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DEGENERATE SERIES REPRESENTATIONS FOR $GL\,(2n, \mathbf{R})$ AND FOURIER ANALYSIS

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1. INTRODUCTION

In this paper we present an example of close relationship between a filtration of an induced representation by invariant subspaces and the orbit structure of the action of a subgroup on a vector space. Such a relationship was first observed in the case $SO(4,2)$ in the work of Jakobsen, Vergne and one of the authors [8] [11]. They considered the representations induced from a one dimensional representation of a maximal parabolic subgroup Q with abelian nilradical N. The Levi subgroup L of Q is isomorphic to a product of $SO(3,1)$ and $\mathbf{R}^+$. If the induced representation has a finite dimensional subrepresentation, then there is a bijective correspondence between the orbits of L on N and the lattice of invariant subspaces of the induced representation. Furthermore there is a one to one correspondence between closed L-invariant subsets and invariant subspaces. This correspondence preserves inclusion relations.

A similar correspondence between invariant subspaces and orbits is also true for $G = GL(2n, \mathbf{R})$, the general linear group of $2n \times 2n$ real matrices. Let P be its «middle» maximal parabolic subgrouup. It has the Levi decomposition $P = LN$ with Levi factor $L = Gl(n, \mathbf{R}) \times GL(n, \mathbf{R})$ and $N = M_{(n,n)}(\mathbf{R})$ (the additive groups of all $n \times n$ real matrices). In this paper we consider the representations $\mathrm{Ind}_P^G(\chi)$ of G induced from an arbitrary one dimensional character χ of P. Using analytic techniques we determine a filtration by invariant subspaces as follows.

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We consider $\mathrm{Ind}_P^G(\chi)$ as realized in a certain space V of functions on $\bar{N} \approx M_{(n,n)}(\mathbf{R})$. In particular if it a K-finite function in V, its Fourier transform f always makes sense as a tempered distribution on $N \approx M_{(n,n)}(\mathbf{R})$. Now $M_{(n,n)}(\mathbf{R})$ has a natural filtration by closed L-stable subvarieties given by

$$M_{(n,n)}(\mathbf{R}) = \Omega_n \supseteq \dots \supseteq \Omega_1 \supseteq \dots \supseteq \Omega_0 = \{0\} \supseteq \Omega_{-1} = \emptyset$$

where $\Omega_l = \{T \in M_{(n,n)}(\mathbf{R}), \mathrm{Rank}(T) \leq l\}$. We may consider the corresponding filtration on V

$$V = V_n \supseteq \dots \supseteq V_l \dots \supseteq V_0 \supseteq V_{-1}$$

where $V_l = \{f \in V, \mathrm{supp}(\hat{f}) \supseteq \Omega_l\}$. Since V_l is $(\mathbf{g}, K)$-invariant this defines a filtration by invariant subspaces.

In the first part we obtain necessary conditions for this filtration to be nontrivial. The key ingredient here is the Radon transform from $(n+l) \times (n-l)$ matrices to $M_{(n,n)}(\mathbf{R})$. In the second part we use the parametrization of irreducible (g, K)-modules with a fixed infinitesimal character to get a priori control on possible composition factors of $\mathrm{Ind}_P^G(\chi)$ with a given wave front set. This together with the results of the first section shows that if χ satisfies a certain positivity condition, then the quotients $V^l = V_l/V_{l-1}$ are either irreducible or zero. A precise statement is given in theorem VII.1.

It is possible to obtain the composition series of the representation $\mathrm{Ind}_P^G(\chi)$ using only the techniques introduced in the second part. We use Fourier analysis in this paper because it allows us to obtain a very natural analytic description of the lattice of invariant subspaces. Furthermore we also obtain an interpretation of the wave front set of subrepresentations which is useful for application to automorphic forms and number theory.

From now on we will always assume that $n > 0$.

II. DEGENERATE SERIES REPRESENTATIONS

Let $G = GL(2n, \mathbf{R})$ be the group of $2n \times 2n$ real matrices with nonzero determinant. Its Lie algebra $\mathbf{g}$ is the Lie algebra of $2n \times 2n$ matrices. Its maximal compact subgroup K is isomorphic to $O(2n)$, the group of orthogonal matrices. Its Lie algebra $\mathbf{k}$ is equal to the Lie algebra $o(2n)$ of skew symmetric matrices.

We write a typical element g in $GL(2n, \mathbf{R})$ as

$$g = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

where $a, b, c, d \in M_{(n,n)}(\mathbf{R})$ and $\det g \neq 0$. Also, let us write

$$P = \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \right\},$$

$$L = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \right\},$$

$$\bar{N} = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \right\},$$

for the indicated subgroups of G. Notice that N and $\bar{N}$ may be identified with the additive group $M_{(n,n)}(\mathbf{R})$.

One dimensional characters of $GL(n,\mathbf{R})$ are parametrized by elements in $\mathbf{Z}/2 \times \mathbf{C}$, where (ϵ, s) corresponds to the character

$$a \to (\text{sign}\det a)^{s} \cdot |\det a|^{s}.$$

Characters of L consist of pairs $\chi = (\chi_1, \chi_2)$ with each $\chi_i = (\epsilon_i, s_i)$ in $\mathbf{Z}/2 \times \mathbf{C}$. (Here we write $\mathbf{Z}/2$ additively).

The representation $\operatorname{Ind}_G^P(\chi)$ is realized by right translations on the space of $f \in C^\infty(G)$ satisfying

$$f\left(\begin{pmatrix} a & 0 \\ b & d \end{pmatrix} g\right) = |a|^{-n/2}\ |\mathrm{d}|^{n/2} \chi_1(a)\chi_2(d) f(g)$$

for all $g \in G$ and $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in P$. Here $|a|$ etc. is shorthand for $|\det a|$ etc.

If χ_0 is the character (ϵ_0, s_0) of G, we have

$$\chi_0 \otimes \operatorname{Ind}_P^G \chi = \operatorname{Ind}_P^G \tilde{\chi}$$

where

$$\tilde{\chi} = ((\epsilon_0 + \epsilon_1, s_0 + s_1), (\epsilon_0 + \epsilon_2, s_0 + s_2))$$

Also, if $(\operatorname{Ind}_P^G(\chi))^*$ is the hermitian dual of $\operatorname{Ind}_P^G(\chi)$, then

$$(\operatorname{Ind}_P^G(\chi))^* = \operatorname{Ind}_P^G(\chi^*),$$

where $\chi^* = ((\epsilon_1, -\bar{s}_1), (\epsilon_2, -\bar{s}_2))$. Hence to determine the structure of the induced representation we may assume from now on that $\chi((\epsilon, s), (0, -s))$ with $\epsilon \in \mathbf{Z}/2$.

PROPOSITION II.1. *The representation* $\operatorname{Ind}_P^G(\chi)$ *is irreducible unless* $2s-(n-1)\in \mathbf{Z}$.

PROOF. The subgroup

$$H^0=\{\operatorname{diag}(\pi_1\cdot e^{y1},\dots,\pi_{2n}\cdot e^{y2n}),y_i\in\mathbf{R},\pi_i\in\mathbf{Z}/2\}$$

is a Cartan subgroup. A character γ of H^0 is given by $((\delta_1,\nu_1),\dots,(\delta_{2n},\nu_{2n}))$ with characters (δ_i,ν_i) of $GL(1,\mathbf{R})$ determined by

$$\gamma((\pi_1,e^{y1}),\dots,(\pi_i,e^{yi}),\dots,(\pi_{2n},e^{y2n}))=\delta_i(\pi_i)\cdot e^{\nu_i y_i}$$

Let

$$((\epsilon,(n-1)/2+s),(\epsilon,(n-3)/2+s),\dots,(\epsilon,-(n-1)/2+s),$$
$$(0,(n-1)/2-s),\dots,(0,-(n-1)/2-s)).$$

be a Langlands parameter and $X(\chi)$ be the corresponding standard module (principal series representation) corresponding to this parameter. The claim follows from factoring the long intertwining operator for $X(\chi)$. The part corresponding to roots outside $GL(n)\times GL(n)$ is formed of isomorphisms. The other just gives $\operatorname{Ind}_P^G(\chi)$ as its image.
Q.E.D.

From now on we will use the same notation for the $(\mathbf{g},K)$*-module associated to a representation as for the representation of* G *itself*.

III. THE RANK FILTRATION

Let χ be a character of L. Since $G=P\cdot\bar N$ up to a set of measure zero, functions in $\operatorname{Ind}_P^G(\chi)$ are determined by their restriction to $\bar N=M_{(n,n)}(\mathbf{R})$. We write $V(\chi)$ for the corresponding realization of $\operatorname{Ind}_P^G(\chi)$. For $\chi=((\epsilon,s),(0,-s))$ the action Π_χ of G on $V(\chi)$ is «formally» given by

$$\Pi_\chi(g)f(x)=(\operatorname{sgn}(\tilde a))^\epsilon|\tilde a|^{-n/2+s}|\tilde d|^{2/2-s}f(\tilde x) \tag{3.1}$$

where $\tilde a,\tilde d$ and $\tilde z$ are obtained by solving

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}a & c\\ b & d\end{pmatrix}=\begin{pmatrix}\tilde a & 0\\ \tilde b & \tilde d\end{pmatrix}\cdot\begin{pmatrix}1 & \tilde x\\ 0 & 1\end{pmatrix},$$

and $\operatorname{sgn}(\tilde a)=\operatorname{sgn}(\det(\tilde a))$. This gives

$$\tilde a=a+xb, \tag{3.2}$$

$$\tilde x=(a+xb)^{-1}\cdot(c+xd), \tag{3.3}$$

and

$$|\tilde d|=|g|\cdot|\tilde a|^{-1}=|g|\cdot|a+xb|^{-1}. \tag{3.4}$$

Note that these formulas are valid only if all terms are defined.

A translation functor

Let F_l = { polynomials on $M_{(n,n)}(\mathbb{R})$ degree $\leq l$ in the *minors* of $x \in M_{(n,n)}(\mathbb{R})$}. Then we have

LEMMA III.1. F_l is an invariant subspace of $V(\epsilon(l), \frac{n+l}{2})$, where $\epsilon(l) \equiv l(\text{mod } 2)$.

PROOF. The invariance is immediate on the subgroup $\bar{P} = L \cdot \bar{N}$, thus it suffices to show that F_l is invariant under $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. From 3.2, 3.3 we see that for $f \in F_l$ and $\chi = (\epsilon(l), \frac{n+l}{2})$,

$$\Pi_\chi(w) f(x) = (\det x)^l \cdot f(x^{-1}).$$

Now suppose I and J are two subsets of $\{1, \ldots, n\}$ of size k and that f is the $k \times k$ minor function corresponding to rows from I and columns from J. Then by Leibniz's rule

$$f(x^{-1}) = \pm(\det x)^{-1} \cdot f^c(x)$$

where f^c is the complementary $(n-k) \times (n-k)$ minor corresponding to rows and columns *not* in I and J.

Thus if $f = f_1 \ldots f_m$, where each f_i is a minor and $m \leq l$ we get

$$w \cdot f = \pm(\det x)^{l-m} f_1^c \ldots f_m^c$$

which is again in F_l. Q.E.D.

LEMMA III. 2. *The multiplication map*

$$\mu : F_l \otimes V(\epsilon, s) \to V\left(\epsilon + \epsilon(l), s + \frac{l}{2}\right)$$
$$f \otimes v \to f \cdot v$$

is a $(\mathfrak{g}, K)$*-homomorphism.*

PROOF. We have to check only that μ is a homomorphism. This is best verified for the G-action as in the proof of III.1, and then it is immediate. Q.E.D.

REMARK. The correspondence $(F_l \otimes -) \mapsto V\left(\epsilon + \epsilon(l), s + \frac{l}{2}\right)$ is related to the Zuckerman translation functors.

Construction of a subrepresentation

For $l \leq n$, let P_l be the parabolic subgroup of G with Levi factor $GL(l) \times GL(2n-l)$. The unitarily induced representation $I_l = \operatorname{Ind}_{P_l}^{G}(1)$ is irreducible by an old result of Gelfand and Naimark [6]. It may be identified with a certain space of smooth functions on the space $M_{(l,2n-l)}(\mathbf{R})$ of $l \times (2n-l)$ real matrices.

We introduce the Radon transform Φ from $M_{(l,2n-1)}(\mathbf{R})$ to $M_{(n,n)}(\mathbf{R})$ as follows: Let x, y, z be $l \times n$, $(n-l) \times n$ and $l \times (n-l)$ real matrices respectively. Then (z, x) and $\binom{x}{y}$ are elements of $M_{(l,2n-l)}(\mathbf{R})$ and $M_{(n,n)}(\mathbf{R})$. If f is a function on $M_{(l,2n-1)}(\mathbf{R})$, we write

$$\Phi f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \int f((z, x + zy))\,\mathrm{d}z. \tag{3.5}$$

The key properties of Φ are as follows:

LEMMA III.3. *The integral defining Φ converges absolutely for f in I_l and defines an injective homomorphism from I_l into $V(0, (n-l)/2)$. Furthermore*

$$\cup\{\operatorname{supp}(\widehat{\Phi(f)})) : f \in I_l\} = \Omega_l.$$

PROOF Let us write $g \in G$ as $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where a, b, c and d are $l \times l$, $(2n-l) \times l$, $l \times (2n-l)$ and $(2n-l) \times (2n-l)$ respectively. Then $P_l = \left\{\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}\right\}$ and the representation I_l is realized on $L^2(M_{(l,2n-1)}(\mathbf{R}))$ by

$$I_l(g) f(u) = |\breve{a}|^{-(2n-l)/2}\, |\breve{d}|^{l/2} f(\breve{u}),$$

where $\breve{a}, \breve{d}$ and $\breve{u}$ are obtained by solving

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \breve{a} & 0 \\ \breve{b} & \breve{d} \end{pmatrix} \cdot \begin{pmatrix} 1 & \breve{u} \\ 0 & 1 \end{pmatrix}.$$

Now if f is a function in I_l, then in particular f is the restriction of a smooth function on G. Thus there is a constant $a \in R$ such that $a \geq |f(k)|$ for all $k \in K$. Writing $k \in K$ as

$$k = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},$$

we get $|a| = |d|^{-1}$ and

$$\begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \cdot \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ u^t & 1 \end{pmatrix} \cdot \begin{pmatrix} a^t & b^t \\ 0 & d^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} a(1+u\cdot u^t)a^t & * \\ * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $\det(1+u\cdot u^t) = |\det a|^{-2}$ and we have

$$|f(u)| \leq A\cdot\det(1+u\cdot u^t)^{-n/2} \tag{3.6}$$

for $f\in I_l$. Writing u as (z,x) we get

$$\det(1+u\cdot u^t) = \det(1+z\cdot z^t + x\cdot x^t) \geq (1+z\cdot z^t).$$

So to prove the absolute convergence, it suffices to check the absolute convergence of

$$\int \det(1+z\cdot z^t)^{-n/2}\,\mathrm{d}z.$$

Let $z_1,\dots,z_{n-l}$ be the columns of z. Then this integral becomes

$$\int \det(1+z_1\cdot z_1^t+\ldots+z_{n-l}^t\cdot z_{n-l})^{-n/2}\,\mathrm{d}z_1\ldots\mathrm{d}z_{n-l} =$$

$$=\prod_{i=1}^{n-l}\left(\int \det(1+z_i\cdot z_i^t)\right)^{-(n-i+1)/2}\mathrm{d}z_i) =$$

$$=\prod_{i=1}^{n-l}\left(\int (1+|z_i|^2)^{-(n-i+1)/2}\mathrm{d}z_i\right).$$

Since $(n-i+1)\geq l+1$, each factor in the product converges. This proves the absolute convergence.

Next, we show that Φ is an intertwining operator from I_l to $V\left(0,\frac{n-1}{2}\right)$. We first introduce some useful notation (due to Zelevinski):

Let us write ν_p for the character $|\det(\,)|$ of $GL(p,\mathbf{R})$ and denote by $\nu_p^s\times\nu_q^t$ the representation $\mathrm{Ind}_{P_p}^{GL(p+q)}(\nu_p^s\times\nu_q^t)$. Here P_p is the parabolic $\left\{\begin{pmatrix} a & 0 \\ b & d\end{pmatrix}\right\}$ where a,b,d are $p\times p, q\times p$ and $q\times q$ matrices respectively.

The character ν_{p+q}^s is a a subrepresentation of $\nu_p^{s+q/2}\times\nu_q^{s-p/2}$, and a quotient of $\nu_p^{s-q/2}\times\nu_q^{s+p/2}$. If these representations are realized in the noncompact picture, the maps are given by

$$c\rightarrow f_c,$$

where $f_c(x)=c$ for all $x\in M_{p\times q}(\mathbf{R})$ and

$$f\rightarrow\int f(x)\,\mathrm{d}x.$$

Using induction by stages, we get an intertwining map as follows:

$$\begin{aligned} I_l &= \nu_l^0 \times v_{2n-l}^0 \\ &\xrightarrow{T} \nu_l^0 \times \left(\nu_{n-l}^{n/2} \times \nu_n^{-(n-l)/2}\right) \\ &\xrightarrow{\sim} \left(\nu_l^{(n-l)/2-(n-l)/2} \times \nu_{n-l}^{(n-l)/2+l/2}\right) \times \nu_n^{-(n-l)/2} \\ &\xrightarrow{U} \nu_n^{(n-l)/2} \times \nu_n^{-(n-l)/2} \\ &= V(0,(n-l)/2). \end{aligned}$$

In the noncompact picture the intermediate representation above is realized on functions on the nilpotent group $\left\{\begin{pmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right\}$, where x, y and z are as in 3.5.

Since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & z & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & x - zy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

it follows that the operators T and U are

$$Tf\left(\begin{pmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 1 & 1 \end{pmatrix}\right) = f((z\ x))$$

and

$$U(\phi)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \int \phi\left(\begin{pmatrix} 1 & z & x + zy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right) \mathrm{d}z.$$

Since $\Phi = U \cdot T, \Phi$ is a $(\mathfrak{g}, K)$-homomorphism from I_l to $V(0,(n-l)/2)$. Furthermore, the argument leading to 3.6 shows that the spherical vector in I_l is $f(u) = \det(1 + u \cdot u^t)^{-n/2}$. Since this is a positive function, 3.5 shows that Φ is not identically zero. Now the irreducibility of I_l implies that Φ is injective.

For $f \in I_l, \Phi f$ is in $V(0, n-l)$. By an argument similar to 3.6. it follows that $|\Phi(f(v))| \leq A \det(1 + v \cdot v^t)^{-(n-l)/2}$ for $v \in M_{(n,n)}(\mathbf{R})$. In particular Φf is a tempered distribution and we may consider its Fourier transform $\widehat{\Phi f}$.

To calculate $\operatorname{supp}(\widehat{\Phi f})$, we proceed as follows: Write $v = \binom{x}{y}$ and let $w = (\zeta, \eta)$ be the dual variables. Suppose ϕ is a Schwartz function on $M_{(n,n)}(\mathbf{R})$.

Then

$\langle \widehat{\Phi f}, \phi \rangle \equiv \int ((\zeta$

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$\Phi(I_l))\} = \Omega_l$

Let $I_l(\chi)$

algebra. We d

Then

$$\begin{aligned}
\langle\widehat{\Phi f},\phi\rangle &\equiv \int (\Phi f)\hat{\phi} \\
&= \int \left[\int f((z, x + z\cdot y))\,\mathrm{d}z\right]\left[\int e^{-\mathrm{itr}(\zeta x+\eta\cdot y)}\phi((\zeta,\eta)\,\mathrm{d}\zeta\,\mathrm{d}\eta\right]\mathrm{d}x\,\mathrm{d}y \\
&= \int e^{-\mathrm{itr}(\zeta(x-zy)+\eta y)} f((z,x)\phi((\zeta,\eta))\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int\left(\int e^{-\mathrm{itr}\zeta x} f(z,x))\,\mathrm{d}x\right)\left(\int e^{-\mathrm{itr}\eta y} e^{\mathrm{itr}\zeta z y}\phi((\zeta,\eta))\,\mathrm{d}y\,\mathrm{d}\eta\right)\mathrm{d}\zeta\,\mathrm{d}z \\
&= \int \mathcal{F}_x f((z,\zeta)\phi((\zeta,\zeta\cdot z))\,\mathrm{d}\zeta\,\mathrm{d}z.
\end{aligned}$$

where $\mathcal{F}_x$ is the Fourier transform in the x variable.

This shows that $\mathrm{supp}(\widehat{\Phi(f)} \subset \overline{\{(\zeta,\zeta z) \in M^n(\mathbf{R})\}} = \Omega_l$, and so we have

$$\bigcup\{\mathrm{supp}(\widehat{\Phi f}) : f \in I_t\} \subset \Omega_l.$$

To prove equality, it suffices to find for each z_0 and ζ_0 a function $f \in I_l$ such that $\mathcal{F}_x f(z,\zeta) \neq 0$ in neighborhood of (z,ζ). But if there were no such function, then I_l could not be dense in $L^2(M_{(l,2n-1)}(\mathbf{R}))$ and we would get a contradiction. Q.E.D.

The filtration and its characteristic variety

Combining lemma III.2 and III.3 we obtain a filtrations of $V(\chi)$ by invariant subspaces.

PROPOSITION III.4. *Let* $\chi = (\epsilon, s)$ *for* $2s \in \mathbf{Z}$ *and* $0 < 2s \leq n$. *For* $0 \leq l \leq n$ *put* $V_l(\chi) = \{f \in V(\chi), \mathrm{supp}(\hat{f}) \subset \Omega_l\}$. *Put* $r = \left[\frac{n-2s}{2}\right] - 1$.

- *Suppose* $\epsilon = 0$. *Then*

$$0 \neq V_{n-2s}(\chi) \neq V_{n-2s+2}(\chi) \neq \ldots \neq V_{n-2(s-r)}(\chi) \neq V_n(\chi). \tag{3.7}$$

- *Suppose* $\epsilon = 1$. *Then*

$$0 \neq V_{n-2s+1}(\chi) \neq V_{n-2s+3}(\chi) \neq \ldots V_{n-2(s-r)+1} \neq V_n(\chi). \tag{3.8}$$

PROOF. By lemma III.2 and III.3 $\mu(F_q \otimes \Phi(I_l))$ is a submodule of $V\left(\epsilon(q), \frac{n-1+q}{2}\right)$. Multiplication by a polynomial does not increase the support of the Fourier transform and since $1 \in F_l$, we see that the union of $\mathrm{supp}\{\hat{f}, f \in \mu(F_q \otimes \Phi(I_t))\} = \Omega_l$. Q.E.D.

Let $I_l(\chi) = \mathrm{Ann}_{U(\mathfrak{g})}(V_l(\chi))$ be the annihilator of $V_l(\chi)$ in the enveloping algebra. We denote its associated variety by $\mathrm{Ass}(I_l(\chi))$.

PROPOSITION III.5. *Suppose* $V_l(\chi) \neq V_{l-1}(\chi)$. *Then* $\Omega_l \subset \mathrm{Ass}(I_l(\chi))$.

PROOF. The Fourier transform provides us with a «fake» Whittaker functional since for $T \in U(\mathbf{n})$ and $f \in V_l(\chi)$

$$\widehat{Tf} = \sigma(T)\hat{f}.$$

Here $\sigma(T)$ is the polynomial in $S(\mathbf{n})$ corresponding to T. The equation $\sigma(T)$ $\hat{f} = 0$ for all $f \in V_l(\chi)$ implies that $\Omega_l \subseteq \{\lambda \in n, \sigma(T)(\lambda) = 0\}$.

The proposition follows now from the proof of theorem 1 in [10]. Q.E.D.

IV. THE COHERENT CONTINUATION REPRESENTATION

In order to find the possible candidates for the composition factors of the degenerate series representation, we will need the notion of coherent continuation representation and cells as described in sections 12-14 in [13]. The notions introduced will be illustrated in section VI in the case of $U(p,q)$ and $GL(n,\mathbf{R})$.

Recall that a Langlands parameter for an irreducible representation with regular integral infinitesimal character is a G-conjugacy class of data

$$\gamma = (\Gamma, \bar{\gamma}, H), \tag{4.1}$$

where $H = TA$ is a θ-stable Cartan subgroup with Lie algebra $\mathbf{h}$ such that

$$T = \{x \in H \,|\, \theta x = x\}, \quad \mathbf{t} = \{x \in \mathbf{h} \,|\, \theta x = x\},$$
$$A = \exp \mathbf{a}, \qquad \mathbf{a} = \{x \in \mathbf{h} \,|\, \theta x = -x\},$$

and $\Gamma \in \hat{H}, \bar{\gamma} \in \mathbf{h}^*$, satisfy

$$d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi). \tag{4.2}$$

Here Ψ is the unique positive system of imaginary roots such that $(\bar{\gamma}, \alpha) > 0$ for $\alpha \in \Psi$, Ψ_c are the compact roots and

$$\rho(\Psi) = 1/2 \sum_{\alpha \in \Psi} \alpha,$$
$$\rho_c(\Psi) = 1/2 \sum_{\alpha \in \Psi_c} \alpha.$$

Then $X(\gamma)$ is the standard module (induced from dicrete series) and $\bar{X}(\gamma)$ is the unique irreducible quotient corresponding to the parameter γ. We will denote by

$\mathcal{P}(\chi)$ the parameters with infinitesimal character χ, and by $\mathcal{P}_H(\chi)$ the subset corresponding to the Cartan subgroup H. Let

$$[X(\gamma) : \bar{X}(\delta)] = \text{multiplicity of } \bar{X}(\delta) \text{ in } X(\gamma). \tag{4.3}$$

Recall the equivalence relation $\sim_B$ defined in 1.14 [13] or Chapter 9 in [14]. This is the equivalence relation generated by

$$\gamma \sim_B \delta \text{ if } [X(\gamma) : \bar{X}(\delta)] \neq 0 \text{ or } [X(\delta) : \bar{X}(\gamma)] \neq 0. \tag{4.4}$$

The equivalence classes are called *blocks*.

We will denote by subscript a, the abstract objects attached to the complex Lie algebra $\mathfrak{g}_a$ isomorphic to $\mathfrak{g}$.

Given a regular integral infinitesimal character χ, let $\mathcal{G}(\chi)$ be the Grothendieck group over $\mathbb{Q}$ generated by characters of irreducible $(\mathfrak{g}, K)$-modules with infinitesimal character χ. This has as bases

$$\{\bar{X}(\gamma)\}_{\gamma\in\mathcal{P}(\chi)}, \{X(\gamma)\}_{\gamma\in\mathcal{P}(\chi)}. \tag{4.5}$$

Then

$$\mathcal{G}(\chi) = \bigotimes_{\mathcal{B}} \mathcal{G}_{\mathcal{B}}(\chi), \tag{4.6}$$

where $\mathcal{B}$ are the various blocks for the infinitesimal character χ.

The Weyl group W_a acts on $\mathcal{G}(\chi)$ by the coherent continuation representation $t(w)$ defined in sections 12-13 of [13] (with u specialized to 1) and preserves the direct sum decomposition (4.6). We will describe the properties of $t(\ )$ that we need.

The τ-invariant of γ is defined as follows. Let $R^+(\gamma)$ be the positive system in $\Delta(\mathfrak{g},\mathfrak{h})$ determined by $\bar{\gamma}$ and let $\Pi(\gamma) \subset R^+(\gamma)$ be the set of simple roots. Then

$$\tau(\gamma) = \left\{ \alpha \in \Pi(\gamma) \;\; \begin{array}{l} \alpha \text{ is compact imaginary or} \\ \alpha \text{ is complex, and } \theta\alpha \notin R^+(\gamma) \text{ or} \\ \alpha \text{ real, satisfying the parity condition.} \end{array} \right\} \tag{4.7}$$

Recall also, the relation $\gamma \to_s \gamma'$ and the Bruhat order $<$ defined in sections 12-13 in [13]. Then the formulas for $t(s)$ with $s = s_\alpha, \alpha \in \Pi(\gamma)$ on the basis $\bar{X}(\gamma)$ are given by

$$t(s)\bar{X}(\gamma) = \begin{cases} -\bar{X}(\gamma) & \text{if } \alpha \in \tau(\gamma), \\ \bar{X}(\gamma) + \sum_{\gamma\to_s\gamma'} \bar{X}(\gamma') + \sum_{\phi<\gamma, \alpha\in\tau(\phi)} \mu(\phi,\gamma)\bar{X}(\phi) & \text{if } \alpha \notin \tau(\gamma). \end{cases} \tag{4.8}$$

The action of $t(s)$ on the basis $X(\gamma)$ are given by the following formulas.

a) $t(s)X(\gamma) = -X(\gamma)$, if α is compact imaginary.

a)* $t(s)X(\gamma) = \gamma$, if α is real not satisfying the parity condition.

b) $t(s)X(\gamma) = s \times X(\gamma)$, if α is complex, $\alpha \in R^+(\gamma)$.

b)* $t(s)X(\gamma) = s \times X(\gamma)$, if α is complex, $\alpha \notin R^+(\gamma)$.

c) $t(s)X(\gamma) = -s \times X(\gamma) + c^\alpha(X(\gamma))$ if α is noncompact imaginary type II.

c)* $t(s)X(\gamma) = s \times X(\gamma)$, if α is real type II, satisfying the parity condition.

d) $t(s)X(\gamma) = -X(\gamma) + X(\gamma)_+^\alpha + X(\gamma)_-^\alpha$ if α is noncompact imaginary type I.

d)* $t(s)X(\gamma) = X(\gamma)$, if α is real type I, satisfying the parity condition.

In these formulas, $\times$ is the cross action and $c^\alpha(X,(\gamma)), X(\gamma)_+^\alpha, X(\gamma)_-^\alpha$ come from the Cayley transform of γ. These are defined in sections 4 and 7 of [13].

DEFINITION IV.1. *The $<_{LR}$ -preorder on $\mathcal{B}$ is the smallest order relation with the following property.*

Fix $w \in W, \gamma \in \mathcal{B}$ and write

$$t(w)\bar{X}(\gamma) = \sum_{\phi\in\mathcal{B}} a_\phi \bar{X}(\phi). \tag{4.9}$$

Then $a_\phi \neq 0 \Rightarrow \gamma <_{LR} \phi$. The cone over γ is defined to be

$$\bar{C}^{\mathrm{R}}(\gamma) = \{\phi \in \mathcal{B} | \gamma <_{LR\Phi}\}, \tag{4.10}$$

$$\bar{\mathcal{V}}^{\mathrm{R}}(\gamma) = \mathrm{span}\{\bar{X}(\phi) | \phi \in \bar{C}^{\mathrm{R}}(\gamma)\}. \tag{4.11}$$

DEFINITION IV.2. *The relation $\sim_{LR}$ is the smallest equivalence relation generated by*

$$\gamma \sim_{LR} \phi \Leftrightarrow \gamma <_{LR} \phi \text{ and } <_{LR} \gamma. \tag{4.12}$$

Then we define

$$C^{\mathrm{R}}(\gamma) = \{\phi \in \mathcal{B} | \phi \sim_{LR} \gamma\}, \tag{4.13}$$

$$\mathcal{V}^{\mathrm{R}}(\gamma) = \text{span } \{\bar{X}(\phi) | \phi \in \bar{C}^{\mathrm{R}}(\gamma)\}, \tag{4.14}$$

$$C_+^{\mathrm{R}}(\gamma) = \{\phi \in \mathcal{B} | \phi <_{LR} \gamma, \phi \not\sim_{LR} \gamma\}, \tag{4.15}$$

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$$\mathcal{V}^{\mathbb{R}}_{+}(\gamma) = \text{span } \{\bar{X}(\phi) \in \mathcal{C}^{\mathbb{R}}_{+}(\gamma)\}, \tag{4.16}$$

$$\mathcal{V}^{\mathbb{R}}(\gamma) = \{\bar{\mathcal{V}}^{\mathbb{R}}(\gamma)/\mathcal{V}^{\mathbb{R}}_{+}(\gamma).\}, \tag{4.17}$$

By their definition, $\bar{\mathcal{V}}^{\mathbb{R}}$, $\mathcal{V}^{\mathbb{R}}_{+}(\gamma)$ and $\mathcal{V}^{\mathbb{R}}(\gamma)$ are invariant under the action $t(\,)$.

Recall that the wavefront set $WF(\bar{X}(\gamma))$ of a representation $\bar{X}(\gamma)$ is a nilpotent orbit in g [7]. Most of its properties that we will use use are found in [1], [2] and [3].

Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$. Then we can also define

$$\bar{\mathcal{C}}^{\mathbb{R}}(\mathcal{O}) = \{\bar{X}(\gamma) \in \mathcal{B} | WF(\bar{X}(\gamma)) \subset \bar{\mathcal{O}}\}, \tag{4.18}$$

$$\mathcal{C}^{\mathbb{R}}_{+}(\mathcal{O}) = \{\bar{X}(\gamma) \in \bar{\mathcal{C}}^{\mathbb{R}}(\mathcal{O}) | \text{Dim}(\bar{X}(\gamma)) < 1/2 \dim \mathcal{O}\}, \tag{4.19}$$

$$\mathcal{C}^{\mathbb{R}}(\mathcal{O}) = \{\bar{X}(\gamma) \in \bar{\mathcal{C}}^{\mathbb{R}}(\mathcal{O}) | \text{Dim}(\bar{X}(\gamma)) = 1/2 \dim \mathcal{O}\}. \tag{4.20}$$

The corresponding objects in $\mathcal{G}$ will be denoted by the letter $\mathcal{V}^{\mathbb{R}}$.

Let $\mathcal{H}(\bar{\gamma}_a)$ be the category of highest weight modules with respect to the Borel subalgebra $\mathfrak{b}_a$ determined by $\bar{\gamma}_a$ and with (generalized) infinitesimal character $\bar{\gamma}_a$. Then the Grothendieck group $\mathcal{G}_{\mathcal{H}}(\bar{\gamma}_a)$ has as bases

$$\{M(w\bar{\gamma}_a)\}_{w \in W_a}, \quad \{L(w\bar{\gamma}_a)\}_{w \in W_a}, \tag{4.21}$$

where $M(w\bar{\gamma}_a)$ is the Verma module with highest weight $w\bar{\gamma}_a$ and $L(w\bar{\gamma}_a)$ is its unique irreducible quotient.

$W_a \times W_a$ acts by the formula

$$(w_L, w_R) \cdot M(w\bar{\gamma}_a) = M(w_L^{-1} w w_R \bar{\gamma}_a). \tag{4.22}$$

The left and right τ-invariants are defined as

$$\tau_L(w) = \{\alpha \in R^+(\bar{\gamma}_a) | w^{-1}\alpha \notin R^+(\bar{\gamma}_a)\}, \tag{4.23}$$

$$\tau_R(w) = \{\alpha \in R^+(\bar{\gamma}_a) | w\alpha \notin R^+(\bar{\gamma}_a)\}, \tag{4.24}$$

Let $s = s_\alpha$. Similar to 4.8, there are $\mu(y, w) \in \mathbb{N}$ such that

$$(s,1) \cdot L(w\bar{\gamma}_a) = \begin{cases} -L(w\bar{\gamma}_a) & \text{if } \alpha \in \tau_L(w), \\ L(sw\bar{\gamma}_a) + L(w\bar{\gamma}_a) + \sum\limits_{y<w, s \in \tau_L(y)} \mu(y,w) L(y\bar{\gamma}_a) & \text{if } \alpha \notin \tau_L(w). \end{cases} \tag{4.25}$$

Similar formulas hold for $(1, s)$ with the same $\mu(y, w)$. Here $<$ is the usual Bruhat order. As before, we can define cones and cells. This time there are left, right, double cones and double cells depending on whether we are using the action of $1 \times W_a, W_a \times 1$, or $W_a \times W_a$. We will denote them by the same notation as the objects coming from the $(\mathfrak{g}, K)$-modules, with a superscript L, R, or LR.

REMARK. The double cells can be interpreted as real cells for the complex group with Lie algebra $\mathfrak{g}$, viewed as a real group.

The two types of cells are related by the family of Jacquet functors

$$\mathcal{I}_H^i ; \mathcal{P}_H(\chi) \to \mathcal{H}(\chi) \tag{4.26}$$

defined in [5]. These functors have the following properties.

- $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\mathcal{I}_H^i(\bar{X}(\gamma))) \supset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\bar{X}(\gamma))$.
- Define

$$\mathcal{I} : \mathcal{G}(\chi) \to \bigoplus_H \mathcal{G}_H(\chi) \tag{4.27}$$

by $\mathcal{I}(\bar{X}(\gamma)) = \oplus_H \mathcal{I}_H(\bar{X}(\gamma)) = \oplus_H(\sum(-1)^i \mathcal{I}_H^i(\bar{X}(\gamma)))$. Then $\mathcal{I}$ is an injective intertwinining map for the actions of W_a and $1 \times W_a$.

In [13], to each pair $(G, \mathcal{B})$, a dual pair $(\check{G}, \check{\mathcal{B}})$ and a correspondence $\gamma \to \check{\gamma}$ from $\mathcal{B}$ to $\check{\mathcal{B}}$ is attached. We will use the following properties of this duality to estimate the number of parameters with certain properties as well as to write them down explicitly.

1. $\gamma <_{LR} \phi$ if and only if $\check{\phi} <_{LR} \check{\gamma}$.
2. Define a pairing

$$\langle \bar{X}(\gamma), \bar{X}(\check{\phi}) \rangle = (-1)^{l^I(\gamma)} \delta_{\phi\gamma}.$$

Then

$$\langle X(\gamma), X(\check{\phi}) \rangle = (-1)^{l^I(\gamma)} \delta_{\gamma\phi}.$$

3. The form $\langle , \rangle$ is invariant under the action $t(\ )$ in the sense that

$$\langle t(w)\bar{X}(\gamma), \bar{X}(\check{\phi}) \rangle = (-1)^{l(w)} \langle \bar{X}(\gamma), t(w)\bar{X}(\check{\phi}) \rangle. \tag{4.28}$$

In particular,

$$\mathcal{V}^{\mathrm{R}}(\gamma) \approx [\mathcal{V}^{\mathrm{R}}(\check{\gamma})]^* \otimes \mathrm{sgn}. \tag{4.29}$$

Here $l^I(\gamma)$ is the length function defined in 12.1 in [13].

COROLLARY IV.3. *$\mathcal{I}_H$ maps $\bar{\mathcal{V}}^{\mathrm{R}}(\mathcal{O})$ to $\bar{\mathcal{C}}^{LR}(\mathcal{O})$ and intertwines the action of W_a and $1 \times W_a$. By passage to the quotient, it also maps $\mathcal{V}^{\mathrm{R}}(\mathcal{O})$ to $\mathcal{V}^{LR}(\mathcal{O})$. The induced map $\mathcal{I} : \mathcal{V}^{\mathrm{R}}(\mathcal{O}) \to \otimes_H \mathcal{V}^{LR}(\mathcal{O})$ is also injective.*

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V. AN ESTIMA

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PROOF. The first part follows from property (1) of $\mathcal{I}_H^i$.

We thank D. Vogan for providing the following proof of the second assertion.

Suppose $\mathcal{I}$ is not injective. Then there is a combination $Z = \sum a_\phi \bar{X}(\phi)$ with $a_\phi \neq 0$ such that $WF(\bar{X}(\phi)) \subset \bar{\mathcal{O}}$ and $\mathrm{Dim}(\bar{X}(\phi)) = \dim \mathcal{O}$, but Z transforms according to a representation σ such that $\sigma \otimes \sigma$ belongs to a $\bar{C}^{LR}(\mathcal{O}')$ with $\dim \mathcal{O}' < \dim \mathcal{O}$. Consider $\check{Z}$. This transforms according to $\sigma \otimes \mathrm{sgn}$. This does not come from $\bar{C}^{LR}(W, F(\bar{X}(\check{\phi}))$. By the first part of Corollary, this is a contradiction. Q.E.D.

V. AN ESTIMATE ON THE NUMBER OF PARAMETERS

The results in this section are a variant of some general results on the number of special unipotent representations of a real reductive group, obtained by D. Vogan and one of the authors some time ago. A detailed account of these aforementioned results will appear elsewhere.

Fix a block $\mathcal{B}$ and two orthogonal sets $S_1, S_2 \subset \Pi(\bar{\gamma}_a)$. Let

$$\mathcal{B}(S_1, S_2) = \{\gamma \in \mathcal{B} | S_1 \subset \tau(\gamma), S_2 \not\subset \tau(\gamma)\} \tag{5.1}$$

Given a nilpotent orbit $\mathcal{O}$, we are interested in the cardinality of

$$\mathcal{B}(S_1, S_2) = \{\gamma \in \mathcal{B}(S_1, S_2) | WF(x(\gamma)) \subset \bar{\mathcal{O}}\}. \tag{5.2}$$

PROPOSITION V.1. *Let*

$$\mathcal{A} = \left\{ Z = \sum a_\phi \bar{X}(\phi) \;\middle|\; \begin{array}{ll} t(w)Z = (-1)^{l(w)} Z & \text{for } w \in W(S_1), \\ t(w)Z = Z & \text{for } w \in W(S_2). \end{array} \right\}$$

Then

$$|\mathcal{B}(S_1, S_2)| = \dim \mathcal{A}.$$

PROOF. Denote by $\mathcal{G}_\mathcal{B}(S_1, S_2)$ the linear span of $\mathcal{B}(S_1, S_2)$. Then by 4.8, any Z satisfying the condition for $W(S_1)$ is a combination of elements in $\mathcal{B}(S_1, \emptyset)$. Define a map

$$\Phi_{S_2} : \mathcal{G}_\mathcal{B}(S_1, S_2) \to \mathcal{A}, \tag{5.3}$$

$$\Phi_{S_2}(Z) = \sum_{w \in W(S_2)} t(w) Z. \tag{5.4}$$

Since $W(S_1)$ and $W(S_2)$ commute, Φ_{S_2} maps $\mathcal{G}_\mathcal{B}(S_1, S_2)$ to $\mathcal{A}$. Using formula 4.28, 4.29 that Φ_{S_2} is injective. To see that it is also surjective, consider the map Φ which projects $\mathcal{A}$ onto $\mathcal{G}_\mathcal{B}(S_1, S_2)$. The proof follows from formula 4.8. Q.E.D.

COROLLARY V.2. *Let*

$$\mathcal{A}(\mathcal{O}) = \left\{ Z = \sum a_\phi \bar{X}(\phi) \in \mathcal{G}_{\mathcal{B}}(S_1, S_2) \,|\, WF(\bar{X}(\phi)) \subset \bar{\mathcal{O}} \right\}.$$

Then

$$|\mathcal{B}(S_1, S_2)(\mathcal{O})| = \dim A(\mathcal{O}).$$

PROOF. This follows from V.1 and 4.27. Q.E.D.

PROPOSITION V.3.

$$|\mathcal{B}(S_1, S_2)| = \dim \ \mathrm{Hom}_{W_a} \left[\mathcal{G}_{\mathcal{B}} : \mathrm{Ind}_{W(S_1) \times W(S_2)}^{W_a} \epsilon_1 \otimes \epsilon_2 \right],$$

where $\epsilon_1 = \mathrm{sgn}, \epsilon_2 = \mathrm{triv}$.

PROOF. This is a standard fact about representations of compact groups and Proposition V.1. Q.E.D.

PROPOSITION V. 4. *Let*

$$\mathcal{G}_{\mathcal{B}}(S_1, S_2)(\mathcal{O}) = \mathrm{span}\{X(\gamma) | X(\gamma) \in \mathcal{B}(S_1, S_2)(\mathcal{O})\}.$$

Then

$$|\mathcal{B}(S_1, S_2)(\mathcal{O})| = \dim \ \mathrm{Hom}_{W_a}(\mathcal{G}_{\mathcal{B}}(S_1, S_2)(\mathcal{O}) : \mathrm{Ind}_{W_{\mathcal{B}}(H)}^{W_a} [\epsilon_{\mathcal{B}}(H)]).$$

PROOF. This follows from V.1-V.3. Q.E.D.

We summarize the results obtained in this section. For $\sigma \in \check{W}_a$, let

$$m_{\mathcal{B}}(\sigma) = [\sigma : \mathrm{Ind}_{W_{\mathcal{B}}(H)}^{W_a} \{\epsilon_{\mathcal{B}}(H)\}], \tag{5.5}$$

$$m_{\mathcal{S}}(\sigma) = [\sigma : \mathrm{Ind}_{W_{\mathcal{S}} \times W(S_2)}^{W_a} \{\epsilon_1 \otimes \epsilon_2\}]. \tag{5.6}$$

THEOREM V.5.

$$|\mathcal{B}(S_1, S_2)(\mathcal{O})| = \sum_{\sigma \otimes \sigma \in \bar{\mathcal{C}}^{LR}(\mathcal{O})} m_{\mathcal{B}}(\sigma) m_s(\sigma). \tag{5.7}$$

VI. THE CAS

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VI. THE CASE $GL(2n, R)$ and $U(p,q)$

In this section we illustrate the notions introduced in section 4 and 5, and compute $|\mathcal{B}(S_1, S_2)(\mathcal{O})|$ for the case relevant to the degenerate series in section 1-3 explicitly.

Fix the Cartan involution to be

$$(6.1) \qquad \theta(x) = (x^{-1})^{-t} \text{ for } x \in G,$$

$$(6.2) \qquad \theta(X) = -X^t \text{ for } X \in \mathbf{g}.$$

Then a set of representatives for the conjugacy classes of Cartan subgroups are given by $H^r = T^r A^r$,where

$$(6.3) \qquad T^r = \left\{ \mathrm{diag}[\, t(\phi_1, \alpha_1), \ldots, t(\phi_r, \alpha_r), \epsilon_1, \ldots, \epsilon_{2n-2r}] | \phi_i \in \mathbf{R} \right\},$$

$$(6.4) \qquad A^r = \{\mathrm{diag}[\, e^{x_1}, e^{x_1}, \ldots, e^{x_r}, e^{x_r}, e^{y_1}, \ldots, e^{y_{2n-2r}}] | x_i, y_i \in \mathbf{R} \},$$

Then $\epsilon_j = \pm 1$ and

$$(6.5) \qquad t(\phi_i, \alpha_i) = \begin{pmatrix} \alpha_i & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \cos\phi_i & \sin\phi_i \\ -\sin\phi_i & \cos\phi_i \end{pmatrix}.$$

The complexified Lie algebra $\mathbf{h}^r = \mathbf{t}^r + \mathbf{a}^r$ can be identified with $\mathbf{C}^{2n} \cong \mathbf{C}e_1 \otimes \ldots \otimes \mathbf{C}e_{2n}$ such that

$$(6.6) \qquad \mathbf{t}^r = \left\{ \sum_{1 \le i \le r} \phi_i (e_{2i-1} - e_{2i}) | \phi_i \in \mathbf{C} \right\},$$

$$(6.7) \qquad \mathbf{a}^r = \left\{ \sum_{1 \le i \le r} x_i (e_{2i-1} - e_{2i}) + \sum_{1 \le j \le 2n-2r} y_j e_{j+2r} | x_i, y_j \in \mathbf{C} \right\}.$$

The Cartan involution is in this notation,

$$(6.8) \qquad \theta^r(e_{2i-1}) = -e_{2i} \text{ for } 1 \le i \le r,$$

$$\theta^r(e_j) = -e_j \text{ for } 2r < j. \tag{6.9}$$

The roots $e^*_{2i-1} - e^*_{2i}$ are noncompact imaginary, $e^*_k - e^*_l$ for $k, l > 2r$ real and all others complex. Let

$$M^r = \mathrm{Cent}(G; A^r) \cong GL(2,\mathbf{R})^r \times GL(1,\mathbf{R})^{2n-2r} \tag{6.10}$$

be the Levi component attached to A^r. A Langlands parameter for an irreducible (g, K)-module in $\mathcal{P}(H)$ with not necessarily regular integral infinitesimal character is given by $\{\pi_i|\det|^{\zeta_i}\}$ where π_i are discrete series for $GL(2,\mathbf{R})$ and by (δ_j, ν_j), character of $GL(1,\mathbf{R})$ determined by

$$(\delta_j, \nu_j)\ (\epsilon_j e^{y_j}) = \delta_j(\epsilon_j)\, e^{\nu_j y_j}. \tag{6.11}$$

The parameter $\bar{\gamma}$ in 4.1 is given by

$$\bar{\gamma} = (\mathrm{d}\pi_1 + \zeta_1, \dots, \mathrm{d}\phi_r + \zeta_r, \nu_1, \dots, \nu_{2n-2r}). \tag{6.12}$$

We write it as (λ^G, ν), where

$$\lambda^G = (\mathrm{d}\phi_1, \dots, \mathrm{d}\pi_{2r}, 0, \dots, 0), \tag{6.13}$$

$$\nu = \left(\zeta_1, \dots, \zeta_{2r}, \dots, \nu_1, \nu_{2n-2r}\right). \tag{6.14}$$

Γ in 4.1 is determined by 4.2 and the δ_j.

Let $P^r = M^r N^r$ be the parabolic subgroup such that the roots of N^r are dominant for ν. Then the standard module $X(\gamma)$ is

$$X(\gamma) = \mathrm{Ind}_{P_r}^{G}[\,\pi_i|\det|^{\zeta_i} \otimes (\delta_j, \nu_j) \otimes \mathbf{1}\,]. \tag{6.15}$$

In the case of $GL(2n,\mathbf{R})$, this has a unique irreducible quotient (even at nonintegral singular infinitesimal character) denoted $\bar{X}(\gamma)$.

The Langlands quotient is also determined by the lowest K-type as follows. Parametrize $\widehat{O(2n)}$ as in [15] by

$$\mu = [(\mu_1, \dots, \mu_n), \eta], \quad \mu_1 \geq \dots \geq \mu_n \geq 0, \tag{6.16}$$

where $\mu_i \in \mathrm{N}$ give the highest weight and $\eta = 0$ or 1 if $\mu_n = 0$, $\eta = 1/2$ otherwise.

The relation of μ to γ is as follows. Let r be the largest integer such that $\mu_r \geq 2$. Then

$$\lambda^G = (\mu_1 - 1, \dots, \mu_r - 1, 0, \dots, 0). \tag{6.17}$$

Let $q = 2n - 2r = 2(n-r)$ and $p = |\{j | \mu_j = 1\}|$. Define q_0, q_1 such that $q = q_0 + q_1$ by the formula

$$q_0 = \begin{cases} q - p & \text{if } \eta = 0, 1/2 \\ p & \text{if } \eta = 1, 1/2. \end{cases} \tag{6.18}$$

Then $GL(1,\mathbf{R})^{2n-2r}$ is the split Cartan subgroup of

$$L(\mu) = GL(q_0) \times GL(q_1). \tag{6.19}$$

$L(\mu)$ is the intersection of the Levi component of a complex θ-stable parabolic subgroup with G. The δ_j are obtained by restricting $1 \otimes \det$ of $L(\mu)$ to $GL(1,\mathbf{R})^{2n-2r}$.

We will write the parameter γ as

$$\gamma = (\underline{\gamma_1, \gamma_2}, \dots, \gamma_{2r-1}, \gamma_{2r}, \gamma_{2r+1}^{\epsilon_{2r+1}}, \dots, \underline{\gamma_{2n}}^{\epsilon_{2n}}), \tag{6.20}$$

where

$$\gamma_{2i-1} - \gamma_{2i} = 2\pi_i, \quad \gamma_{2i-1} + \gamma_{2i} = 2\zeta_i,$$

$$\gamma_j = \nu_j, \qquad \epsilon_j = \begin{cases} + & \text{for } \delta_j = \text{triv}, \\ - & \text{for } \delta_j \neq \text{triv}. \end{cases}$$

γ is integral if $\gamma_k - \gamma_l \in \mathbf{Z}$ for all k, l.

Let $U(p,q)$ with $p \geq q$ be the group of $m \times m$ matrices which leave the hermitian form

$$\sum_{i \leq p} |x_i|^2 - \sum_{i > p} |x_i|^2$$

invariant. Its Lie algebra is the subalgebra of skew-hermitian matrices that leave this form invariant.

The conjugacy classes of Cartan subalgebras are parametrized as follows. For each $r \leq q$ let $H^r = T^r A^r$ be the Cartan subgroup such that

$$\begin{aligned} T^r = \{(&\mathrm{d}(i\phi_1), \dots, \mathrm{d}(i\phi_{p-r}), \mathrm{e}(i\phi_{p-r+1}), \dots, \mathrm{e}(i\phi_p), \\ &\mathrm{d}(i\phi_{p+r+1}), \dots, \mathrm{d}(i\phi_{p+q}))\}, \end{aligned} \tag{6.21}$$

$$A^r = \{(a(x_{p-r+1}), \dots, a(x_p))\}, \tag{6.22}$$

where

$$d(i\phi) = \exp[i\phi E_{i,i}], e(i\phi) = \exp[i\phi(E_{i,i} + E_{p+i,p+i}], \tag{6.23}$$

$$a(x_j) = \exp[x_j(E_{p-j,p+j} + E_{p+j,p_j}] \tag{6.24}$$

and $E_{i,j}$ is the matrix with a 1 in the (i,j) entry, 0 otherwise. The Lie algebra is written

$$t^r = \{(i\phi_1, \dots, i\phi_{p-r}, i\phi_{p-r+1}, \dots, i\phi_p | i\phi_{p-r+1}, \dots, i\phi_p, i\phi_{p+r+1}, \dots, i\phi_{p+q})\},$$

$$a^r = \{(0, \dots, 0, x_{p-r+1}, \dots, x_p | -x_{p-r+1}, \dots, -x_p, 0, \dots, 0)\}.$$

Since the Cartan subgroups are connected, we replace the parameter γ by $\bar{\gamma} = (\lambda^G, \nu)$. In coordinates, (λ^G, ν) is given by

$$\begin{aligned} &(\eta_1, \dots, \eta_{p-r}, \eta_{p-r+1} + \nu_1, \eta_p + \nu_r \\ &|\eta_{p-r+1} - \nu_1, \dots, \eta_p - \nu_r, \eta_{p-r+1}, \dots, \eta_{p+q}) \end{aligned} \tag{6.25}$$

This parameter satisfies the parity condition with respect to a real root $\alpha_j(\nu) = 2\nu_j$ if and only if $\lambda^G(m_\alpha) = \exp[2\pi i \eta_{p-r+j}] = -(-1)^{2\nu_j}$. ($m_\alpha$ is defined as in 4.3.6 of [14]). Note also that the infinitesimal character is integral if and only if the coordinates in 6.3.4 are all integers or half-integers. At the infinitesimal character ρ of the trivial representation, there are two blocks, one with dual block in $GL(2n, \mathbb{R})$ and the other occuring in the case $p = q$ only, with dual block in $U^*(2n)$.

In the following, $\mathcal{G}$ will refer to the block with dual in $GL(2n, \mathbb{R})$.

Recall that the nilpotent orbits in $GL(2n, \mathbb{C})$ are parametrized by their Jordan canonical form, so they are in 1-1 correspondence with partitions of $2n$. The representations of the Weyl group W_a are also parametrized by partitions of $2n$. Tensoring with sgn changes the partition to its transpose. Given a nilpotent orbit $\mathcal{O}$, we denote by $\sigma(\mathcal{O})$ the representation corresponding to the partition defining the Jordan canonical form of $\mathcal{O}$. Recall also that, the (real) nilpotent orbits in $GL(2n, \mathbb{R})$ are also parametrized by partitions. In $U(p,q)$ the nilpotent orbits are parametrized by signed partitions. This means that, if we picture each partitions as rows of squares, each square gets a + or a − subject to the condition that they alternate in rows. The number of +'s must equal p, the number of −'s must equal q.

DEFINITION VI.1. *We will denote a partition by $1^{k_1} 2^{k_2} \dots$, where k_i is the number of times row i occurs in the partition. (The terms with $k_i = 0$ will of course be*

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omitted.) For a signed partition we will write $1^{+p_1}1^{-q_1}\dots$, *where* p_i *is the number of rows of size* i *starting with* + *and* q_i *is the number of rows of size* i *starting with* −.

Given a partition η, *we will denote by* $\mathcal{O}(\eta)$ *and* $\sigma(\eta)$ *the nilpotent and representation attached to* η *as before.*

From [4], we recall the following result.

THEOREM VI.2.

$$m[\sigma(\mathcal{O}) : \mathcal{G}] = |\{\textit{orbits in } U(p,q) \textit{ corresponding to the partition of } \mathcal{O}\}|.$$

This computes $m_{\mathcal{B}}(\sigma)$ in 5.6.

Write the infinitesimal character of $V(\chi)$

$$\begin{aligned}\gamma(s) = \Big(&\frac{n-1}{2}+s,\dots,\frac{n-1}{2}-s,\frac{n-1}{2}-s,\dots,\\ &-\frac{n-1}{2}+s,-\frac{n-1}{2}+s,\dots,-\frac{n-1}{2}-s\Big).\end{aligned} \tag{6.26}$$

The condition in Proposition II.1 translates into $2s \in \mathbf{N}$.

The WF-set of $V(\chi)$ is $\overline{\mathcal{O}(2n)}$. By [2], [3], the cone $\bar{\mathcal{V}}^{LR}$ is formed of representations corresponding to the partitions $2^{n-i}1^{2i}$.

The set S_2 is formed of the roots $\epsilon_l - \epsilon_{l+1}$ for which the infinitesimal character is singular (i.e. $\langle\gamma, \epsilon_l - \epsilon_{l+1}\rangle = 0$.) This is justified in view of the following proposition.

PROPOSITION VI.3. *Let* $\mathcal{L}(\gamma)$ *be the category of Harish-Chandra modules with integral infinitesimal character* γ *as before. Let* γ_{reg} *be a regular integral infinitesimal character. Then*

$$\mathcal{L}(\gamma) = \bigcup_{\mathcal{B}} \mathcal{B}(\gamma)$$

where each $\mathcal{B}(\gamma)$ *is in 1-1 correspondence with a* $\mathcal{B}(\emptyset, S_2)(\gamma_{\text{reg}})$ *via translation functors to singular infinitesimal character (e.g. [12]), given in terms of tensoring with finite dimensional representations and projecting onto generalized infinitesimal character.*

To find S_1 we argue as follows. Let $R^+(\gamma)$ be a positive system such that

$$\alpha \in R^+(\gamma) \text{ if } \langle\alpha,\gamma\rangle > 0. \tag{6.27}$$

If α is simple, let $\gamma_\alpha = \gamma - \langle\gamma,\check{\alpha}\rangle\alpha$ and $F(\gamma_\alpha)$ be the finite dimensional irreducible module with extremal weight $-\langle\gamma,\check{\alpha}\rangle\alpha$.

We define a functor

$$T_\alpha(X) = P_{\gamma_\alpha}[\,X \otimes F(\gamma_\alpha)\,] \tag{6.28}$$

where P_{γ_α} is projection onto infinitesimal character γ_α. This functor is exact.

Using the aforementioned equivalence with $\mathcal{B}(\emptyset, S_2)$ we find that $T_\alpha(\bar{X}(\gamma)) = 0$ if and only if $\alpha \in \tau(\gamma_{\text{reg}})$.

Let now α be one of the simple roots such that $\langle \alpha, \gamma \rangle > 0$. Then we find that $T_\alpha(V(\chi)) = 0$. Thus every irreducible composition factor of $V(\chi)$ has the same property.

We have proved the following result.

THEOREM VI.4. *The only possible factors of $V(\chi)$ are the ones coming from the irreducible representations in $\mathcal{B}(S_1, S_2)(\mathcal{O})$.*

PROPOSITION VI.5. *Let ϵ_1, ϵ_2 be the representations attached to S_1, S_2 as in section 5. Then*

$$m_S(\sigma(2^{n-i}1^{2i})) = \begin{cases} 1 & \text{if } 2s \le i. \\ 0 & \text{if } 2s < i. \end{cases}$$

PROOF. This follows by the Littlewood-Richardson rule. See for example [9] for the description of the rule. We omit the details. Q.E.D.

We determine the parameters in $\mathcal{B}(S_1, S_2)(\mathcal{O})$.

Assume that the WF-set is $\mathcal{O}(2^n)$. and the infinitesimal character is as in 6.26. Then the WF-set in the dual is either $\{n^+n^+\}, \{n^+n^-\}$, or $\{n^-n^-\}$.

To $\{n^+n^-\}$ we attached the parameter

$$\left(\underline{\frac{n-1}{2}+s, \frac{n-1}{2}-s}, \underline{\frac{n-1}{2}+s-1, \frac{n-1}{2}-s-1}, \dots\right). \tag{6.29}$$

This has lowest K-type$[\,(2s+1, \dots, 2s+1), 1/2]$. This is a derived functor induced reopresentation in $GL(2n, \mathbf{R})$.

Now consider the case $\{n^+n^+\}, \{n^-n^-\}$. The parameters in $GL(2n, \mathbf{R})$ are obtained from the parameter in 6.29 corresponding to $GL(2n-2, \mathbf{R})$ of the form

$$\left(\underline{\frac{n-1}{2}+s-1, \frac{n-1}{2}-s}, \underline{\frac{n-1}{2}+s-2, \frac{n-1}{2}-s-1}, \dots\right). \tag{6.30}$$

by adding $\left(\underline{\frac{n-1}{2}+s}\right)^{\epsilon}, \left(\underline{-\frac{n-1}{2}-s}\right)^{\epsilon}$, if $2s$ is even, $\left(\underline{\frac{n-1}{2}+s}\right)^{\epsilon}$, $\left(\underline{-\frac{n-1}{2}-s}\right)^{-\epsilon}$, if $2s$ is odd. Their lowest K-types are $[\,(2s, \dots, 2s, 0), 0]$

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The reason for this is that in the dual group at regular infinitesimal parameter, these correspond to derived functor induced representations from the parabolic subalgebra determined by

$$\xi = \begin{cases} (1,0,\ldots,0|0,\ldots,0,-1) & \text{for } n \text{ even}, \\ (1,0,\ldots,0,-1|0,\ldots,0) & \text{for } n \text{ odd}, \end{cases}$$

and the trivial representation tensored with the representation corresponding to $\{(n-1)^+,(n-1)^-\}$. The relevant calculation of WF-set are in [4].

Assume that the WF-set is strictly smaller than $\mathcal{O}(2^n)$. Then the parameters are obtained as follows.

Start with the parameter

$$(6.31) \qquad \left(\frac{n-1}{2}+s-j, \frac{n-1}{2}-s, \ldots, \frac{-n-1}{2}+s, \frac{-n-1}{2}-s+j,\right)$$

and add to it

$$\left(\frac{n-1}{2}+s\right)^{\epsilon}, \left(-\frac{n-1}{2}-s\right)^{\epsilon}, \ldots, \left(\frac{n-1}{2}+s-j+1\right)^{\epsilon},$$
$$\left(-\frac{n-1}{2}-s+j-1\right)^{\epsilon}, \text{ or } \left(\frac{n-1}{2}+s\right)^{\epsilon}, \left(-\frac{n-1}{2}-s\right)^{-\epsilon}, \ldots,$$
$$\left(\frac{n-1}{2}+s-j+1\right)^{\epsilon}, \left(-\frac{n-1}{2}-s+j-1\right)^{-\epsilon}.$$

The K-types are obtained by the formulas in 6.2.

The reason why these representations have the right WF-set is the same as for the case $\mathcal{O}(2^n)$.

The following lemma is well known.

LEMMA VI.6. *The K-types in $V(\chi)$ are of the form*

$$[(2s-j+1,\ldots,2s-j+1,0,\ldots,0),0 \text{ or } 1] \quad \text{for } 2s-j+1 \text{ even},$$
$$[(2s-j+1,\ldots,2s-j+1,1,\ldots,1),1/2], \quad \text{for } 2s-j+1 \text{ odd}.$$

All K-types have multiplicitly one.

The computations of the minimal K-type and VI.6 show

PROPOSITION VI.7. *For $i > 0$ there are exactly 2 representations in $\mathcal{B}(S_1,S_2)(\mathcal{O}(2^{n-i}1^{2i}))$ whose minimal K-type is in $\mathrm{Ind}_P^G(\chi)$ for some character χ.*

VII. THE MAIN THEOREM

Let $\chi = ((\epsilon, s), (0, -s))$ be a character of $GL(n) \times GL(n)$.

THEOREM VII.1. *Suppose* $2s \in \mathbb{Z}$ *and* $0 < 2s \leq n$. *For* $0 \leq k \leq n$ *put* $V_k(\chi) = \{f \in V(\chi), \operatorname{supp}(\check{f}) \subseteq \Omega_k\}$. *Put* $r = \left[\frac{n-2s}{2}\right] - 1$.

- *Suppose* $\epsilon = 0$. *Then*

$$0 \neq V_{n-2s}(\chi) \neq V_{n-2s+2}(\chi) \neq \ldots \neq V_{n-2(s-r)}(\chi) \neq V_n(\chi).$$

Futhermore all quotients are irreducible.

- *Suppose* $\epsilon = 1$. *Then*

$$0 \neq V_{n-2s+1}(\chi) \neq V_{n-2s+3}(\chi) \neq \ldots \neq V_{n-2(s-r)+1}(\chi) \neq V_n(\chi).$$

Furthermore all quotients are irreducible.

REMARK. Since the other two families of degenerate series representations are obtained by tensoring with a one dimensional representation we also have similar results for those other families.

PROOF. By III.4 this defines a filtration by invariant subspaces. By III.5 all the quotient representations are distinct. Since all the quotient representations are in $B(S_1, S_2)(\overline{\mathcal{O}(2^n)})$ proposition VI.7 and the previous remark imply the theorem.

Q.E.D.

REFERENCES

[1] D. BARBASCH and D. VOGAN: The local structure of characters. *Journal of Functional Analysis,* **37**, 27-55, 1980.

[2] D. BARBASCH and D. VOGAN: Primitive ideals and orbital integrals in complex classical Lie groups. *Math. Annalen,* **259**, 153-199, 1982.

[3] D. BARBASCH and D. VOGAN: Primitive ideals and orbital integrals in complex exceptional Lie groups. *J. of Algebra,* **80**, 350-382, 1983.

[4] D. BARBASCH and D. VOGAN: Weyl group representations and nilpotent orbits. In *Representation theory of Reductive Groups,* pp. 21-33, Birkhäuser, 1983.

[5] L. CASIAN: Primitive ideals and representation theory. *J. of Algebra,* **101**, 497-515, 1986.

[6] I. GELFAND: *Unitary representations of the Classical Groups.* Volume 36, Trudy Mat. Inst. Steklov, 1950.

[7] R. HOWE: Wave front sets of representations of the Classical Groups. In *Automorphic forms, Representation theory and Arithmetic,* Tata Institute Bombay 1979.

[8] H. JABOSON and M. VERGNE: The Wave and Dirac operators and representations of the conformal group. *Journal of Functional Analysis,* **24**, 52-106, 1977.

[9] G. JAMES
Not

[10] H. MATU
89,

[11] B. SPEH:
SU

[12] B. SPEH
Act

[13] D. VOGA
49,

[14] D. VOGA

[15] D. VOGA
449

[9] G. JAMES: *The representation theory of Symmetric groups.* Volume 682 of *Lecture Notes in Mathematics*, Springer Verlag, 1978.

[10] H. MATUMOTO: Whittaker vectors and associated varities. *Inventiones Mathematicae*, 89, 219-224, 1987.

[11] B. SPEH: Degenerate series representations of the universal covering group of $SU(2,2)$. *Journal of Functional Analysis,* 33, 95-118, 1979.

[12] B. SPEH and D. VOGAN: Reducibility of generalized principal series representations. *Acta Math.*, **145**, 227-299, 1980.

[13] D. VOGAN: Irreducible characters of semisimple Lie groups IV. *Duke Math. Journal*, 49, 943-1073, 1982.

[14] D. VOGAN: *Representations of Real Reductive Lie Groups.* Birkhäuser, 1981.

[15] D. VOGAN: The unitary dual of $GL(n)$ over an archimedean field. *Invent. Math.*, **83**, 449-505, 1986.