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A Strategic Market Game with Complete Markets*

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An exchange economy is modeled as a strategic market game with all pairwise markets available. Existence of noncooperative equilibria is proved. It is shown that if resources are distributed in a skewed manner, in equilibrium prices may not satisfy the no arbitrage condition. One round of trade even with all goods serving as money is not sufficient to provide enough liquidity for efficient trade *Journal of Economic Literature* Classification Numbers: 021, 026, 311.

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1. The Economic Model and Its Interest

An economic process approach to the study of exchange is by means of a strategic market game (see Shubik [8]; Shapley, [6]; Shapley and Shubik, [7]; Dubey and Shubik, [3]). Price is formed by the simultaneous actions of all agents. One of the simplest models distinguishes one of m commodities as a money. Then $m-1$ markets are considered in which the money can be exchanged directly for one of the other commodities. The direct exchange of other commodities for each other is ruled out. In contrast, when all commodities can be exchanged directly, then for m commodities there are $m(m-1)/2$ markets instead of $m-1$.

A simple geometric representation of trade and markets can be given. Let goods be points and markets be arcs connecting them. Figure 1a shows the market structure for an exchange economy with four goods where the fourth good acts as a money. Figure 1b shows the structure with all markets. Under this description a money is a good which can be exchanged directly for all other goods. In an economy with complete markets all goods are monies.

Suppose that there are m goods and that the mth serves as a money. Dubey and Shubik [3] considered a strategic market game where the strategy of a player α was of the form

$$(q_1^\alpha, ..., q_{m-1}^\alpha; b_1^\alpha, ..., b_{m-1}^\alpha),$$

where $0 \leqslant q_j^\alpha \leqslant \alpha_j^\alpha$ for $\alpha = 1, ..., n$, $j = 1, ..., m-1$, where n is the number of traders and a_j^α is the initial endowment of individual α of good j

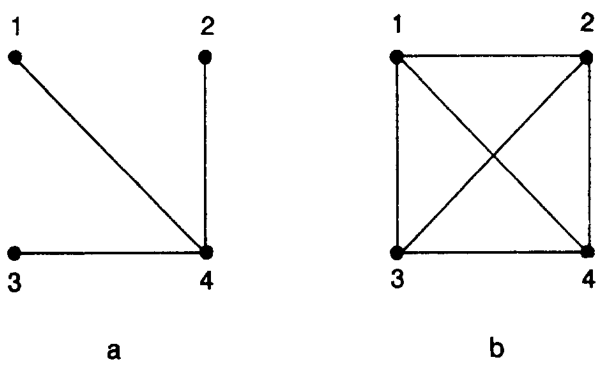

FIGURE 1

$(j=1, \ldots, m-1)$; q_j^α is the amount of good j offered for sale by individual α and

$$b_j^\alpha \geqslant 0 \qquad \text{and} \qquad \sum_{j=1}^{m-1} b_j^\alpha \leqslant a_m^\alpha,$$

where b_j^α is the amount of money committed by trader α to purchase good j and a_m^α is the amount of money in the initial endowment of α. In this game, as can be seen from the bounds on the strategies, there is no trading on credit.

Price (in terms of the money) is formed for good j by dividing the amount of money committed to j by the amount of j placed on sale:

$$p_j = \begin{cases} \sum_{\alpha=1}^{n} b_j^\alpha \Big/ \sum_{\alpha=1}^{n} q_j^\alpha & \text{if } \sum_{\alpha=1}^{n} q_j^\alpha \neq 0 \\ 0 & \text{if } \sum_{\alpha=1}^{n} q_j^\alpha = 0. \end{cases}$$

The second line in the above definition is just one of several possible trading conventions.

The type of price formation mechanism offered by Dubey and Shubik [3] is a symmetric two sided analogue of the Cournot market mechanism. It is possible to define this mechanism for an exchange economy with complete markets. Here a strategy for an individual will have a dimension of $m(m-1)$. There are $m(m-1)/2$ markets and an individual can be on either or both sides of each market. For example in the apples for oranges market an individual could supply both apples and oranges. In the stock market, when an individual simultaneously buys and sells the same shares, this is known as a wash sale.

For the strategic market game with m goods, $m-1$ markets, and one good serving as a money, Dubey and Shubik [3] were able to prove the existence of a pure strategy noncooperative equilibrium (NE)[1] where each player could be on both sides in each market.

In this paper we have extended the model with $(m-1)$ markets to one with $m(m-1)/2$ and have been able to prove the existence of an NE with active trade in all markets when there are at least two j-furnished individuals for every good j (no monopolists) and at least two individuals who have positive marginal utility for j. Our approach remains valid when the strategic exchange economy is endowed with any number of markets between $(m-1)$ and $m(m-1)/2$. In particular, our existence results (Section 2) subsume those in Shapley and Shubik [7] and Dubey and Shubik [3].

[1] We use the abbreviation NE to stand for a pure strategy noncooperative equilibrium.

2. Description of the Model and Existence Results

Let $I_n = \{1, 2, ..., n\}$ and $I_m = \{1, 2, ..., m\}$ be the sets of traders and commodities, respectively. We shall use Greek letters for traders and Roman letters for commodities.

We assume that each trader α has an initial endowment $a_i^\alpha > 0$ of each commodity i. The traders' utility functions are assumed to be strictly concave, increasing, and continuously differentiable functions $u^\alpha : \Omega^m \to \Omega^1$, $\alpha \in I_n$, where Ω^m, Ω^1 are the nonnegative orthants in R^m and R^1, respectively.[2]

A pair $\{i, j\} \subseteq I_m$ is to be thought of as a market between commodities i and j.

A move by a trader α is an $m \times m$ matrix B^α such that:

(i) $b_{ij}^\alpha \geqslant 0$,

(ii) $\sum_j b_{ij}^\alpha \leqslant a_i^\alpha$,

where b_{ij}^α is the amount of commodity i that α sends to trade in the market $\{i, j\}$. We write S^α for the set of possible moves (strategy set) by trader α, and let

$$S = S^1 \times \cdots \times S^n.$$

We shall denote by Γ the game in which the outcome of moves

$$B = (B^1, ..., B^n) \in S$$

is determined as follows:

First, the price matrix P is given by

$$p_{ij} = \begin{cases} \sum_\alpha b_{ij}^\alpha \Big/ \left(\sum_\alpha b_{ji}^\alpha\right) & \text{if } \sum_\alpha b_{ji}^\alpha \neq 0 \\ 0 & \text{if } \sum_\alpha b_{ji}^\alpha = 0. \end{cases} \tag{1}$$

Next, the final holdings are computed by

$$x_i^\alpha = a_i^\alpha - \sum_j b_{ij}^\alpha + \sum_j b_{ji}^\alpha p_{ij}. \tag{2}$$

[2] We may drop the differentiability requirement on the utility functions and only insist that they be continuous. This would entail replacing the infinitesimal argument in the proof of Theorem 1(b) by a more careful "finite" argument as in [1], using the "uniform monotonicity" lemma from [4].

Finally, the payoffs to the traders are

$$\pi^\alpha(B) = u^\alpha(x^\alpha). \tag{3}$$

Given a choice of moves

$$B^{-\alpha} \in S^1 \times \cdots \times S^{\alpha-1} \times S^{\alpha+1} \times \cdots \times S^n$$

by all the traders *except* α, we say that $B^\alpha \in S^\alpha$ is a "best response" by α to $B^{-\alpha}$ if

$$\pi^\alpha(B^{-\alpha}, B^\alpha) = \sup_{T \in S^\alpha} \pi^\alpha(B^{-\alpha}, T).$$

A Nash Equilibrium (NE) of Γ is an n-tuple

$$B = (B^1, ..., B^n) \in S$$

such that each B^α is a best response to $(B^1, ..., B^{\alpha-1}, B^{\alpha+1}, ..., B^n)$.

Nash equilibria of Γ exist for trivial reasons. For instance, the set of moves where $B^\alpha = 0$ for each α clearly constitutes an (autarky) Nash equilibrium; see Peck and Shell [5] for further discussion. To get around this, we consider (as in Dubey and Shubik [3]) a slight modification of Γ:

For each $m \times m$ matrix $\varepsilon = (\varepsilon_{ij})$, $0 < \varepsilon_{ij} \leqslant 1$, we consider the game Γ_ε, where some external agency supplies the fixed amounts ε_{ij} in the various markets, so that the prices are now given by

$$p_{ij} = \left(\sum_\alpha b^\alpha_{ij} + \varepsilon_{ij}\right) \Big/ \left(\sum_\alpha b^\alpha_{ji} + \varepsilon_{ji}\right)$$

and the other computations are as before.

We define an *equilibrium point* (EP) of Γ to be a pair (B, P) such that

(1) B is an NE of Γ

(2) $p_{ij} > 0$ for all i, j

(3) there is a sequence of matrices $\varepsilon_l \to 0$ and NEs $B(\varepsilon_l)$ of $\Gamma(\varepsilon_l)$ with prices $P(\varepsilon_l)$ such that $B(\varepsilon_l) \to B$ and $P(\varepsilon_l) \to P$.

By ${}^k\Gamma_\varepsilon$ (respectively ${}^k\Gamma$) we shall understand the k-fold replication of Γ_ε (respectively Γ) in which each trader is replaced by k copies of himself. A type-symmetric Nash equilibrium (TSNE) is an NE wherein identical players employ identical strategies. A type-symmetric equilibrium point TSEP is a TSNE of ${}^k\Gamma$ which is the limit in the above sense of a sequence of TSNEs of ${}^k\Gamma_{\varepsilon_l}$.

We say that a good is *active* at an equilibrium if some trader offers a non-zero amount of that good in exchange for some other good.

The first theorem asserts

THEOREM 1. *For each* k, ${}^k\Gamma$ *has a TSEP with at least* $m-1$ *goods active. Furthermore there is a constant* D (independent *of* k) *such that if* (B, P) *is* any *TSEP of* ${}^k\Gamma$, *then*

$$D \geqslant p_{ij} \geqslant 1/D \qquad \text{for all} \quad i, j.$$

We start with a lemma which clarifies the underlying geometry of the model.

LEMMA 1. *In the game* Γ_ε, *fix a choice of strategies by all traders except* α; *and let* H *and* Π *denote, respectively, the set of possible final holdings by* α *and the prices which form as* α *varies his strategy in* S^α. *Then*

(a) *If* P *and* Q *are in* Π, *so is* R, *where* $r_{ij} = (p_{ij} q_{ij})^{1/2}$ *for all* i, j.

(b) H *is compact; and if* x *and* y *are in* H, *then there is a* z *in* H *such that*

$$z_i \geqslant \tfrac{1}{2}(x_i + y_i) \qquad \text{for all} \quad i.$$

Proof. Let $c_{ij} = \sum_{\beta \neq \alpha} b_{ij}^\beta + \varepsilon_{ij}$. Then if α makes the move $T = (t_{ij}) \in S^\alpha$, the prices will be

$$p_{ij} = (c_{ij} + t_{ij})/(c_{ji} + t_{ji}) \tag{4}$$

and his final holding will be

$$\begin{aligned} x_i &= a_i^\alpha - \sum_j t_{ij} + \sum_j t_{ji} p_{ij} \\ &= a_i^\alpha + \sum_j (t_{ji} p_{ij} - t_{ij}) \\ &= a_i^\alpha + \sum_j (c_{ij} - c_{ji} p_{ij}) \qquad \text{(by (4))} \\ &= a_i^\alpha + \sum_j c_{ij} - \sum_j c_{ji} p_{ij}. \end{aligned} \tag{5}$$

Now, suppose strategies T and U in S^α achieve x and y in H with prices P and Q in Π. Let

$$r_{ij} = (p_{ij} q_{ij})^{1/2} \qquad \text{and} \qquad v_{ij} = [(c_{ij} + t_{ij})(c_{ij} + u_{ij})]^{1/2} - c_{ij}.$$

Then

$$\begin{aligned} (c_{ij} + v_{ij})/(c_{ji} + v_{ji}) &= [(c_{ij} + t_{ij})(c_{ij} + u_{ij})]^{1/2} / [(c_{ji} + t_{ji})(c_{ji} + u_{ji})]^{1/2} \\ &= (p_{ij} q_{ij})^{1/2} = r_{ij}. \end{aligned}$$

So to complete the proof of part(a), it remains only to show that $V \in S^\alpha$; and the only condition to be checked is that

$$\sum_j v_{ij} \leqslant a_i^\alpha.$$

But

$$\begin{aligned}
\sum_j v_{ij} &= \sum_j (c_{ij} + v_{ij}) - \sum_j c_{ij} \\
&= \sum_j [(c_{ij} + t_{ij})(c_{ij} + u_{ij})]^{1/2} - \sum_j c_{ij} \\
&\leqslant \left[\sum_j (c_{ij} + t_{ij})\right]^{1/2} \left[\sum_j (c_{ij} + u_{ij})\right]^{1/2} - \sum_j c_{ij} \\
&\leqslant \max\left\{\sum_j (c_{ij} + t_{ij}), \sum_j (c_{ij} + u_{ij})\right\} - \sum_j c_{ij} \\
&= \max\left\{\sum_j t_{ij}, \sum_j u_{ij}\right\} \\
&\leqslant a_i^\alpha,
\end{aligned}$$

where the first inequality is to Cauchy–Schwartz inequality; and the last inequality holds since T and U lie in S^α.

Part (b) is a simple consequence of part (a) and the arithmetic–geometric mean inequality.

Thus, if V is as before and z is the final holding resulting from V, we have

$$\begin{aligned}
z_i &= a_i^\alpha + \sum_j c_{ij} - \sum_j c_{ji} r_{ij} \qquad \text{(by (5))} \\
&= a_i^\alpha + \sum_j c_{ij} - \sum_j c_{ji} (p_{ij} q_{ij})^{1/2} \\
&\geqslant a_i^\alpha + \sum_j c_{ij} - \sum_j c_{ji} [1/2(p_{ij} + q_{ij})] \\
&= \frac{1}{2}\left[a_i^\alpha + \sum_j c_{ij} - \sum_j c_{ji} p_{ij}\right] + \frac{1}{2}\left[a_i^\alpha + \sum_j c_{ij} - \sum_j c_{ji} q_{ij}\right] \\
&= \frac{1}{2}(x_i + y_i).
\end{aligned}$$

Finally, since S^α is compact and the map from S^α to H is clearly continuous, H must be compact. Q.E.D.

The next step is

LEMMA 2. *For each* $\varepsilon > 0$, Γ_ε *has an* NE.

Proof. Given a fixed set of moves by the other traders, since u^α is strictly concave and continuous, there exists a unique $x \in H$ which maximizes u^α.

It is easily seen that there must be a *unique* price P associated with x. For if Q is also associated with x and $q_{ij} \neq p_{ij}$ for some i, j, then write $r_{ij} = (p_{ij} q_{ij})^{1/2}$ and let z be the final holding corresponding to R. Then by the analysis in the proof of part (b) of Lemma 1 we see that $z \geqslant x$; and by the *strict* artihmetic–geometric mean inequality for (p_{ij}, q_{ij}) in that proof, it follows that

$$z_i > x_i.$$

This contradicts the maximality of x.

Consider all the (best) responses by α which yield the price matrix P. This is clearly a compact, convex, and non-empty set. By the Theorem of the Maximum in Berge [1], it follows that the best response correspondence is upper semi-continuous from

$$S^1 \times \cdots S^{\alpha-1} \times S^{\alpha+1} \times \cdots \times S^n \qquad \text{to subsets of } S^\alpha.$$

As before, let $S = S^1 \times \cdots \times S^n$ and consider $\Phi: S \to$ subsets of S given as follows: If $B = (B^1, ..., B^n)$, then $\Phi(B) = \{(T^1, ..., T^n)$: each T^α is a best response by α to $(B^1, ..., B^{\alpha-1}, B^{\alpha+1}, ..., B^n)\}$.

Then Φ is upper semi-continuous, S is compact and convex, and $\Phi(B)$ is compact, convex, and not empty for each $B \in S$. By Kakutani's Theorem, Φ has a fixed point, which is easily seen to be a Nash Equilibrium for Γ_ε.

Q.E.D.

Lemma 2 can be refined to yield

LEMMA 3. *For each* $k = 1, 2, 3, ...$, *the game* ${}^k\Gamma_\varepsilon$ *has a TSNE.*

Proof. Let ${}^kS$ be the Cartesian product of the strategy sets of all the traders in ${}^k\Gamma_\varepsilon$; and let ${}^kS^*$ $(\approx S)$ be the set of type-symmetric strategies in ${}^kS$. Then it is clear that for each $B \in {}^kS^*$, the set $\Phi^*(B) = \Phi(B) \cap {}^kS^*$ is compact, convex and non-empty.

Let Φ^* be the correspondence $B \to \Phi^*(B)$. Then Φ^* is upper semi-continuous; and by Kakutani's theorem, it has a fixed point, which is easily seen to be a TSNE of ${}^k\Gamma_\varepsilon$. Q.E.D.

This can be strengthened to

LEMMA 4. *For each k, ${}^k\Gamma_\varepsilon$ has a TSNE with at least $m-1$ goods active.*

Proof. Given a fixed choice of moves by all traders except one, consider the optimization problem faced by that reader. Let C be the aggregate bid matrix of the other traders (plus ε). Then if the remaining trader makes the move T, the resulting price is P where

$$p_{ij} = (c_{ij} + t_{ij})/(c_{ji} + t_{ji}) \tag{6}$$

Since the best response price is unique, there is a unique T which satisfies (6) for this price, and also the condition $t_{ij} \cdot t_{ji} = 0$ for all i, j. Let us call this *the* "minimal best response." By the theorem of the maximum in Berge [1], the minimal best response map is a continuous function from ${}^kS$ to ${}^kS$.

We can also define the "maximal best response" (with respect to a fixed ordering of the goods) as follows: Let T be a minimal best response by a trader to moves by the others. Suppose that at T, the trader has non-zero amounts of at least two commodities left over after his move. Choose $\{i, j\}$ to be the smallest such pair (with respect to the ordering), and let the trader increase his bids on *both* sides of the (i, j)th market in such a manner that p_{ij} remains unchanged. This can be done until the trader has used up all of at least one of the two commodities i and j. Now let him select the next lowest pair and proceed in this fashion until he reaches a strategy T' in which he has used up all of at least $m-1$ of his goods. Since T' also gives the best response price, it is a best response, which we can call the "maximal best response."

Let Φ': ${}^kS \to {}^kS$ be the maximal best response map; then Φ' is easily seen to be continuous. Furthermore, Φ' maps ${}^kS^*$ (indeed all of ${}^kS$) into ${}^kS^*$. By Brouwer's Theorem, Φ' has a fixed point on ${}^kS^*$.

Since we assumed $a_i^\alpha > 0$ for all α and i, any fixed point of Φ' has the property that *each* trader sends a positive amount (a_i^α) of *at least* $m-1$ goods to the market. So the fixed points of Φ' are TSNEs of the desired kind. Q.E.D.

We proceed next to the proof of boundedness of prices. In the proof of the next lemma and subsequent discussion, we shall abuse the notation slightly and use the index α to represent both a trader and a trader type in ${}^k\Gamma_\varepsilon$.

LEMMA 5. *There is a constant $D \geq 1$ (independent of k and ε) such that if P is the price matrix at* any *TSEP of any ${}^k\Gamma_\varepsilon$ (with $k = 1, 2, 3, \ldots$ and $D < \varepsilon_{ij} \leq 1$), then*

$$D \geq p_{ij} \geq 1/D \qquad \text{for all} \quad i, j.$$

Proof. Let $A=\{x\in\Omega^m: x_i\leqslant L_i \text{ for each } i\}$, *where* $L_i=m+a_i^\alpha$ *for each i.* It is easy to see that if $\varepsilon\leqslant 1$ and x^α is the final holding by any trader α at a TSNE in the game Γ_ε, then x^α lies in the (compact) set A.

For each α, let ψ_i^α be the partial derivative of u^α in the ith commodity (if $x_i=0$, ψ_i^α is the right-handed derivative). By virtue of the assumptions on u^α, ψ_i^α is continuous and positive on A. So we may choose a constant $D_1>1$ such that

$$D_1>\psi_j^\alpha(x)/\psi_i^\alpha(x) \qquad \text{for all} \quad x\in A. \tag{7}$$

Now, let $B=(B^1, ..., B^n)$ be a TSNE of ${}^k\Gamma_\varepsilon$ with prices P and final holdings $x^1, ..., x^n$ by the various trader types.

Let α, β be two traders; we may assume that

$$b_{ji}^\alpha\leqslant\frac{1}{2}\left(\sum_\gamma b_{ji}^\gamma+\varepsilon_{ij}\right). \tag{8}$$

For convenience we shall divide the argument into three cases:

Case 1. $b_{ji}^\alpha\geqslant(1/m)a_j^\alpha$

Case 2. $b_{ji}^\alpha<(1/m)a_j^\alpha$ and $\sum_i b_{ji}^\alpha<a_j^\alpha$

Case 3. $b_{ji}^\alpha<(1/m)a_j^\alpha$ and $b_{jh}^\alpha>(1/m)a_j^\alpha$ for some $h\in I_m$.

It is easy to see that these cases exhaust all the possibilities.

In *Case* 1, we have

$$p_{ij}=\left(\sum_\alpha b_{ij}^\alpha+\varepsilon\right)\Big/\left(\sum_\alpha b_{ji}^\alpha+\varepsilon\right)\leqslant L_i\Big/\frac{1}{m}\,a_j^\alpha,$$

so if

$$D_2=\max_{\alpha,\, i}\ \max_{j:\, a_j^\alpha>0}\ (m\cdot L_i)/a_j^\alpha, \tag{9}$$

then $p_{ij}<D_2$.

In *Case* 2, we consider the effect on α's utility if he increases b_{ji}^α slightly. Thus, let

$$e=b_{ij}^\alpha, \qquad f=b_{ji}^\alpha, \qquad E=\sum_{\beta\neq\alpha} b_{ij}^\beta+\varepsilon_{ij}, \qquad F=\sum_{\beta\neq\alpha} b_{ji}^\beta+\varepsilon_{ji}.$$

Then

$$p_{ij}=\frac{E+e}{F+f};$$

and if α changes b_{ji}^{α} to $b_{ji}^{\alpha}+\eta$, the price changes to

$$q_{ij}=\frac{E+e}{F+f+\eta},$$

and α's final holding of i increases by

$$\begin{aligned}(f+\eta)\cdot q_{ij}-fp_{ij}&=\frac{(f+\eta)\cdot(E+e)}{(F+f+\eta)}-\frac{f(E+e)}{(F+f)}\\&=\frac{E+e}{F+f}\left(\frac{(f+\eta)(F+f)-f(F+f+\eta)}{(F+f+\eta)}\right)\\&=\eta\cdot\frac{E+e}{F+f}\cdot\frac{F}{F+f+\eta}=\eta\cdot p_{ij}\cdot\frac{F}{F+f+\eta}.\end{aligned}$$

Now by (8), we know that $e<E$; if in addition, $n<E$, then the increase in α's holding of commodity i is greater than $\frac{1}{3}\eta\cdot p_{ij}$. Also the decrease in his holding of j is η.

So if δu^{α} is the increase in α's utility, we must have

$$\begin{aligned}\delta u^{\alpha}&>(\tfrac{1}{3}\eta\cdot p_{ij})\,\psi_i(x^{\alpha})-\eta\psi_j(x^{\alpha})+O(\eta^2)\\&=\tfrac{1}{3}\eta\psi_i(x^{\alpha})\cdot(p_{ij}-3\psi_j(x^{\alpha})/\psi_i(x^{\alpha}))+O(\eta^2).\end{aligned}$$

Since b_{ij}^{α} is a best response, δu^{α} must be negative. Therefore, we must have

$$p_{ij}\leqslant 3\psi_j(x^{\alpha})/\psi_i(x^{\alpha});$$

and if D_1 is as in (7), then

$$p_{ij}\leqslant 3D_1. \tag{10}$$

Finally, in *Case* 3, we consider the change in α's utility if he diverts a small amount η from b_{jh}^{α} to b_{ji}^{α}.

As in Case 2, for sufficiently small η, the increase in his holding of commodity i is greater than $\frac{1}{3}\eta p_{ij}$. It remains to estimate the decrease in his holding of h. Let

$$v=b_{ij}^{\alpha},\qquad V=\sum_{\beta\neq\alpha}b_{hj}^{\beta}+\varepsilon$$

$$w=b_{jh}^{\alpha},\qquad W=\sum_{\beta\neq\alpha}b_{jh}^{\beta}+\varepsilon.$$

Then the decrease in α's holding of h is

$$w \cdot \frac{V+v}{W+w} - (w-\eta)\,\frac{V+v}{W+(w+\eta)} = \frac{V+v}{W+w} \cdot \frac{W}{W+w-\eta} \cdot \eta.$$

Now in the present situation, we know that $w > (1/m)a_j^\alpha$ and so if $\eta < (1/2m)a_j^\alpha$, the above decrease is less than

$$\frac{L_h \cdot L_j}{((1/m)a_j^\alpha)((1/2m)a_j^\alpha)} \cdot \eta < 2 \cdot D_2^2 \cdot \eta,$$

where D_2 is as in (9).

Therefore if δu^α is the change in α's utility,

$$\delta u^\alpha > (\tfrac{1}{3} p_{ij} \eta)\, \psi_i^\alpha(x^\alpha) - 2D_2^2 \eta \psi_h^\alpha(x^\alpha) + O(\eta^2)$$

for sufficiently small η.

Since δu^α must be negative, we must have

$$p_{ij} < 6D_2^2 \psi_h^\alpha(x^\alpha)/\psi_i^\alpha(x^\alpha) \leqslant 6D_1 D_2^2. \tag{11}$$

So, let $D = D_2 + 3D_1 + 6D_1 D_2^2$. Then, by (9), (10), and (11), we see that, in all cases,

$$p_{ij} \leqslant D \qquad \text{for all} \quad i, j.$$

Since $p_{ji} = 1/p_{ij}$, the other half of the inequality follows. Q.E.D.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Fix k and choose a sequence $\{\varepsilon_l\}_{l=1}^\infty$ decreasing to zero. By Lemma 4, we can find TSNEs $\{(B_{\varepsilon_l}, P_{\varepsilon_l})\}$ of the ${}^k\Gamma_{\varepsilon_l}$ with $m-1$ goods active. Now, by compactness of the strategy spaces and Lemma 5, we may assume (passing to a subsequence, if necessary) that $\{(B_{\varepsilon_l}, P_{\varepsilon_l})\}_{l=1}^\infty$ converges to a limit (B, P) with $m-1$ goods active. Then $D \geqslant p_{ij} \geqslant 1/D$, and it remains only to show that (B, P) is a TSNE of ${}^k\Gamma$.

Let x_{ε_l} (resp. x) be the allocation resulting from B_{ε_l} in ${}^k\Gamma_{\varepsilon_l}$ (resp. B in ${}^k\Gamma$). First of all, we establish that $x_{\varepsilon_l} \to x$, as $l \to \infty$. For this we examine the contribution at B_{ε_l} of each market (i, j) in ${}^k\Gamma_{\varepsilon_l}$ as $l \to \infty$. Clearly, the aggregate (and individual) amounts sent to trade on both sides of the market converge to those at B in ${}^k\Gamma$. So if we aggregate amounts on both sides of the market tend to non-zero limits, then it is clear from (1) and (2) that individual trades converge to those at B in ${}^k\Gamma$. On the other hand, if the aggregate amount on one side tends to zero, so must the other side (by Lemma 5). Consequently the net trade itself tends to zero, and is zero in the limit.

Now, suppose B is not a TSNE of ${}^k\Gamma$. Then there is a trader of type α who can improve on his payoff at B. Let B' be the new set of moves resulting from this trader's change and let

$$\delta = \pi^\alpha(B') - \pi^\alpha(B) > 0.$$

Let $\pi^\alpha_{\varepsilon_l}$ be the payoff function in ${}^k\Gamma_{\varepsilon_l}$; then we can choose l large enough so that

$$|\pi^\alpha_{\varepsilon_l}(B') - \pi^\alpha(B')| < \delta/3.$$

Also, by the continuity of u^α, and the first part of this argument ($x_{\varepsilon_l} \to x$), we may further ensure that

$$|\pi^\alpha_{\varepsilon_l}(B_{\varepsilon_l}) - \pi^\alpha(B)| < \delta/3.$$

So

$$\begin{aligned}\pi^\alpha_{\varepsilon_l}(B') - \pi^\alpha_{\varepsilon_l}(B_{\varepsilon_l}) &> (\pi^\alpha(B') - \delta/3) - (\pi^\alpha(B) + \delta/3)\\ &= \delta - 2\delta/3 = \delta/3 > 0,\end{aligned}$$

which is not possible since B_{ε_l} is an NE for ${}^k\Gamma_{\varepsilon_l}$. So B must be a TSNE of ${}^k\Gamma$.[3]

3. The Relationship between Noncooperative and Competitive Equilibrium

3.1. *The Inequivalence*

We recall the notion of a *competitive equilibrium* (CE) for the *exchange economy* corresponding to Γ. With notation as in the first two paragraphs of Section 2, we proceed as follows:

Given any price *vector* $p \in \Omega^m$, the *budget set* of the trader α is defined (as usual) to be

$$B^\alpha(p) = \{x \in \Omega^m : p \cdot x \leqslant p \cdot a^\alpha\}$$

[3] The case $n = 2$, $m = 2$, $k = 1$ is special; Shapley [6] has provided a construction of an open set of NEs without using a fixed point theorem. It may be possible to give a similar existence argument in general, but this is not clear. In particular, Dubey and Rogawski [2] have shown that for $m > 2$, the set of NE allocations has *strictly* smaller dimension than the space of all allocations.

and a CE for Γ consists of a price vector $p \in \Omega^m$ and allocations $x^\alpha \in \Omega^m$ such that

$$u^\alpha(x^\alpha) = \max\{u^\alpha(y) : y \in B^\alpha(p)\} \qquad \text{for all } \alpha \tag{12}$$

and

$$\sum_{\alpha=1}^{n} x^\alpha = \sum_{\alpha=1}^{n} a^\alpha. \tag{13}$$

It is well known that any exchange economy which satisfies the conditions in the first two paragraphs of Section 2 has a competitive equilibrium.

A strategic market game, in contrast with the CE model of exchange, is "process and transactions"-oriented. Liquidity constraints, which have no role to play in general equilibrium, play a fundamental role in a strategic market game. Thus, in such a game, even with a continuum of traders, there may be CEs which are not NEs and vice versa.

Since, in our model, direct exchanges are permitted between all commodity pairs, it may seem at first that the lack of liquidity will never be a constraint on the actions of the traders. However, this is not so; and we give two examples to show this.

EXAMPLE 1. Consider the case when there are three goods and four trader types. We assume a continuum of each type (the interval $[0, 1]$), characterized by the following utility functions and endowments:

$$\varphi^0(x, y, z) = (xyz)^{1/3} \quad \text{and} \quad (1, 1, 1)$$

$$\varphi^1(x, y, z) = (yz^2)^{1/3} \quad \text{and} \quad (0, 3, 0)$$

$$\varphi^2(x, y, z) = (xy^2)^{1/3} \quad \text{and} \quad (3, 0, 0)$$

$$\varphi^3(x, y, z) = (x^2 z)^{1/3} \quad \text{and} \quad (0, 0, 3)$$

There is a TSNE with prices $p_{xy} = p_{yz} = p_{zx} = 2$, where traders 1, 2, and 3 each offer two units of y, x, and z, respectively, in exchange for one unit of goods z, y, and x, respectively, from the traders of type 0. The final endowments of the trader types are $(2, 2, 2)$, $(0, 1, 1)$, $(1, 1, 0)$, and $(1, 0, 1)$, respectively.

On the other hand, the unique CE for the related exchange economy has prices $p_x = p_y = p_z = 1$ and final holdings $(1, 1, 1)$, $(0, 1, 2)$, $(1, 2, 0)$ and $(2, 0, 1)$.

Strictly speaking, this example does not satisfy the positiveness conditions imposed in Section 2. The next example does satisfy these conditions, and is furthermore robust.

EXAMPLE 2. Consider the case of three goods and three players. We assume a continuum of each type (the interval [0, 1]), with the utility functions and endowments

$$\varphi^1(x, y, z) = (x^\lambda yz)^{1/(\lambda+2)} \quad \text{and} \quad (1, 1, \lambda)$$
$$\varphi^2(x, y, z) = (xy^\lambda z)^{1/(\lambda+2)} \quad \text{and} \quad (\lambda, 1, 1)$$
$$\varphi^3(x, y, z) = (xyz^\lambda)^{1/(\lambda+2)} \quad \text{and} \quad (1, \lambda, 1),$$

where λ is some large, positive constant.

By virtue of the symmetry of the model, it is clear that there will be an NE with $p_{xy} = p_{yz} = p_{zx} = p$, say. We would like to deduce a relationship between p and λ. First of all, it is easy to see that if λ is large, then $p > 1$; and also that at an NE strategy, the bids by player 1 must be of the form where $a > 0$, $b > 0$, and $\lambda > a + b$.

If (x, y, z) are the final holdings by player 1, we have

$$x = p + a/p + 1, \qquad y = bp, \qquad z = \lambda - (a + b).$$

Now, by considering the effects of small changes in a and b on the payoff to player 1, we see that

$$p = \lambda z/x = y/z.$$

Finally, since the bids by the other two types are symmetrically determined, we have

$$p = a(b + 1).$$

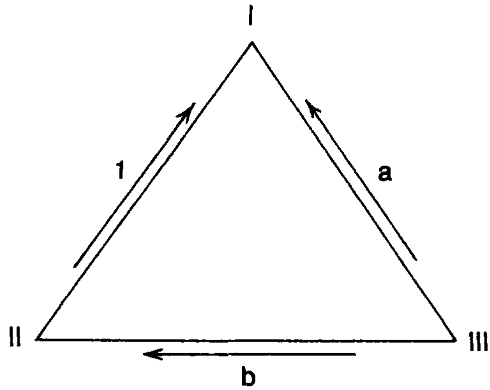

FIGURE 2

Using these six relations, we quickly obtain

$$(\lambda - p)^2 = p(p+2)^2.$$

In particular, if $p=4$ then $\lambda = 16$; and (with a little more calculation) $b=2$, $a=12$, $x=8$, $y=8$, $z=2$. So the final holdings (for $\lambda=16$) are (8, 8, 2), (2, 8, 8), and (8, 2, 8) by the different players.

Also, the unique CE has prices $p_x = p_y = p_z = 1$, and final holdings are (16, 1, 1), (1, 16, 1), and (1, 1, 16). It is of some interest to note that in Example 2 the CE allocations are actually achievable through a particular choice of strategies by the various traders. However, such choices will not constitute an NE.

3.2. *Conditions for Equivalence*

Let (B, P) be a TSEP for ${}^k\Gamma$; we say that the TSEP is *price consistent* if for any three goods $\{h, i, j\}$ we have $p_{hi} p_{ij} = p_{hj}$.

If (B, P) is price consistent, then the *matrix* P can be expressed in terms of a price *vector* q such that

$$p_{ij} = q_i / q_j. \tag{14}$$

Price inconsistencies at a TSEP imply the existence of arbitrage possibilities which are not being exploited because of liquidity constraints. In fact, we have

LEMMA 6. *Let (B, P) be a TSEP of ${}^k\Gamma$ such that $p_{i_1 i_2} p_{i_2 i_3} \neq p_{i_1 i_3}$ for a triple $\{i_1, i_2, i_3\}$ of goods. Then for each trader type α, we must have*

$$\sum_j b_{ij}^\alpha = a_i^\alpha$$

for some $i \in \{i_1, i_2, i_3\}$.

Proof. It suffices to prove the lemma for TSNEs of ${}^k\Gamma(\varepsilon)$; but there it must hold, since otherwise such a player would be able to increase his holding of all three goods i_1, i_2, and i_3 by an obvious modification of his bid. Q.E.D.

The examples in Section 3.1 show that it is possible for *all* TSEPs of a game to be price inconsistent.

THEOREM 2. *Let $({}^kB, {}^kP)$ be a sequence of TSEPs of ${}^k\Gamma$; then this sequence has a limit point, say (B, P). Suppose*

(1) *$({}^kB, {}^kP)$ are price consistent;*

(2) *(B, P) has all markets active, i.e., for any two goods i, j, there is* $\alpha \in I_n$ *with* $b_{ij}^{\alpha} \neq 0$.

Then

(a) *P is consistent and we may choose q as in* (14);

(b) *If* ${}^k x^{\alpha}$ *are the final holdings at* $({}^k B, {}^k P)$, *then* ${}^k x^{\alpha}$ *converges, to* x^{α}, *say,*

(c) (q, x^{α}) *is a CE for the exchange economies corresponding to* Γ^k, $j = 1, 2, 3, \ldots$.

Proof. We omit the proof since it is completely analogous to Dubey and Shubik [3]. Q.E.D.

We conclude with some remarks and conjectures.

Remark 1. For the game with a continuum of players, fix players' utility functions and the *total* amount of each commodity in the game. Consider the different games that arise as the total commodities are distributed across the players. If a distribution is on the Pareto-optimal surface, then clearly this is the unique CE and TSNE of the corresponding game. Now Lemma 6 and a Cobb–Douglas like condition on the utility functions are sufficient to show that, for distributions in a *neighborhood* of the Pareto-optimal surface, there are active TSNEs which "coincide" with the CEs.

Conjecture 1. For the game with a continuum of players, if all CEs are active TSNEs (with consistent prices) then these are the only TSNEs.

Remark 2. We can show that with three goods, and Cobb–Douglas utility functions, the game ${}^k\Gamma$ always has a TSNE with *all* markets active.

Conjecture 2. With Cobb–Douglas utility functions, ${}^k\Gamma$ has a TSNE with all markets active.

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