langle , \rangle$.*

This is well known for $m = 0$, and will be proved in Section 3 for $m \geqq 1$. We devote the rest of this section to various consequences of this result.

First of all, let us write $\bar{S}_m$ for the closure of S_m in $\check{H}_m$. Then by a well known theorem of Harish-Chandra (see [H]), the representation (π_m, S_m) extends to a unitary representation δ_m on the Hilbert space $\bar{S}_m$.

Theorem 2. (a) $\bar{S}_m = \check{H}_m$.

(b) *$(\delta_m, \check{H}_m)$ is irreducible, even upon restriction to R (see (8)).*

(c) *$(\delta_m, \check{H}_m)$ is equivalent to the m-th Speh representation of G_{2n}.*

We shall prove this in a moment; but first we deduce a corollary. Let P_m be the subgroup of G_m consisting of those matrices whose last row is $(0, 0, \ldots, 1)$. Then Kirillov's conjecture asserts that any irreducible unitary representation of G_m stays irreducible upon restriction to P_m. Since $P_{2n} \supseteq R$, we have the following

Corollary. *Speh's representations satisfy Kirillov's conjecture.*

In §4, we will prove a "finer" version of this corollary and calculate the "adduced" representation of a Speh representation (see [S1] and Theorem 3 below).

We conclude this section with the

Proof of Theorem 2. Let us write A_m for the operator on M_n given by

$$A_m \varphi(\xi) = |\xi|^{m/2} \varphi(\xi) .$$

Then $A_m \mathscr{F}$ gives an isometric imbedding of S_m into $L^2(M_n)$; and if we write $\hat{\pi}_m(g)$ for the operator on $A_m \mathscr{F}(S_m)$ given by

$$\hat{\pi}_m(g) f = (A_m \mathscr{F}) \pi_m(g) (A_m \mathscr{F})^{-1} f ,$$

then, specializing (19) to $\begin{bmatrix} a & c \\ 0 & 1 \end{bmatrix} \in R$, we have

$$\hat{\pi}_m\left(\begin{bmatrix} a & c \\ 0 & 1 \end{bmatrix}\right) f(\xi) = \chi(c\xi) |a|^{n/2} f(a\xi) . \tag{20}$$

But it is a classical fact, (first observed by Gelfand and Naimark) that the right-hand side of (20) gives an *irreducible* representation of R on $L^2(M_n)$. Indeed any operator commuting with the action of $\left\{\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\right\}$ consists of multiplication by a bounded, Borel function. The condition of commuting with $\left\{\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right\}$ implies that this function is a constant (a.e.).

Consequently, $A_m \mathscr{F}(\bar{S}_m) = L^2(M_n)$, and so $\bar{S}_m = \check{H}_m$. This proves part (a) and (b).

Part (c) is a simple consequence of the results in [BSS]. Let $W_m \subseteq V_m$ be the Harish-Chandra modules of S_m and I_m. In [BSS] all the composition factors of V_m are computed. In particular it is shown that the Harish-Chandra module Z_m of the m-th Speh representation is a submodule of V_m.

Now if X_m is as in Lemma 3, then $\pi_m(X_m)Z_m \subseteq W_m$; consequently $Z_m \cap W_m \neq 0$. But by part (b), W_m is irreducible; so it must be equal to Z_m. □

§2. Discrete series for $GL(2, \mathbb{R})$

The material of this section is classical. We merely record it here in a form suitable for use in Section 3.

Proposition 0. *Let I_m and S_m be as in Section 2, for $GL(2, \mathbb{R})$* (i.e., *the case $n = 1$*), *and let*

$$S_m^{\pm} = \{ f \in S_m \mid \hat{f}(\xi) \equiv 0 \text{ on } \mathbb{R}^{\mp} \} .$$

Then

(1) $S_m = S_m^+ \oplus S_m^-$;

(2) *S_m^+ and S_m^- are $SL(2, \mathbb{R})$-invariant subspaces;*

(3) *$\int_0^\infty |\hat{f}(\xi)|^2 \xi^{-m} d\xi$ (respectively $(-1)^m \int_{-\infty}^0 |\hat{f}(\xi)|^2 \xi^{-m} d\xi$) is a positive definite, $SL(2, \mathbb{R})$-invariant inner product on S_m^+ (resp. S_m^-).*

Proof. This is essentially contained in Chapter VII of [GGV]. □

Corollary. *Theorem 1 holds for $n = 1$.*

Proof. The above proposition shows that the norm $\int_{-\infty}^\infty |\hat{f}|^2 |\xi|^{-m} d\xi$ is $SL(2, \mathbb{R})$-invariant. Since $SL(2, \mathbb{R})$ and Q generate $GL(2, \mathbb{R})$, the result follows from Lemma 4. □

§3. Proof of the main theorem

We shall prove Theorem 1 by induction on n; that is, we shall assume the result for n and prove it for $n + 1$. This means that we will need to use the notation of section 1 for both n and $n + 1$. To forestall confusion, we adopt the following conventions:

Notation 1. a) We will write π_m (resp. π_m^-) for the action of G_{2n+2} (resp. G_{2n}) on $I_m(n + 1)$ (resp. $I_m(n)$).

b) The norms on $S_m(n + 1)$ and $S_m(n)$ will be denoted by $\| \quad \|$ and $\| \quad \|_-$, respectively.

c) x, y, z, t will denote typical elements of the spaces M_n, $\mathbb{R}^{n \times 1}$, $\mathbb{R}^{1 \times n}$ and $\mathbb{R}$ respectively; ξ, η, ζ, τ will denote elements of their dual spaces M_n, $\mathbb{R}^{1 \times n}$, $\mathbb{R}^{n \times 1}$, $\mathbb{R}$. Thus $\begin{bmatrix} x & z \\ y & t \end{bmatrix}$ is a typical element of M_{n+1} and $\begin{bmatrix} \xi & \eta \\ \zeta & \tau \end{bmatrix}$ is a typical element of its dual space. We will write $\mathscr{M}(x, y, \zeta, \tau)$, $\mathscr{M}(\xi, y, \zeta, \tau)$, etc., for the spaces of all measurable functions of the indicated variables.

d) Matrices in M_{2n+2} will be decomposed into blocks in accordance with the decomposition $\mathbb{R}^{2n+2} = \mathbb{R}^n \oplus \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}$. Thus a typical element of M_{2n+2} is written as

$$\begin{array}{c} \\ n \\ 1 \\ n \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} n & 1 & n & 1 \end{array} \\ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array}$$

We define the following subgroups of G_{2n+2}

$$P_{2n+2} = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \quad G_{2n+1} = \left\{ \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\},$$

$$V_{2n+1} = \left\{ \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\},$$

$$P_{2n+1} = \left\{ \begin{bmatrix} * & * & * & 0 \\ 0 & 1 & 0 & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \quad G_{2n} = \left\{ \begin{bmatrix} * & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \tag{21}$$

$$V_{2n} = \left\{ \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}.$$

Then $P_{2n+2} \approx G_{2n+1} \ltimes V_{2n+1}$ and $P_{2n+1} \approx G_{2n} \ltimes V_{2n}$.

Let V^*_{2n+1} be the group of unitary characters of V_{2n+1}. Then $V^*_{2n+1} \approx \mathbb{R}^{2n+1}$ and under the action of P_{2n+2}, there are exactly two orbits: a) the trivial orbit, and *b*) $\mathcal{O}$ = everything else.

Let $\omega_0 \in \mathcal{O}$ denote the character

$$\omega_0 \left(\begin{bmatrix} 1 & 0 & 0 & j \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \chi(k) \equiv e^{ik} . \tag{22}$$

Then the stabilizer of ω_0 is $P_{2n+1} \ltimes V_{2n+1}$, where these groups are as in (21).

By the inductive hypothesis, we have the representation $\delta_m(n)$ of $GL(2n, \mathbb{R})$ as described in Theorem 2. Using this we construct a unitary representation τ_m of P_{2n+2} as follows:

First extend $\delta_m(n)$ (trivially on V_{2n}) to a representation $E\delta_m(n)$ of P_{2n+1}; and then define the representation $\tau_m = IE\delta_m(n)$ to be

$$\mathrm{Ind}_{P_{2n+1} \ltimes V_{2n+1}}^{P_{2n+2}} (E\delta_m(n) \otimes \omega_0) . \tag{23}$$

Now, by Mackey theory, this is an irreducible unitary representation of P_{2n+2} on the space $L^2(\mathcal{O}; \check{H}_m(n)) \approx L^2(V_{2n+1}^*; \check{H}_m(n))$.

We wish to write down explicit formulas for τ_m on a certain dense subspace $\mathscr{S}_m$ of $L^2(V_{2n+1}^*; \check{H}_m(n))$.

Definition 1. A subset Ω of $\mathbb{R}^k$ will be called a *full set* if it is open and dense, and its complement has Lebesgue measure zero.

Examples. If p is a polynomial on $\mathbb{R}^k$, then the set $\{x \mid p(x) \neq 0\}$ is a full set. Also, the intersection of a finite number of full sets is a full set.

Let us identify V_{2n+1}^* with $\mathbb{R}^n \oplus \mathbb{R} \oplus \mathbb{R}^n$ by

$$(y, \tau, \zeta) \mapsto \omega \text{ where } \omega\left(\begin{bmatrix} 1 & 0 & 0 & j \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = \chi(yj + \tau k + \zeta l) . \tag{24}$$

Definition 2. Let $\mathscr{S}_m$ be the space of $S_m(n)$-valued measurable functions $f = f(x, y, \zeta, \tau)$ on V_{2n+1}^* such that

a) There is a full set $V(f) \subseteq V_{2n+1}^*$ on which f is continuous.
b) $\int \| f(\cdot, y, \zeta, \tau)\|_-^2 \, dy \, d\zeta \, d\tau < \infty$

Let B_m be the operator acting on $\mathscr{M}(\xi, y, \zeta, \tau)$ by the formula

$$B_m \varphi(\xi, y, \zeta, \tau) = |\xi|^{-m/2} \varphi(\xi, y, \zeta, \tau) ; \tag{25}$$

and let $B_m^\vee$ be the operator acting on $\mathscr{M}(\xi, \eta, \zeta, \tau)$ by the formula

$$B_m^\vee \varphi(\xi, \eta, \zeta, \tau) = |\xi|^{-m/2} \varphi(\xi, \eta, \zeta, \tau) . \tag{26}$$

Then we may rewrite b) as

b′) $\int |B_m \mathscr{F}_x f|^2 d\xi \, dy \, d\zeta \, d\tau < \infty$

where $\mathscr{F}_x$ is the Fourier transform in the x-variables.

Now the action τ_m is given as follows:
If

$$p = \begin{bmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \varphi \in \mathscr{S}_m \tag{27}$$

then

$$\tau_m(p)\varphi(x, y, \zeta, \tau) = |p|^{1/2}\chi(-k'')|a''|^{-(n+m)/2}\varepsilon(i'')^{m+1}|i''|^{(n+m)/2}\varphi(x'', y'', \zeta'', \tau'') \tag{28}$$

where

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \zeta & \tau & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & d & g & i \\ b & e & h & k \\ c & f & i & l \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \tilde{a} & \tilde{d} & \tilde{g} & \tilde{j} \\ 0 & 1 & 0 & k'' \\ \tilde{c} & \tilde{f} & \tilde{i} & \tilde{l} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \zeta'' & \tau'' & y'' & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

and

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{a} & \tilde{g} \\ \tilde{c} & \tilde{i} \end{bmatrix} = \begin{bmatrix} a'' & 0 \\ c'' & i'' \end{bmatrix} \begin{bmatrix} 1 & x'' \\ 0 & 1 \end{bmatrix}. \tag{30}$$

The action given by formulas (28)–(30) extends also to a larger space of functions.

Definition 3. Let $\mathscr{I}'_m$ be the space of those functions f in $\mathscr{M}(x, y, \zeta, \tau)$ for which there is a full set $U(f) \subseteq M_n \times V^*_{2n+1}$ such that $f | U(f)$ is continuous.

Then $\mathscr{I}'_m$ is stable under the action described by formulas (28)–(30). In fact, if p is as in (27), let us define

$$U^p = \{(x, y, \zeta, \tau) | \tau \neq 0, \tau'' \neq 0, \det(a'') \neq 0\} \tag{31}$$

where τ'' and a'' are as in (29) and (30). Then we may choose

$$U(\tau_m(p)f) = U(f) \cap U^p. \tag{32}$$

The key ingredient of the inductive argument is the "change-of-variables" operator C_m acting on $\mathscr{M}(x, y, \zeta, \tau)$ by the formula

$$C_m f(x, y, \zeta, \tau) = |\tau|^{-(n+m)/2} f\left(\begin{bmatrix} 1 & 0 \\ \zeta & \tau \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}, \zeta, \tau\right). \tag{33}$$

We also define $C_m^\vee$, which acts on $\mathscr{M}(\xi, \eta, \zeta, \tau)$ by

$$C_m^\vee f(\xi, \eta, \zeta, \tau) = |\tau|^{(n-m)/2} f\left([\xi\ \eta] \begin{bmatrix} 1 & 0 \\ \zeta & \tau \end{bmatrix}, \zeta, \tau\right). \tag{34}$$

Let us write T_m for the operator $C_m \mathscr{F}_{zt}$, where $\mathscr{F}_{zt}$ is the partial Fourier transform in the z, t variables.

Proposition 1. *Suppose $F \in I_m(n+1)$, then $T_m F \in \mathscr{I}'_m$; and for all $p \in P_{2n+2}$*

$$T_m \pi_m(p) F = \tau_m(p) T_m F \quad \text{a.e.} \tag{35}$$

Proof. Start with $F \in I_m(n+1)$; note that by (14) and the elementary identity

$$\det\left(1 + \begin{bmatrix} x & z \\ y & t \end{bmatrix}^t \begin{bmatrix} x & z \\ y & t \end{bmatrix}\right) \geqq 1 + \|z\|^2 + t^2$$

we get that $F(x, y, \cdot, \cdot) \in L^1(dz\,dt)$ for each x and y. Then making the change of variables implied by C_m shows that we may take $U(T_m F) = M_n \times \mathcal{O}'$, where $\mathcal{O}' = \{(y, \zeta, \tau) | \tau \neq 0\} \leqq \mathcal{O}$.

The proof of (35) is identical with the argument in the appendix of [S2]. The following comments may help to clarify the situation. The right-hand side of (35) is made up of the operator $\mathscr{F}_{zt}$ (acting on F) followed by change-of-variables and multiplication operators. However as far as the (ζ, τ) variables are concerned, this change is linear (see (23) in [S2]). Now in the left-hand side of (35), the operator $\pi_m(p)$ consists effectively in a change of variables and multiplication operator, where the (z, t) variables are changed in an affine manner (the linear part is dual to the transformation in the (ζ, τ) variables), and the translation contributing the character factor $\chi(-k)$ in the definition of τ_m. Note of course that $F \in I_m(n+1)$ implies that $F \in L^2(M_{n+1})$, (see Lemma 2). □

The "almost everywhere" part of Proposition 1 can be sharpened considerably. In fact, let U^p be as in (31), then

Corollary. *With the hypotheses of Proposition* 1,

$$T_m \pi_m(p) F(x, y, \zeta, \tau) = \tau_m(p) T_m F(x, y, \zeta, \tau) \tag{36}$$

for all $(x, y, \zeta, \tau) \in U^p$.

Proof. Since $F \in I_m(n+1)$, $\pi_m(p) F \in I_m(n+1)$ as well. Now by (32) and the proof of Proposition 1, both sides of (36) are continuous functions on U^p. Since they agree almost everywhere, they must be equal on U^p. □

Proposition 2. *Suppose* $F \in I_m(n+1)$. *Then, for each* (y', ζ', τ') *in* $\mathcal{O}'$,

$$(T_m F)(\cdot, y', \zeta', \tau') \in I_m(n) . \tag{37}$$

Proof. Let us fix $(y', \zeta', \tau') \in \mathcal{O}'$, and let ψ denote the function

$$\psi(x) = (T_m F)(x, y', \zeta', \tau') .$$

In view of Lemma 1 it suffices to check that $\pi_m^-(\gamma)\psi$ is in $C^\infty(M_n)$ for all $\gamma = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ in G_{2n} (where π_m^- is the representation of G_{2n} on $I_m(n)$, as in Notation 1). Let

$$p = \begin{bmatrix} a & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \zeta' & \tau' & y' & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{38}$$

and consider (29) with $(y, \zeta, \tau) = (0, 0, 1)$. Replacing the second matrix in the left-hand side of (29) by (38), we see that $\tau'' = \tau'$, $\tilde{a} = a$ and $\tilde{c} = c$. Then (30) gives $a'' = (a + xc)$. Consequently for p as in (38), we see that $U^p \supset \{(x, 0, 0, 1) | \det(a + xc) \neq 0\}$. Thus, by the Corollary to Proposition 1,

$$T_m \pi_m(p) F(x, 0, 0, 1) = \tau_m(p) T_m F(x, 0, 0, 1) \text{ for all } x \text{ with } \det(a + xc) \neq 0 . \tag{39}$$

Now the formula for τ_m implies that

$$\tau_m(p) T_m F(\cdot, 0, 0, 1) = |p|^{1/2} \pi_m^-(\gamma)\psi . \tag{40}$$

Replacing $\pi_m(p)F$ by F, it suffices to show that for each $F \in I_m(n+1)$, the function $x \mapsto T_m F(x, 0, 0, 1)$ is a smooth function. But by (33),

$$T_m F(x, 0, 0, 1) = \int F(x, 0, z, t) dz dt \tag{41}$$

Now if $\mathscr{D}$ is any constant coefficient differential operator in the x variables, then $\mathscr{D}$ belongs to $\pi_m(\mathscr{U}(\mathfrak{n}))$, where $\mathfrak{n}$ is the Lie algebra of $N = M_{n+1}$; hence $\mathscr{D}F(x, 0, \cdot, \cdot) \in L^1(dz dt)$. Thus, the result follows by differentiating under the integral in (41). □

Let us write D for the operator $C_0^\vee$. Then D acts on $\mathscr{M}(\xi, \eta, \zeta, \tau)$ by the formula

$$D\varphi(\xi, \eta, \zeta, \tau) = |\tau|^{n/2} \varphi\left([\xi \ \eta] \begin{bmatrix} 1 & 0 \\ \zeta & \tau \end{bmatrix}, \zeta, \tau \right). \tag{42}$$

Also let us write A_m for the operator on $\mathscr{M}(\xi, \eta, \zeta, \tau)$ given by

$$A_m \varphi\left(\begin{bmatrix} \xi & \eta \\ \zeta & \tau \end{bmatrix} \right) = \begin{vmatrix} \xi & \eta \\ \zeta & \tau \end{vmatrix}^{-m/2} \left(\begin{bmatrix} \xi & \eta \\ \zeta & \tau \end{bmatrix} \right). \tag{43}$$

Recall that $\mathscr{F}_{zt}$ is the partial Fourier transform in the z, t variables. The function $\mathscr{F}_{zt} f(x, y, \zeta, \tau)$ can be viewed in two equivalent ways (when $f \in L^2(M_{n+1})$). The first is as a function $(x, y) \mapsto L^2(\mathbb{R}^{n \times 1} \times \mathbb{R})$, which is defined for almost every (x, y); the second is as a function $(\zeta, \tau) \mapsto L^2(M_n \times \mathbb{R}^{n \times 1})$, defined for almost all (ζ, τ). The partial Fourier transform $\mathscr{F}_{xy}$ can similarly be defined in two equivalent senses. If we use the second definition of $\mathscr{F}_{zt} f$ (as a function of (ζ, τ)) and the first definition of $\mathscr{F}_{xy} g$ (as a function of (ξ, η)), then we get that $\mathscr{F}(f) = \mathscr{F}_{xy}(\mathscr{F}_{zt} f)$ almost everywhere. Below it will also be necessary to decompose $\mathscr{F}_{xy}$ as $\mathscr{F}_y \mathscr{F}_x$ in a similar fashion.

Proposition 3. *Suppose $f \in L^2(dx dy d\zeta d\tau)$, then*

$$B_m^\vee \mathscr{F}_{xy} C_m \mathscr{F}_{zt} f = D A_m \mathscr{F} f \quad \text{a.e.} \tag{44}$$

Proof. The right-hand side is of course defined almost everywhere. As far as the left-hand side is concerned, $\mathscr{F}_{zt} f$ is in $L^2(dx dy)$ for almost all ζ, τ, and so $C_m \mathscr{F}_{zt} f$ is in $L^2(dx dy)$ for almost all ζ, τ; hence $\mathscr{F}_{xy}$ acts on this function for those ζ, τ, giving for those ζ, τ an element in $L^2(d\xi d\eta)$. Applying the change of variables and multiplication operators (34) and (26), shows that the left side of (44) is defined almost everywhere. To prove the resulting identity, note that $\mathscr{F}_{xy} C_m = C_m^\vee \mathscr{F}_{xy}$ (see (33) and (34)).

Therefore (44) follows because first, $\mathscr{F}_{xy} \mathscr{F}_{zt} = \mathscr{F}$; and second, by comparing the definitions (26), (34), (42) and (43) we see that $B_m^\vee C_m^\vee = D A_m$. □

Proposition 4. *Suppose $F \in I_m(n+1)$. Then $F \in S_m(n+1)$ if and only if $T_m F(\cdot, y, \zeta, \tau) \in S_m(n)$, for almost all (y, ζ, τ), and*

$$\int \| T_m F(\cdot, y, \zeta, \tau) \|_-^2 \, dy d\zeta d\tau < \infty \ . \tag{45}$$

Proof. Recall that according to the definition of $S_m(n+1)$, $F(x, y, z, t) \in S_m(n+1)$, if $F \in I_m(n+1)$, and $A_m \mathscr{F} F \in L^2(M_{n+1})$. Similarly, given $G(x, y, \zeta, \tau)$, then for each

(y, ζ, τ), $G(\cdot, y, \zeta, \tau) \in S_m(n)$ if and only if $G(\cdot, y, \zeta, \tau) \in I_m(n)$, and $B_m \mathscr{F}_x G(\cdot, y, \zeta, \tau) \in L^2(\xi)$. Indeed

$$\| G(\cdot, y, \zeta, \tau) \|_-^2 = \int | B_m \mathscr{F}_x G(\xi, y, \zeta, \tau)|^2 d\xi \ . \tag{46}$$

We shall need the following simple variant of (44) which holds for $F \in S_m(n+1)$.

$$B_m \mathscr{F}_x T_m F = \mathscr{F}_y^{-1} D A_m \mathscr{F} F \ . \tag{47}$$

In fact when $F \in S_m(n+1)$, we have seen that $A_m \mathscr{F} F$ belongs to $L^2(M_{n+1})$, and D^{-1} and $\mathscr{F}_y^{-1}$ are unitary operators (respectively from $L^2(\xi, \eta, \zeta, \tau)$ to $L^2(\xi, \eta, \zeta, \tau)$ to $L^2(\xi, y, \zeta, \tau)$). Thus the right side of (47) is well defined for a.e. (ξ, y, ζ, τ). For the left-side we observe that, by Proposition 2, $T_m F \in L^2(x)$ for almost every (y, ζ, τ) and so (47) is again defined for a.e. (ξ, y, ζ, τ). The identity (47) then follows from (44) by observing that $\mathscr{F}_{xy} = \mathscr{F}_y \mathscr{F}_x$, and $B_m^\vee \mathscr{F}_y = \mathscr{F}_y B_m$. Once (47) is established we see that if $F \in S_m(n+1)$, then

$$\int | B_m \mathscr{F}_x T_m F(\xi, y, \zeta, \tau)|^2 \, d\xi dy d\zeta d\tau < \infty \ . \tag{48}$$

but since $(T_m F)(y, \zeta, \tau) \in I_m(n)$ for a.e. (y, ζ, τ) we get (45).

The converse follows in the same way, proving the proposition. □

Corollary. *If $F \in S_m(n+1)$, then $T_m F \in \mathscr{S}_m$ and*

$$\| F \|^2 = \int \| T_m F \|_-^2 \, dy d\zeta d\tau \ . \tag{49}$$

This follows immediately from the proof of the proposition.

Proof of Theorem 1. We proceed by induction on n. The case $n = 1$ was treated in Section 2. For $n+1$ (with $n \geqq 1$) we observe that Q_{2n+2} and P_{2n+2} generate G_{2n+2}. So, by Lemma 4, it suffices to show that if $p \in P_{2n+2}$ and $F \in S_m(n+1)$, then $\pi_m(p) F \in S_m(n+1)$, and $\| \pi_m(p) F \| = \| F \|$.

Now if $F \in S_m(n+1)$, then $F \in I_m(n+1)$ and so by the Corollary to Proposition 1

$$T_m \pi_m(p) F(\cdot, y, \zeta, \tau) = \tau_m(p) T_m F(\cdot, y, \zeta, \tau) \quad \text{a.e.}$$

for a.e. (y, ζ, τ). However $T_m F(\cdot, y, \zeta, \tau)$ belongs to $S_m(n)$ for a.e. (y, ζ, τ), so, for a.e. (y, ζ, τ), $\tau_m(p) T_m F$ also belongs to $S_m(n)$, by the inductive step. Hence $T_m \pi_m(p) F$ belongs to $S_m(n)$ for a.e. (y, ζ, τ). However $\pi_m(p) F$ is in $I_m(n+1)$ for all $p \in P$, since $F \in I_m(n+1)$. Thus by Proposition 4, $\pi_m(p) F \in S_m(n+1)$, as soon as we see that $\int \| T_m \pi_m(p) F \|_-^2 \, dy d\zeta d\tau < \infty$. This is equivalent with

$$\int \| \tau_m(p) T_m F \|_-^2 \, dy d\zeta d\tau = \int \| T_m F \|_-^2 \, dy d\zeta d\tau < \infty$$

(by the inductive step), and the second integral is finite, again, by Proposition 4. Thus if $F \in S_m(n+1)$, then $\pi(p) F \in S_m(n+1)$. The fact that $\| \pi(p) F \| = \| F \|$ follows immediately from Corollary to Proposition 4 (identity (49)). □

§4. Concluding remarks

For the first result of this section we recall some ideas from [S1].

As before, let P_n be the group $G_{n-1} \ltimes \mathbb{R}^{n-1}$. Then if $\hat{P}_n$ is the unitary dual of P_n, then by Mackey theory

$$\hat{P}_n \approx \hat{G}_{n-1} \amalg \hat{G}_{n-2} \amalg \ldots \amalg \hat{G}_0 \tag{50}$$

A unitary representation ρ of G_n is called *adducible* of *depth* k, if $\rho|P_n$ corresponds to a representation σ of G_{n-k}. Also, σ is called the *adduced* representation of ρ, and is denoted by $A\rho$.

Now the proof of Theorem 1 shows that

$$\delta_m(n+1)|P_{2n+2} \approx \tau_m = IE\delta_m(n)\,.$$

This means that the adduced representation of $\delta_m(n+1)$ is $\delta_m(n)$. Thus we have

Theorem 3. *The m-th Speh representation $\delta_m(n+1)$ of G_{2n+2} is adducible, of depth 2; and*

$$A\delta_m(n+1) = \delta_m(n)\,.$$

The next objective of this section is to prove the following result.

Theorem 4. *Let $(\delta_m, \check{H}_m)$ be the m-th Speh representation of $GL(2n, \mathbb{R})$ (realized as in Theorem 2). Then $\delta_m|SL(2n, \mathbb{R})$ is a direct sum of two irreducible representations $\delta_m^{\pm}$, realized on $\check{H}_m^{\pm}$.*

Proof. The proof of part b) in Theorem 2 shows that for R as in (8), $\delta_m|(SL(2n, \mathbb{R}) \cap R)$ decomposes exactly as asserted. It remains only to show that $\delta_m|SL(2n, \mathbb{R})$ is reducible. Let W_m be the Harish-Chandra module of $(\delta_m, \check{H}_m)$ and let $\mathfrak{g}'$ be the Lie algebra of $SL(2n, \mathbb{R})$; then it suffices to prove that $W_m|\mathfrak{g}'$ is reducible. Now it is quite easy to see this for the derived functor realization, but to keep our development self-contained, we give an alternative proof.

We claim that it suffices to produce a nonzero K-finite function φ in S_m such that

$$\operatorname{supp}(\mathscr{F}\varphi) \subseteq M_n^{\pm}\,.$$

For then, since (13) shows that the Lie algebra $\mathfrak{g}'$ acts by (polynomial coefficient) differential operators, it follows that the set $W_m^+ \equiv \{\varphi \in W_m | \operatorname{supp}(\mathscr{F}\varphi) \subseteq M_n^+\}$ is a $\mathfrak{g}'$-submodule of W_m.

It is not too hard to write down all the K-finite functions in S_m. In particular, the functions

$$\varphi_1(x) = |1 + x^t x|^{-(n+1)/2} \quad \text{and} \quad \varphi_2(x) = \det(x)|1 + x^t x|^{-(n+1)/2}$$

are in W_0; and if $\Box$ is as in (17), then $\Box^m(\varphi_1)$ and $\Box^m(\varphi_2)$ are in W_m.

Now the Fourier transforms of φ_1 and φ_2 were computed by C. Herz ([He]) in 1954! It follows from his work (formulas 5.2 and 5.7) that

$$\mathscr{F}\varphi_1(\xi) = i^n \varepsilon(\xi)\mathscr{F}\varphi_2(\xi)$$

Consequently, if $\varphi_{\pm} = \varphi_1 \pm i^n \varphi_2$ then $\operatorname{supp}(\Box^m \varphi_{\pm}) = \operatorname{supp}(\varphi_{\pm}) \subseteq M_n^{\pm}$. $\Box$

We would like to close this section with some remarks which may serve to clarify the relation between our construction and the derived functor realization of Speh's representations.

Let $\mathscr{G}_n^{2n}(\mathbb{C})$ be the Grassmannian of complex n-planes in $\mathbb{C}^{2n}$. Under the action of $GL(2n, \mathbb{R})$ there is an open (and dense) orbit $X \approx GL(2n, \mathbb{R})/GL(n, \mathbb{C})$, and also a closed orbit $GL(2n, \mathbb{R})/Q \approx \mathcal{G}_n^{2n}(\mathbb{R})$, where Q is a maximal parabolic $\approx (GL(n, \mathbb{R}) \times GL(n, \mathbb{R})) \rtimes M_n(\mathbb{R})$.

Let us write χ_m for the character of $GL(n, \mathbb{C})$ given by $\chi_m(g) = ((\det g)/|g|)^m$. Then χ_m can be used to define a line bundle λ_m on $X \simeq GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.

According to the "philosophy" of cohomological induction (see [VI]), the m-th Speh representation should be realized on the (Dolbeault) cohomology of λ_m in degree $n(n-1)/2$ which is half the (real) dimension of the compact, complex subvariety $(O(2n)/U(n))$.

If this could be done rigorously, then one ought also to be able to define the "boundary values" of such cohomology classes to be certain hyperfunction sections of a (corresponding) line bundle on $\mathcal{G}_n^{2n}(\mathbb{R})$. In particular, for C^∞-vectors in the Speh representation, one should get C^∞-sections of the bundle L_m, in other words, functions in I_m.

Now using the "flat" description of $\mathcal{G}_n^{2n}(\mathbb{R})$ (i.e., the full set $M_n(\mathbb{R})$) one ought to be able to obtain the connection with the spaces H_m and $\check{H}_m$. In fact, one should be able to define the analogue of the Fourier-Laplace transform from the spaces $\check{H}_m^{\pm}$ into the Dolbeault cohomology groups in the appropriate degree.

The program just described makes perfect sense in much greater generality. We intend to take it up in a future paper.

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