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The Capelli Identity, Tube Domains, and the Generalized Laplace Transform*

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INTRODUCTION

The key result in this paper is a proof of a large class of identities, generalizing the Capelli identity [C].

The Capelli identity is a centerpiece of 19th century invariant theory. It asserts the equality of two differential operators on the n^2-dimensional space $M(n, \mathbf{R})$ of $n \times n$ real matrices. Let E_{ij} denote the ijth elementary matrix, let x_{ij} denote the linear functions on $M(n, \mathbf{R})$ dual to E_{ij}, and write ∂_{ij} for the partial differential operator $\partial/\partial x_{ij}$. Then this identity says

$$\det(\pi_{ij} + (n-i)\,\delta_{ij}) = \det(x_{ij})\det(\partial_{ij}), \tag{0.1}$$

where δ_{ij} is the Kronecker delta, and π_{ij} is the vector field

$$\pi_{ij} = \sum_{k=1}^{n} x_{ki}\partial_{kj}. \tag{0.2}$$

By the determinant of an $n \times n$ matrix (A_{ij}) of (possibly) noncommuting variables, we mean the expression

$$\sum_{w \in S_n} \operatorname{sgn}(w)\, A_{w(1),1} \cdots A_{w(n),n}; \tag{0.3}$$

and the left side of (0.1) is to be understood in this sense.

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The vector field in (0.2) arises as follows. Let π be the action of $GL(n, \mathbf{R})$ on the space of functions on $M(n, \mathbf{R})$ given by

$$\pi(g)\, f(x) = f(xg). \tag{0.4}$$

The corresponding action (also denoted by π) of the Lie algebra $\mathfrak{g} = M(n, \mathbf{R})$ of $GL(n, \mathbf{R})$ is by differential operators, and if E_{ij} is as above, then we have

$$\pi(E_{ij}) = \pi_{ij}. \tag{0.5}$$

The differential operator on the left side of (0.1) is the image of an element in the center of the enveloping algebra $\mathscr{U}(\mathfrak{g})$. (One recognizes the entries $n-i$ as a "ρ-shift".)

The identity is intriguing from a number of points of view. (It is called "mysterious" in the Atiyah–Bott–Patodi paper [ABP].) For example, we understand the right side of (0.1) in terms of the additive structure of $M(n, \mathbf{R})$ (hence the use of constant coefficient vector fields), whereas the left side is understood in terms of the multiplicative structure of $GL(n, \mathbf{R})$ (left-invariant vector fields).

In [W], Hermann Weyl makes essential use of this identity in his approach to invariant theory. However, the classical proof of the identity is not very revealing. An alternative approach to the identity is described in [Ho], whedre the connection of the left side of (0.1) with the center of $\mathfrak{gl}_n$ is also made explicit. It is remarkable that, in some sense, Capelli was already aware of this connection! (See [B, p. 77].)

The approach we take—and one which leads to our generalization—is to regard $M(n, \mathbf{R})$ *not* as a Lie algebra, but rather as a *Jordan* algebra ($A \circ B = (AB + BA)/2$)! The determinant is the norm, and the multiplicative action of $GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$ (left and right) is the action of the "norm-preserving" group [J].

We observe next that (0. 1) is equivalent to a *holomorphic* differential identity on the *complex* Jordan algebra $M(n, \mathbf{C})$. This Jordan algebra has another real form, namely the Hermitian $n \times n$ matrices. This real form is a "formally real" Jordan algebra. After this observation, Jordan algebras serve merely as a background for this paper and we reconnect with the Lie theory associated with formally real Jordan algebras. In particular we consider the equivalence of categories between simple, formally real Jordan algebras on the one hand, and irreducible, symmetric tube domains on the other. This equivalence is due to [K], and is very highly developed in [KW].

Let us recall some elementary facts about the Jordan side of the above equivalence. Most of these may be found in Chapter XI of [BK]. A real

Jordan algebra is a finite-dimensional vector space $\mathfrak{u}$ over $\mathbf{R}$, with a multiplication $x \circ y$ satisfying $x \circ y = y \circ x$ and $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$. The algebra is called formally real if $x^2 + y^2 = 0$ implies $x = y = 0$. Simplicity is defined in an obvious way. Assume $\mathfrak{u}$ is simple and formally real. Then $\mathfrak{u}$ has an identity element e. Let $\{e_1, ..., e_n\}$ be a maximal set of orthogonal idempotents giving the Peirce decomposition (see [BK, especially XI.5])

$$e = e_1 + \cdots + e_n. \tag{0.6}$$

The cardinality n of this set is called the rank of the Jordan algebra.

Any element $x \in \mathfrak{u}$ generates an associative algebra and in fact there exist $\psi_j \in \mathscr{S}^j(\mathfrak{u}^*)$ (i.e., polynomial functions of degree j), $j = 1, ..., n$, such that x satisfies the identity

$$x^n + \sum_{j=1}^{n} (-1)^j \psi_j(x)\, x^{n-j} = 0. \tag{0.7}$$

Let $S = \sum \mathbf{R} e_j$. In case $x \in S$, so that

$$x = \sum_j r_j e_j, \tag{0.8}$$

then

$$\lambda^n + \sum_{j=1}^{n} (-1)^j \psi_j(x)\, \lambda^{n-j} = \prod_{j=1}^{n} (\lambda - r_j). \tag{0.9}$$

Let us write φ for the polynomial function $\psi_n \in \mathscr{S}^n(\mathfrak{u}^*)$. Then φ is called the norm of the Jordan algebra. Note that if x is given by (0.8), then $\varphi(x) = r_1 \cdots r_n$.

Jacobson [J] (also see [BK, II.5]) has associated two Lie algebras to $\mathfrak{u}$. For any $x \in \mathfrak{u}$, let $L(x)$ denote the operator of left multiplication by x. Let $\mathfrak{p} = \{L(x) \mid x \in \mathfrak{u}\}$ and let $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$. Then $\mathfrak{k}$ is the Lie algebra of the "automorphism group" of $\mathfrak{u}$. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}. \tag{0.10}$$

Then $\mathfrak{g}$ is a reductive Lie algebra and (0.10) is the Cartan decomposition of $\mathfrak{g}$.

Let G (resp. K) be the analytic subgroup of $GL(\mathfrak{u})$ corresponding to $\mathfrak{g}$ (resp. $\mathfrak{k}$). Then G is the identity component of the "norm-preserving group" and K is a maximal compact subgroup. There is a character $\chi: G \to \mathbf{R}^+$ such that with respect to the natural action of G on $\mathscr{S}^n(\mathfrak{u}^*)$, we have $g \cdot \varphi = \chi(g)\varphi$. Furthermore, $\mathfrak{u}$ has a K-invariant positive definite bilinear form $(\, , \,)$.

In fact one has

$$K = \{k \in G \mid k \cdot e = e\}, \tag{0.11}$$

so that if C is the open set in $\mathfrak{u}$ defined by

$$C = G \cdot e, \tag{0.12}$$

then C is a symmetric space, i.e.,

$$C = G/K. \tag{0.13}$$

One knows that any element in $\mathfrak{u}$ is K-conjugate to an element of $S = \sum \mathbf{R}e_j$. This, together with the KA_+K-decomposition of G, readily implies (see [BK])

PROPOSITION 1. *For $y \in \mathfrak{u}$, the following are equivalent*:

(a) $y \in C$,

(b) $\varphi(y) \neq 0$ *and* $y = x^2$, *where* $x \in \mathfrak{u}$,

(c) $\varphi(y) \neq 0$ *and* $y = \sum_j x_j^2$, *where* $x_j \in \mathfrak{u}$,

(d) $y = \exp(x)$, *where* $x \in \mathfrak{u}$,

(e) *y is K-conjugate to an element $x \in S$, where in the notation of* (0.8) *all $r_j > 0$.*

We note that (c) implies that C is a self-dual cone in $\mathfrak{u}$.

If we identify $\mathfrak{u}$ with its dual via $(\ ,\ )$, φ corresponds to a differential operator $\partial(\varphi)$ on $\mathfrak{u}$, and for each positive integer m, we may consider the "generalized Cayley operator" $D_m = \varphi^m \partial(\varphi)^m$. Since C is open, we may regard D_m as a differential operator on C. It is easy to check that D_m is a G-invariant differential operator on C.

Let $G = NAK$ be the Iwasawa decomposition of G; then it follows that the submanifold $A \cdot e$ of $C = G/K$ is transversal to each N-orbit. Identifying $A \cdot e$ with A, by virtue of the simple transitivity of the action, we see that the N-radial part (see [H2, II.3.3], also (4.5) below) of D_m is a differential operator on A. This corresponds, via the exponential map, to a polynomial function on $\mathfrak{a}^*$, where $\mathfrak{a}$ is the Lie algebra of A. This polynomial, after a standard ρ-shift, becomes a W-invariant polynomial p_m, where W is the Weyl group of $\mathfrak{a}$ in K (see [H2, Theorem II.5.17]). In our situation, W always turns out to be the symmetric group S_n! Thus p_m is a "symmetric" polynomial.

Our main result, Theorem 1, is an explicit formula for p_m.

To show how this leads to differential identities, we need some results from [H3].

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$. Write

$$\mathfrak{h} = \mathfrak{a} + \mathfrak{t}. \tag{0.14}$$

Then $\mathfrak{h}$ is a maximally split Cartan subalgebra of $\mathfrak{g}$. Using the subscript c to denote complexification, we have that $\mathfrak{h}_c = \mathfrak{a}_c + \mathfrak{t}_c$ is a Cartan subalgebra of $\mathfrak{g}_c$. Dualizing, we have the decomposition $\mathfrak{h}_c^* = \mathfrak{a}_c^* + \mathfrak{t}_c^*$; and identifying $\mathfrak{a}^*$ with the elements of $\mathfrak{a}_c^*$ which are real valued on $\mathfrak{a}$, we have the restriction map res from polynomials on $\mathfrak{h}_c^*$ to polynomials on $\mathfrak{a}^*$.

Let W_c be the Weyl group of $\mathfrak{h}_c$ in $\mathfrak{g}_c$ and write $I(\mathfrak{a})$ and $I(\mathfrak{h}_c)$ for W- and W_c-invariant polynomials on $\mathfrak{a}^*$ and $\mathfrak{h}_c^*$, respectively. Choose a positive root system for $\mathfrak{t}_c$ in $\mathfrak{m}_c$, and let ρ_0 be half the sum of positive roots. For a polynomial f on $\mathfrak{h}_c^*$ let us write Tf for the polynomial given by $Tf(\mu) = f(\mu + \rho_o)$, and write $I_o(\mathfrak{h}_c)$ for the image of $I(\mathfrak{h}_c)$ under T.

Then res maps $I_o(\mathfrak{h}_c)$ into $I(\mathfrak{a})$, although the map need not be surjective if G is not a classical group. However, by the proof of Theorem 2.2 in [H3], res is always surjective at the level of the associated quotient fields. It follows that we can find nonzero polynomials q_m and q'_m in $I(\mathfrak{h}_c)$ such that

$$p_m(\lambda)\, q'_m(\lambda + \rho_o) = q_m(\lambda + \rho_o). \tag{0.15}$$

Write ψ for the Harish–Chandra isomorphism (see [Hm]) from the center of the enveloping algebra of $\mathfrak{g}_c$ to the algebra of W_c-invariant polynomials on $\mathfrak{h}_c^*$. Now every element Z, in the center, gives rise to a G-invariant holomorphic differential operator $\pi^c(Z)$ on $\mathfrak{u}_c$. On the other hand, the differential operator D_m also has a unique extension to a holomorphic differential operator D_m^c on $\mathfrak{u}_c$.

Let $Z_m = \psi^{-1}(q_m)$ and $Z'_m = \psi^{-1}(q'_m)$. Then, as in Theorem 2.2 of [H3], we have

$$D_m^c \pi^c(Z'_m) = \pi^c(Z_m). \tag{0.16}$$

As remarked in [H3], if G is classical, we can always choose q'_m (and hence Z'_m) to be a constant.

The Capelli identity arises as a special case of (0.16), when we consider the cone of $n \times n$ positive definite Hermitian matrices, put $m = 1$, $Z'_m = 1$ and make a particular choice for Z_m. (See Section 5 for details.)

The main ingredient in the proof of Theorem 1 is the Laplace transform on a self dual cone. This theory has its origins in the work of Siegel [Si], and is described in full generality in [Gi]. We give a uniform derivation of some key aspects of the Laplace transform, making an explicit connection with Lie theory. The organization of this paper is as follows.

In the first section we recall various results about symmetric spaces. Section 2 contains some preliminary algebraic calculations which we use, in Section 3, to derive the theory of the Laplace transform. In Section 4, we discuss the operators D_m and calculate their N-radial parts. In Section 5 we specialize (0.16) to different tube domains and, among other things, deduce the Capelli identity.[1]

1. Symmetric Tube Domains

We now turn to the Lie-theoretic side of the equivalence of categories mentioned in the introduction. In what follows, all Lie algebras will be real (i.e., uncomplexified) unless explicitly stated otherwise.

We start with a simple linear Lie group $G^\flat$. Let $\mathfrak{g}^\flat$ be the Lie algebra of $G^\flat$ and let

$$\mathfrak{g}^\flat = \mathfrak{k}^\flat + \mathfrak{a} + \mathfrak{n}^\flat \tag{1.1}$$

be an Iwasawa decomposition of $\mathfrak{g}^\flat$, using the standard notation modulo the superscript $\flat$. (When we approach matters from this point of view, two symmetric spaces appear and we use the superscript $\flat$ to denote the groups and algebras associated with the "$\flat$igger" of two spaces.)

Let $K^\flat$ be the subgroup of $G^\flat$ corresponding to $\mathfrak{k}^\flat$, so that $K^\flat$ is a maximal compact subgroup of $G^\flat$. Extend $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}^\flat$ and, using the Killing form, regard $\mathfrak{a}^*$ as a subspace of $(\mathfrak{h}^\flat)^*$. Then one knows that $\operatorname{rank}(K^\flat) = \operatorname{rank}(G^\flat)$ if and only if there exist n strongly orthogonal roots $\{\gamma_1, ..., \gamma_n\}$ of $\mathfrak{h}^\flat$ in $\mathfrak{g}^\flat$ such that each γ_j lies in $\mathfrak{a}^*$.

Assuming this to be the case, one has n commuting S-triples $\{h_j, e_j, f_j\}$, where the h_i form an orthogonal basis of $\mathfrak{a}$ and—taking the γ_j to be positive with respect to $\mathfrak{n}^\flat$—one has $e_j \in \mathfrak{n}^\flat$. Note also that if

$$h = \sum_{j=1}^n h_j, \qquad e = \sum_{j=1}^n e_j, \qquad f = \sum_{j=1}^n f_j \tag{1.2}$$

then $\{h, e, f\}$ is also an S-triple.

Now if $G^\flat/K^\flat$ is a Hermitian symmetric space, then indeed $\operatorname{rank}(K^\flat) = \operatorname{rank}(G^\flat)$, so that one does have the structure described above. The situation is even nicer in the case that $G^\flat/K^\flat$ is a tube domain. This case is characterized by the fact that the *restricted* root system $\Delta(\mathfrak{g}^\flat, \mathfrak{a})$ is of type C_n (see [M]).

This implies that we may choose a basis $\varepsilon_1, ..., \varepsilon_n$ for $\mathfrak{a}^*$ such that

$$\Delta(\mathfrak{g}^\flat, \mathfrak{a}) = \{\pm\varepsilon_i \pm \varepsilon_j\} \cup \{\pm 2\varepsilon_i\} \tag{1.3}$$

[1] We thank G. Shimura for bringing his paper [Sh] to our attention, which in turn suggested to us the possibility of using the Laplace transform for our purposes. The difference between our approaches is the following: while [Sh] studies the action of an arbitrary differential operator on special functions (φ^n), we consider the action of $\partial(\varphi)^n$ on arbitrary polynomials.

and we may arrange matters so that the root spaces for the roots $\{\varepsilon_i \pm \varepsilon_j \mid i<j\}$ and $\{2\varepsilon_i\}$ are contained in $\mathfrak{n}^\flat$.

In fact the $2\varepsilon_j$ have multiplicity one as restricted roots. (This follows from the Helgason–Cartan condition for a spherical representation—applied to the adjoint representation of $\mathfrak{g}^\flat$.) Thus we may take $\gamma_j = 2\varepsilon_j$.

Now any element $x \in \mathfrak{a}$ defines a parabolic subalgebra $\mathfrak{q}_x$, as the span of the eigenvectors corresponding to the non-negative eigenvalues of $\mathrm{ad}(x)$. Let h be as in (1.2) and put $\mathfrak{q} = \mathfrak{q}_h$.

To describe $\mathfrak{q}$, and also for later use, it is convenient to introduce some notation at this stage. Let us write $\mathfrak{n}_{ij}$ for the $(\varepsilon_i - \varepsilon_j)$-root space with $i<j$; and write $\mathfrak{u}_{ij}$ and $\mathfrak{u}_i$ for the root spaces corresponding to $\varepsilon_i + \varepsilon_j$ and $2\varepsilon_i$, respectively. Let us also write $\bar{\mathfrak{n}}_{ij}$, $\bar{\mathfrak{u}}_{ij}$, and $\bar{\mathfrak{u}}_{ii}$ for the root spaces corresponding to the negatives of these roots.

Put

$$\mathfrak{n} = \bigoplus_{i<j} \mathfrak{n}_{ij}, \qquad \mathfrak{u} = \bigoplus_{i,j} \mathfrak{u}_{ij}$$
$$\bar{\mathfrak{n}} = \bigoplus_{i<j} \bar{\mathfrak{n}}_{ij}, \qquad \bar{\mathfrak{u}} = \bigoplus_{i,j} \bar{\mathfrak{u}}_{ij}. \tag{1.4}$$

Also let $\mathfrak{l}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}^\flat$, and write $\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{l} + \mathfrak{n}$.

Then $\mathfrak{q} = \mathfrak{g} + \mathfrak{u}$ is the Levi decomposition of $\mathfrak{q}$. It easy to see that $\mathfrak{u}$ is abelian and that each e_j (see (1.2)) belongs to $\mathfrak{u}$, and hence $e \in \mathfrak{u}$. Let θ be the Cartan involution of $\mathfrak{g}^\flat$ with respect to $\mathfrak{k}^\flat$; then we may assume that $\theta e_j = -f_j$, so that $e - f \in \mathfrak{k}^\flat$. In fact $e - f$ then spans the one-dimensional center of $\mathfrak{k}^\flat$.

Regarding $\mathfrak{u}$ as a $\mathfrak{g}$-module, let

$$\mathfrak{k} = \{x \in \mathfrak{g} \mid x \cdot e = 0\}. \tag{1.5}$$

Then it is immediate that $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}^\flat$, and that

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \tag{1.6}$$

is an Iwasawa decomposition of $\mathfrak{g}$.

Now $\mathfrak{g}$ is θ-stable and $\theta \mid \mathfrak{g}$ is the Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \tag{1.7}$$

be the corresponding Cartan decomposition. Then by (1.5), one has a $\mathfrak{k}$-module isomorphism

$$x \mapsto [x, e] : \mathfrak{p} \to \mathfrak{u}. \tag{1.8}$$

The point is that $\mathfrak{u}$ inherits the structure of a formally real simple Jordan algebra by defining

$$[x, e] \circ [y, e] = [x[y, e]], \tag{1.9}$$

the multiplication being commutative by virtue of (1.5).

If we identify $\mathfrak{g}$ and $\mathfrak{k}$ with their images in End($\mathfrak{u}$), then we have arrived at the structure defined in the introduction and set up the equivalence of categories. One notes that e is the identity of $\mathfrak{u}$ and that $e = \sum e_j$ (see (1.2)) is the Peirce decomposition (see (0.6)) of e. As in the introduction, one has that $C = G \cdot e$ is an open cone in $\mathfrak{u}$, and is isomorphic to G/K as a G-homogeneous space.

Define an inner product $(\ ,\ )$ on $\mathfrak{u}$ by

$$(x, y) = \frac{B(x, \theta y)}{B(e, \theta e)}, \tag{1.10}$$

where B is the Killing form on $\mathfrak{g}^\flat$. This gives a positive definite K-invariant inner product on $\mathfrak{u}$, and C is self-dual in $\mathfrak{u}$ with respect to $(\ ,\ )$.

For $g \in G$ and $X \in \mathfrak{g}$, let us write

$$g' = \theta(g)^{-1} \qquad \text{and} \qquad X' = -\theta(X). \tag{1.11}$$

Then (1.10) implies that, for all x and y in $\mathfrak{u}$, we have

$$(x, g \cdot y) = (g' \cdot x, y) \qquad \text{and} \qquad (x, X \cdot y) = (X' \cdot x, y). \tag{1.12}$$

(It is interesting to note that under the action of G, $\mathfrak{u}$ is a spherical module with the unusual property that the orbit of the spherical vector is open. Furthermore, under restriction to K, the orthogonal complement of the spherical vector is irreducible.)

Now we wish to consider the action π of G on $\mathscr{P} = \mathscr{P}(\mathfrak{u})$, the algebra of (real) polynomial functions on $\mathfrak{u}$. G acts on $\mathscr{P}$ by

$$\pi(g)\, f(x) = f(g^{-1} \cdot x). \tag{1.13}$$

First of all we recall that if V_c is a complex vector space with an irreducible, finite-dimensional, spherical, holomorphic representation of a reductive group G, then there is a real form V of V_c which is G-stable and, of course, irreducible. Furthermore V has a highest weight vector, which spans the one-dimensional space of MN-fixed vectors in V, where MAN is the Langlands decomposition of a minimal parabolic subgroup of G. Thus in dealing with *spherical* representations of G we may restrict ourselves to real vector spaces. Of course, similar remarks apply to *lowest* weight vectors in V.

Returning to our situation, let us write $\mathfrak{m}$ for the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and $\mathfrak{t}$ for a Cartan subalgebra of $\mathfrak{m}$. Then

$$\mathfrak{h} = \mathfrak{a} + \mathfrak{t} \tag{1.14}$$

is a Cartan subalgebra for $\mathfrak{g}$ and its complexification, $\mathfrak{h}_c$, is one for $\mathfrak{g}_c$, Choose a positive root system $\Sigma^+(\mathfrak{g}_c, \mathfrak{h}_c)$ compatible with $\Delta^+(\mathfrak{g}, \mathfrak{a})$. Now recalling the basis $\{\gamma_j = 2\varepsilon_j\}$ of $\mathfrak{a}^*$ consisting of strongly orthogonal roots, let us write

$$\nu_j = \varepsilon_1 + \cdots + \varepsilon_j. \tag{1.15}$$

Then the ν_j are dominant with respect to Σ^+, and it is easy to see (see [KR] for example) that each $-2\nu_j$ is the *lowest* weight of an irreducible representation of G. In fact, one has

THEOREM 0. *Any irreducible representation of G in $(\pi, \mathscr{P})$ is spherical.*

Consider the $\mathfrak{a}$ (or $\mathfrak{h}_c$) submodule $\mathscr{I} = \mathscr{P}^{M\bar{N}}$ of lowest weight vector. Then for each j there exists a polynomial $\varphi_j \in \mathscr{I}$ of degree j, unique up to scalar, such that $-2\nu_j$ is the corresponding weight.

The most general weight in $\mathscr{I}$ is uniquely of the form

$$-2(s_1 \nu_1 + \cdots + s_n \nu_n), \tag{1.16}$$

where the s_j are non-negative integers. Moreover, the corresponding lowest weight vector is

$$f_{\mathbf{s}} = \varphi_1^{s_1} \cdots \varphi_n^{s_n}. \tag{1.17}$$

In particular, each irreducible representation of G in $\mathscr{P}$ has multiplicity one.

In the notation of the introduction, one has $\varphi_n = \varphi$ (up to a scalar), so that φ_n is the norm of the Jordan algebra $\mathfrak{u}$.

Finally, the irreducible one-*dimensional representations are exactly those with lowest weights $s\nu_n$, for some non-negative integer s.*

Except for the spherical nature of the constituents of $(\pi, \mathscr{P})$, all the statements in the theorem are known. The fact that these constituents are spherical follows from the observation that $G/K = C$ is open in $\mathfrak{u}$, and hence $\mathscr{P}$ restricts faithfully to G/K. But by Frobenius reciprocity, any finite-dimensional irreducible space of functions on G/K has a spherical vector.

The multiplicity one statement and the structure of the space of lowest weight vectors are usually attributed to [S]. However, a much more general theorem was proved earlier by the first named author. (See the footnote on p. 79 in [S].) The more general result is a statement about the structure of the center $\mathfrak{C}$ of $\mathscr{U}([\mathfrak{b}_c, \mathfrak{b}_c])$, where $\mathfrak{b}_c$ is a Borel subalgebra of an arbitrary reductive complex Lie algebra $\mathfrak{g}_c$. The result is that $\mathfrak{C}$ is a polynomial ring in k generators, where k is the maximal number of strongly orthogonal roots of $\mathfrak{g}_c$. Each highest weight is of multiplicity one and may be constructed using the so-called Kostant cascade of orthogonal roots (see [LW]). The paper [LW] gives an interesting representation theoretic construction of some of the highest weight vectors in $\mathfrak{C}$.

The first author had conjectured and subsequently proved that any highest weight vector in $\mathfrak{C}$, say of weight λ, appears as the highest weight vector of that subspace of the λ-harmonic part of $\mathscr{U}(\mathfrak{g}_c)$ for which the generalized exponent $m_i(\lambda)$ is minimal. However, an earlier proof was obtained by [Js] along with a very explicit determination of the weight vectors.

If $\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$ is the Levi decomposition of any parabolic subalgebra of $\mathfrak{g}_c$, then the results about $\mathfrak{C}$ immediately yield the multiplicity one theorem for the action of $\mathfrak{l}_c$ on the center of $\mathscr{U}(\mathfrak{u}_c)$, as well as the determination of highest weight vectors. In particular if $\mathfrak{u}_c$ is abelian (the hermitian symmetric case) one gets this information about the action of $\mathfrak{l}_c$ on $S(\mathfrak{u}_c)$ or $\mathscr{P}(\mathfrak{u}_c)$. Additional references for this case are [Jh] and [U]; the latter expresses the generators φ_j in terms of the norms of smaller Jordan algebras.

Returning, then, to the generators φ_j of $\mathscr{I}$, we have for all $a \in A$ and $\bar{n} \in \bar{N}$,

$$\pi(a\bar{n})\,\varphi_j(x) = \varphi((a\bar{n})^{-1}\cdot x) = a^{-2\nu_j}\varphi_j(x), \tag{1.18}$$

where ν_j is as in (1.15).

2. Algebraic Preliminaries

Let us write $E_{jj} = h_j$ where the h_j are as in (1.2); then the E_{jj} form a basis of $\mathfrak{a}$, dual to $\{\varepsilon_i\}$.

Lemma 1. *For $i<j$, we can choose elements $E_{ij} \in \mathfrak{n}_{ij}$ and $E_{ji} \in \bar{\mathfrak{n}}_{ij}$ such that the set $\{E_{ij} \mid 1 \leqslant i,\ j \leqslant n\}$ forms the standard basis of a reductive subalgebra $\mathfrak{g}_n$ of $\mathfrak{g}$, isomorphic to $\mathfrak{gl}_n(\mathbf{R})$.*

Proof. This follows from Proposition 21 of [KR]. ∎

We wish to investigate the adjoint action of $\mathfrak{g}_n$ on $\mathfrak{g}$ and $\mathfrak{u}$. Let us write π_1 and π_2 for these representations; and, for each dominant integral weight λ of $\mathfrak{a}$, let us write V_λ for the finite-dimensional representation of $\mathfrak{g}_n$ with that highest weight.

Lemma 2. *Let $d = \dim(\mathfrak{n}_{1n})$ (see (1.4)). Then*

(a) *π_1 consists of d copies of $V_{\varepsilon_1 - \varepsilon_n}$ together with some copies of the trivial representation V_0,*

(b) *π_2 consists of* one *copy of $V_{2\varepsilon_1} \approx \mathscr{S}^2(V_{\varepsilon_1})$ and $d-1$ copies of $V_{\varepsilon_1+\varepsilon_2} \approx \Lambda^2(V_{\varepsilon_1})$.*

Proof. First, let us observe that that only nonzero dominant weight for the action of $\mathfrak{g}_n$ on $\mathfrak{g}$ is $\varepsilon_1 - \varepsilon_n$ with $\mathfrak{n}_{1n}$ as the corresponding weight space. This proves (a), and also shows that $\dim(\bar{\mathfrak{n}}) = d(n^2 - n)/2$.

Next, the action of $\mathfrak{g}_n$ has *two* dominant weights, $2\varepsilon_1$ and $\varepsilon_1 + \varepsilon_2$. Since $\dim(\mathfrak{u}_{ii}) = 1$, $V_{2\varepsilon_1}$ occurs with multiplicity 1. Let k be the multiplicity of $V_{\varepsilon_1+\varepsilon_2}$; then we have $\dim(\mathfrak{u}) = \dim(V_{2\varepsilon_1}) + k(\dim(V_{\varepsilon_1+\varepsilon_2}))$. But $\dim(V_{2\varepsilon_1}) = (n^2+n)/2$ and $\dim(V_{\varepsilon_1+\varepsilon_2}) = (n^2-n)/2$. Thus $\dim(\mathfrak{u}) = (n^2+n)/2 + k(n^2-n)/2 = n + (k+1)(n^2-n)/2$.

On the other hand, since C is open in $\mathfrak{u}$, and $\bar{N}A$ acts simply transitively on C, we have $\dim(\mathfrak{u}) = \dim(\mathfrak{a}) + \dim(\bar{\mathfrak{n}}) = n + d(n^2-n)/2$. Comparing this with above we get $k = d-1$, thus proving (b). ∎

COROLLARY. *For $i<j$, each of the root spaces $\mathfrak{n}_{ij}$, $\bar{\mathfrak{n}}_{ij}$, $\mathfrak{u}_{ij}$, $\bar{\mathfrak{u}}_{ij}$ (see (1.4)) is d-dimensional. Thus if ρ is half the sum of the restricted roots of $\mathfrak{a}$ in $\mathfrak{n}$ (counted with multiplicities), we have*

$$\rho = d \sum_{i<j} (\varepsilon_i - \varepsilon_j) = \frac{d}{2} \sum_k (n - 2k + 1)\, \varepsilon_k. \tag{2.1}$$

LEMMA 3. *For Y_{ij} in $\bar{\mathfrak{n}}_{ij}$ let $Y'_{ij} = -\theta(Y_{ij})$ (as in (1.11)). Then $Y'_{ij} \in \mathfrak{n}_{ij}$, and if ε_k is as in (1.2) then we have*

(a) $Y'_{ij} \cdot \varepsilon_k = 0$ *unless* $j = k$.

(b) $Y'_{ik} \cdot e_k \in \mathfrak{u}_{ik}$.

(c) $(Y'_{ik})^2 \cdot e_k = 2\, \|Y_{ik}\|^2\, e_i$.

(d) $(Y'_{ik})^p \cdot e_k = 0$ *for all* $p > 2$.

Proof. All parts are trivial except for (c). For this, let us recall from [Ko, Theorem 2.1.7] that if $d > 1$, then the centralizer of $\mathfrak{a}$ in K acts transitively on the unit sphere in $\bar{\mathfrak{n}}_{ik}$. Therefore it suffices to prove (c) for $Y_{ik} = E_{ki}$. But now $Y'_{ik} = E_{ik}$ and the result follows from Lemma 1, (b) of Lemma 2, the normalization of (,) in (1.10), and the calculation $E_{ik}^2 \cdot (v_k \otimes v_k) = 2(v_i \otimes v_i)$, inside $\mathscr{S}^2(V_{\varepsilon_1})$. ∎

LEMMA 4. *Let $\varphi_1 \cdots \varphi_n$ be as in Section 1; then $\varphi_j(e) \neq 0$ for all j.*

Proof. If $g = \bar{n}ak$ is the Iwasawa decomposition of $g \in G$, we get $\varphi_j(g \cdot e) = \varphi_j(\bar{n}ak \cdot e) = \varphi_j(\bar{n}a \cdot e) = a^{2\nu_j}\varphi_j(e)$ by (1.18). So if $\varphi_j(e) = 0$, φ_j must vanish on all of $C = G \cdot e$; but since C is open, this forces $\varphi_j = 0$, a contradiction. ∎

In view of Lemma 4, we may normalize $\varphi_1, \ldots, \varphi_n$ so that

$$\varphi_j(e) = 1 \qquad \text{for all} \quad j. \tag{2.2}$$

Then we have the following consequence of the proof of Lemma 4:

COROLLARY. *With normalization as above,* $\varphi_j(\bar{n}a \cdot e) = a^{2\nu_j}$. *In particular,* $\varphi_j(y) > 0$ *for all* $y \in C$ *and each* $j = 1, ..., n$.

The converse of this corollary is also true, and follows immediately from the fact that every element in $\mathfrak{u}$ is $M\bar{N}$-conjugate to an element of $\sum_j \mathbf{R}e_j$ as in (0.8). Since we have no use for the converse in what follows, we omit the (easy) proof of this fact.

3. THE LAPLACE TRANSFORM

The results of this section are due to [Si] for the cone of positive definite real matrices, and may be found in [Gi] for the general case. We give here a Lie-theoretic reformulation of some of the results in [Gi].

Continuing with our development, in view of the Corollary to Lemma 4, we may make the following

DEFINITION 1. For $\mathbf{s} = (s_1, ..., s_n) \in \mathbf{C}^n$ we define the complex power function $f_{\mathbf{s}}$ on C by

$$f_{\mathbf{s}}(x) = \prod_{k=1}^{n} \varphi_k(x)^{s_k}. \tag{3.1}$$

DEFINITION 2. Let d^*x be a G-invariant measure on the cone C. Then the gamma function of C is defined by the formula

$$\Gamma_C(\mathbf{s}) = \int_C \exp(-e, x)\, f_{\mathbf{s}}(x)\, d^*x. \tag{3.2}$$

LEMMA 5. *The integral in* (3.2) *is absolutely convergent for* Re(s) *sufficiently large, and for a suitable normalization of* d^*x, *we have*

$$\Gamma_C(\mathbf{s}) = \prod_{k=1}^{n} \Gamma\left(t_k - \frac{d}{2}(k-1)\right) \tag{3.3}$$

where d *is as in Lemma* 2, Γ *is the ordinary gamma function, and*

$$t_k = s_k + \cdots + s_n. \tag{3.4}$$

The proof of this Lemma requires some preparation. Let us define a total order on the positive roots in $\Delta^+(\mathfrak{g}, \mathfrak{a})$ by specifying, for $i < j$ and $p < q$,

$$\varepsilon_p - \varepsilon_q > \varepsilon_i - \varepsilon_j \qquad \text{if either } q \text{ is greater than } j, \text{ or } q = j \text{ and } p \text{ is } \textit{less} \text{ than } i. \tag{3.5}$$

Let $\bar{\mathfrak{n}}_{pq}$ be as in (1.4) and define

$$\bar{\mathfrak{n}}^{ij}=\sum \{\bar{\mathfrak{n}}_{pq} \mid \varepsilon_p-\varepsilon_q \geqslant \varepsilon_i-\varepsilon_j\}. \tag{3.6}$$

Then the $\bar{\mathfrak{n}}^{ij}$ form a chain of ideals in $\bar{\mathfrak{n}}$, and the following lemma in an easy consequence of this observation.

LEMMA 6. *Write Y in $\bar{\mathfrak{n}}$ as the direct sum*

$$\begin{aligned} Y=(Y_{12})+\cdots+(Y_{k-1,k}+\cdots+Y_{1k})+\cdots \\ +(Y_{n-1,n}+\cdots+Y_{1n}), \end{aligned} \tag{3.7}$$

where the Y_{ij} are in $\bar{\mathfrak{n}}_{ij}$. Then the map

$$\begin{aligned} Y\mapsto \eta(Y)=(\exp Y_{12})\cdots(\exp Y_{k-1,k}\cdots\exp Y_{1k})\cdots \\ (\exp Y_{n-1,n}\cdots\exp Y_{1n}) \end{aligned} \tag{3.8}$$

is a diffeomorphism from $\bar{\mathfrak{n}}$ to $\bar{N}$, and takes the Lebesgue measure on $\bar{\mathfrak{n}}$ to the Haar measure on $\bar{N}$.

The crucial calculation involved in the proof of Lemma 5 is contained in

LEMMA 7. *Let Y, Y_{ij} be as in Lemma 6, and let e_k, e be as in (1.2); then*

$$(e_k, \eta(Y)e)=1+\sum_{i=1}^{k-1} \|Y_{ik}\|^2. \tag{3.9}$$

Proof. By (1.12), the right side is $(\eta(Y)'\cdot e_k, e)$ where $\eta(Y)'=\theta(\eta(Y))^{-1}$. Now

$$\begin{aligned} \eta(Y)'\cdot e_k=[(\exp Y'_{1n}\cdots\exp Y'_{n-1,n})\cdots \\ (\exp Y'_{1k}\cdots\exp Y'_{k-1,k})\cdots(\exp Y'_{12})]\cdot e_k. \end{aligned} \tag{3.10}$$

The (adjoint) action of $\exp Y'_{ij}$ on $\mathfrak{u}$ is by $1+Y'_{ij}+(Y'_{ij})^2/2!+\cdots$. So by (a) of Lemma 3, we see that in the right side of (3.10), factors of the form $\exp Y'_{ij}$ with $j<k$, leave e_k unchanged. Now (b), (c), and (d) of Lemma 3 imply that

$$(\exp Y'_{1k}\cdots\exp Y'_{k-1,k})\cdot e_k=e_k+\sum_{i=1}^{k-1}\|Y_{ik}\|^2 e_i \qquad \operatorname{mod}\Big(\sum_{i<j\leqslant k}\mathfrak{u}_{ij}\Big). \tag{3.11}$$

Finally, the remaining factors are all of the form $\exp Y'_{pq}$ where q is *greater* than k. So they leave the last expression fixed. Since the spaces $\bar{\mathfrak{n}}_{ij}$

are mutually orthogonal with respect to (,), (3.9) follows by taking the inner product of (3.11) with $e = e_1 + \cdots + e_n$. ∎

We are now in a position to prove Lemma 5.

Proof of Lemma 5. Recall from Section 1 that $A\bar{N}$ acts simply transitively on C. This sets up a diffeomorphism of $A\bar{N}$ with C which takes the left-invariant measure on $A\bar{N}$ to the G-invariant measure d^*x on C.

For a in A, let us write a_i for the number $a^{2\varepsilon_i}$. Then the map $a \mapsto (a_1, ..., a_n)$ is an isomorphism of A with $(\mathbf{R}_+)^n$. We have $a \cdot e_k = a_k e_k$ and

$$f_{\mathbf{s}}(a\bar{n} \cdot e) = \prod a_k^{t_k}, \tag{3.12}$$

where $t_k = s_k + \cdots + s_n$ as in (3.4).

So, if da and $d\bar{n}$ are Haar measures on A and $\bar{N}$, respectively, the right side of (3.2) becomes

$$\begin{aligned}
\Gamma_C(\mathbf{s}) &= \int_C \exp(-e, x)\, f_{\mathbf{s}}(x)\, d^*x. \\
&= \int_A \int_{\bar{N}} \exp(-e, a\bar{n} \cdot e)\, f_{\mathbf{s}}(a\bar{n} \cdot e)\, d\bar{n}\, da \\
&= \int_A \int_{\bar{N}} \exp(-a \cdot e, \bar{n} \cdot e) \left(\prod_k a_k^{t_k} \right) d\bar{n}\, da \\
&= \int_A \int_{\bar{N}} \prod_k \left[a_k^{t_k} \exp(-a_k e_k, \bar{n} \cdot e) \right] d\bar{n}\, da \\
&= \int_A \int_{\bar{\mathfrak{n}}} \prod_k \left[a_k^{t_k} \exp(-a_k e_k, \eta(Y) \cdot e) \right] dY\, da \\
&= \int_A \int_{\bar{\mathfrak{n}}} \prod_k \left[a_k^{t_k} \exp\left(-a_k - a_k \sum_{i=1}^{k-1} \|Y_{ik}\|^2 \right) \right] dY\, da \\
&= \prod_k \left[\int_{\mathbf{R}_+} a_k^{t_k} \exp(-a_k) \right. \\
&\qquad \left. \times \left\{ \int \exp\left(-a_k \sum_{i=1}^{k-1} \|Y_{ik}\|^2 \right) dY_{1k} \cdots dY_{k-1,k} \right\} d^\times a_k \right] \\
&= \prod_k \left[\int_{\mathbf{R}_+} a_k^{t_k - d(k-1)/2} \exp(-a_k)\, d^\times a_k \right] \\
&= \prod_{k=1}^n \Gamma\left(t_k - \frac{d}{2}(k-1) \right),
\end{aligned}$$

where $d^{\times}a_k$ is a multiplicative Haar measure on $\mathbf{R}_+$, and the various measures have been normalized suitably.

The absolute convergence follows easily from the explicit calculation above. ∎

DEFINITION 3. For a suitable function f on C, we define its Laplace transform $\mathfrak{L}f$ to be the function on C given by the formula

$$\mathfrak{L}f(y) = \int_C \exp(-y, x)\, f(x)\, d^*x. \tag{3.13}$$

Recalling the notation of Section 1, let θ be the Cartan involution of G with respect to K. Then θ gives rise to the usual Cartan "symmetry" (also denoted by θ) on $C = G/K$. Then $\theta \cdot e = e$, and if $a \in A$ and $n \in N$, then

$$\theta(a) = a^{-1} \qquad \text{and} \qquad \theta(n) \in \bar{N}. \tag{3.14}$$

LEMMA 8. *Let $f_{\mathbf{s}}$ be as in Definition* 1; *then the integral defining $\mathfrak{L}f_{\mathbf{s}}$ converges absolutely for* Re(s) *sufficiently large and*

$$\mathfrak{L}f_{\mathbf{s}}(y) = \Gamma_C(\mathbf{s})\, f_{\mathbf{s}}(\theta \cdot y). \tag{3.15}$$

Proof. The group NA acts transitively on C. So we may write $y = na \cdot e$ with $a \in A$ and $n \in N$. Now $\theta \cdot (na \cdot e) = \theta(n)\, \theta(a) \cdot e$. Then by (3.14), we get

$$f_{\mathbf{s}}(\theta \cdot y) = f_{\mathbf{s}}(a^{-1} \cdot e) = \prod a_k^{-t_k}.$$

On the other hand,

$$\begin{aligned}
\mathfrak{L}f_{\mathbf{s}}(y) &= \mathfrak{L}f_{\mathbf{s}}(na \cdot e) \\
&= \int_C \exp(-na \cdot e, x)\, f_{\mathbf{s}}(x)\, d^*x \\
&= \int_C \exp(-e, a'n' \cdot x)\, f_{\mathbf{s}}(x)\, d^*x \\
&= \int_C \exp(-e, x)\, f_{\mathbf{s}}((n')^{-1}(a')^{-1} \cdot x)\, d^*x.
\end{aligned}$$

Since $n' \in \bar{N}$ and $a' = a$ the last integral becomes

$$\left(\prod a_k^{t_k}\right) \int_C \exp(-e, x)\, f_{\mathbf{s}}(x)\, d^*x = f_{\mathbf{s}}(\theta \cdot y)\, \Gamma_C(\mathbf{s}).$$

Again, the absolute convergence is immediate. ∎

Let ρ be as in (2.1). We wish to rewrite (3.15) in a slightly different form, taking into account a shift by ρ.

DEFINITION 4. For $\lambda \in \mathfrak{a}^*$, let f^λ be the function on C satisfying

$$f^\lambda(na \cdot e) = a^{\lambda+\rho} \qquad \text{for all} \quad a \in A,\ n \in N. \tag{3.16}$$

Now the proof of Lemma 8 shows that

$$f^\lambda(y) = f_{\mathbf{s}(\lambda)}(\theta \cdot y), \tag{3.17}$$

where $\mathbf{s}(\lambda)$ is given by solving

$$-2(s_k(\lambda) + \cdots + s_n(\lambda)) = \lambda_k + \frac{d}{2}(n - 2k + 1). \tag{3.18}$$

This gives

$$\begin{aligned} \Gamma_C(\mathbf{s}(\lambda)) &= \prod_{k=1}^{n} \Gamma\left(-\frac{\lambda_k}{2} - \frac{d}{4}(n-2k+1) - \frac{d}{2}(k-1)\right) \\ &= \prod_{k=1}^{n} \Gamma\left(-\frac{\lambda_k}{2} - \frac{d}{4}(n-1)\right). \end{aligned} \tag{3.19}$$

Let us write $\bar{\omega}(\lambda)$ for the last expression. Then (3.15) becomes

$$\mathfrak{L} f_{\mathbf{s}(\lambda)} = \bar{\omega}(\lambda)\, f^\lambda. \tag{3.20}$$

4. INVARIANT DIFFERENTIAL OPERATORS

Let V be a vector space over $\mathbf{R}$, and let V^* be its dual. Write $\mathscr{S}(V)$ and $\mathscr{S}(V^*)$ for the symmetric algebras over these spaces. Then $\mathscr{S}(V^*)$ is naturally isomorphic to $\mathscr{P}(V)$, the space of (real) polynomial functions on V. On the other hand, each v in V defines the differential operator $\partial(v)$ on V by the formula $(\partial(v) f)(x) = (d/dt) f(x + tv)\,|_{t=0}$. This extends to a natural isomorphism, still denoted by ∂, from $\mathscr{S}(V)$ to $\mathbf{D}(V)$, the space of constant (real) coefficient differential operators on V.

For ξ in V^*, let us write e_ξ for the function on V given by $e_\xi(x) = \exp\langle \xi, x\rangle$ where $\langle\ ,\ \rangle$ is the pairing between V^* and V. Then an easy calculation gives for $\tau \in \mathscr{S}(V) \approx \mathscr{P}(V^*)$,

$$\partial(\tau)\, e_\xi = \tau(\xi)\, e_\xi \qquad \text{for all} \quad \xi \in \mathscr{S}(V). \tag{4.1}$$

Let us apply this to the case at hand with $V = \mathfrak{u}$. There the inner product

(,) allows us to identify $\mathfrak{u}$ and $\mathfrak{u}^*$. If τ is a homogeneous polynomial of degree k in $\mathscr{P}(\mathfrak{u})$ and f is a suitable function on C, we get

$$\partial(\tau)(\mathfrak{L}f) = (-1)^k \mathfrak{L}(\tau f). \tag{4.2}$$

This follows by differentiating under the integral in (3.13) and using (4.1).

Let us further specialize this to the case where $f = f_{s(\lambda)}$ and $\tau = \varphi^m$. Applying (3.20) we get

$$\partial(\varphi)^m (\bar{\omega}(\lambda) f^\lambda) = (-1)^{mn} \mathfrak{L}(\varphi^m f_{s(\lambda)}). \tag{4.3}$$

However, $\varphi^m f_{s(\lambda)} = f_{s(\lambda - 2m\varepsilon)}$, where $\varepsilon = v_n = \varepsilon_1 + \cdots + \varepsilon_n$ (see (1.15)). So the right side becomes $(-1)^{mn} \bar{\omega}(\lambda - 2m\varepsilon) f^{\lambda - 2m\varepsilon}$.

Now $f^{\lambda - 2m\varepsilon}(y) = \varphi^m(\theta \cdot y) f^\lambda(y)$; and by an easy calculation $\varphi(\theta \cdot y) = \varphi(y)^{-1}$. Substituting this into (4.3) we get

$$\begin{aligned} \varphi^m \partial(\varphi)^m f^\lambda &= (-1)^{mn} \frac{\bar{\omega}(\lambda - 2m\varepsilon)}{\bar{\omega}(\lambda)} f^\lambda \\ &= \prod_{k=1}^{n} \prod_{j=0}^{m-1} \left(\frac{\lambda_k}{2} + \frac{d}{4}(n-1) - j \right) f^\lambda. \end{aligned} \tag{4.4}$$

Let us recall from [H2] some pertinent facts about invariant differential operators.

If M is a smooth manifold, a linear map D from $C_c^\infty(M)$ to itself is called a differential operator if it decreases supports, i.e., if $\operatorname{supp}(Df) \subseteq \operatorname{supp}(f)$, for all $f \in C_c^\infty(M)$. If D is a differential operator, then D extends canonically to an operator on $C^\infty(M)$ also denoted by D.

Suppose γ is a diffeomorphism of M and $f \in C^\infty(M)$; then the function $f^\gamma = f \circ \gamma^{-1}$ is also in $C^\infty(M)$. If D is a differential operator, then the operator D^γ defined by $D^\gamma f = (Df^{\gamma^{-1}})^\gamma = (D(f \circ \gamma)) \circ \gamma^{-1}$ is also a differential operator. We say that f is γ-invariant if $f^\gamma = f$ and that D is γ-invariant if $D^\gamma = D$.

If G is a Lie group and H is a closed subgroup, then the homogeneous space G/H is a smooth manifolds and G acts on it by diffeomorphisms. We will write $\mathbf{D}(G/H)$ for the set of G-invariant differential operators on G/H.

Suppose now that G is a reductive Lie group. Let $G = KAN$ be its Iwasawa decomposition, so that K is a maximal compact subgroup, A is a maximal real split torus, and N is the nil radical of a minimal parabolic subgroup. Let us write e for the identity coset in G/K; then if $D \in \mathbf{D}(G/K)$, the N-radial part of D is the unique differential operator $\Delta_N(D)$ on the submanifold $A \cdot e$ such that for any N-invariant function $f \in C^\infty(G/K)$, we have

$$\Delta_N(D)(f|_{A \cdot e}) = (Df)|_{A \cdot e}. \tag{4.5}$$

Since A acts simply transitively on $A \cdot e$ we may identify the two spaces and consider $\Delta_N(D)$ to be a differential operator on A. Let us write ρ for the half-sum of the roots of $\mathfrak{a}$ in $\mathfrak{n}$ and a^ρ for the function $a \mapsto \exp(\rho(\log a))$. If W is the Weyl group of A in G, let us write $\mathbf{D}_W(A)$ for the set of W-invariant differential operators in $\mathbf{D}(A)$. Then it is known (see [H2, Theorem II.5.18]) that

$\mathbf{D}(G/K)$ is a commutative algebra and the map

$$\sigma : D \mapsto a^{-\rho} \Delta_N(D) \circ a^\rho \tag{4.6}$$

gives an isomorphism of $\mathbf{D}(G/K)$ with $\mathbf{D}_W(A)$.

Returning to our situation, where G/K is the cone C, let

$$D_m = \varphi^m \partial(\varphi)^m. \tag{4.7}$$

Then it is clear that operators D_m are in $\mathbf{D}(G/K)$. Moreover, the exponential map gives an isomorphism of $\mathfrak{a}$ (with the additive structure) with A. Under this isomorphism, $\mathbf{D}(A)$ corresponds to constant coefficient differential operators, or equivalently, polynomials on $\mathfrak{a}^*$ via the map ∂ described earlier. In our case $\mathfrak{a}^* \approx \mathbf{R}^n$ and W is the symmetric group on n letters, acting in the usual manner on $\mathbf{R}^n$. Thus $\mathbf{D}_W(A)$ corresponds to "symmetric" polynomials on $\mathfrak{a}^*$.

Then our main result is

THEOREM 1. *Let p_m be the function on $\mathfrak{a}^*$ given by*

$$p_m(\lambda) = \prod_{k=1}^{n} \prod_{j=0}^{m-1} \left(\frac{\lambda_k}{2} + \frac{d}{4}(n-1) - j \right). \tag{4.8}$$

Then

$$\sigma(D_m) = \partial(p_m). \tag{4.9}$$

Proof. The functions f^λ are N-invariant and $f^\lambda \mid A \cdot e = a^{\lambda+\rho}$. Clearly it suffices to show that $\sigma(D_m)$ and $\partial(p_m)$ agree on all a^λ. However,

$$\begin{aligned}
\sigma(D_m)^{\lambda} &= (a^{-\rho} \Delta_N(D_m) \circ a^\rho)\, a^\lambda \\
&= a^{-\rho}(\Delta_N(D_m)\, a^{\lambda+\rho}) \\
&= a^{-\rho}(D_m f^\lambda \mid A \cdot e) \\
&= a^{-\rho}(p_m(\lambda) f^\lambda \mid A \cdot e) \\
&= a^{-\rho}(p_m(\lambda)\, a^{\lambda+\rho}) \\
&= p_m(\lambda)\, a^\lambda \\
&= \partial(p_m)\, a^\lambda. \quad \blacksquare
\end{aligned}$$

5. Differential Identities

Let us recall from [H1, p. 528, Example 4] the classification of the irreducible symmetric tube domains. They are the symmetric spaces

(A) **(A III)** $U(n, n)/U(n) \times U(n)$, rank $= n$;

(B) **(BD I)** $O(p, 2)/O(p) \times O(2)$, rank $= 2$;

(C) **(C I)** $\mathrm{Sp}(2n, \mathbf{R})/U(n)$, rank $= n$;

(D) **(D III)** $SO^*(4n)/U(2n)$, rank $= n$;

(E) **(E VII)** $E_{7(-25)}/E_6 \times SO(2)$, rank $= 3$.

In this section we specialize (0.16) to each of these domains, considering the first case in some detail. All unexplained notation is from the introduction, especially from the discussion preceding (0.14), (0.15), and (0.16).

Type A. The group G is $GL(n, \mathbf{C})$, the cone C is the set of all $n \times n$ complex Hermitian, positive definite matrices, and the functions φ_k are the principal $k \times k$ minors. G acts on C by $g \cdot x = gxg^*$, where g^* is the conjugate transpose of g. The root multiplicities are $d = 2$.

Complexifying G and $\mathfrak{u}$ we get the action of $G_c = GL(n, \mathbf{C}) \times GL(n, \mathbf{C})$ on $M(n, \mathbf{C})$ given by $(g_1, g_2) \cdot (z) = g_1 z g_2^*$, so that G imbeds in G_c as the diagonal subgroup. Let us write G^{L} and G^{R} for the two copies (left and right) of $GL(n, \mathbf{C})$, and $\mathfrak{g}^{\mathrm{L}}$ and $\mathfrak{g}^{\mathrm{R}}$ for their Lie algebras. The representations of $G^{\mathrm{L}} \times G^{\mathrm{R}}$ occurring in $\mathscr{P}$ are those of the form $V_c \otimes V_c^*$, where V_c is an arbitrary holomorphic representation of $GL(n, \mathbf{C})$.

As before, let $\mathfrak{h} = \mathfrak{a} + \mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Projection onto each component yields isomorphisms π_{L} and π_{R} of $\mathfrak{h}$ with Cartan subalgebras $\mathfrak{h}^{\mathrm{L}}$ and $\mathfrak{h}^{\mathrm{R}}$ of $\mathfrak{g}^{\mathrm{L}}$ and $\mathfrak{g}^{\mathrm{R}}$, and we note that $\mathfrak{h}_c = \mathfrak{h}^{\mathrm{L}} \oplus \mathfrak{h}^{\mathrm{R}}$ is a Cartan subalgebra for $\mathfrak{g}_c$.

Let $\varepsilon_1, ..., \varepsilon_n$ be as in (1.10) and let $\varepsilon_k^{\mathrm{L}} = \varepsilon_k \circ \pi_{\mathrm{L}}^{-1}$ and $\varepsilon_k^{\mathrm{R}} = \varepsilon_k \circ \pi_{\mathrm{R}}^{-1}$. Then the $\varepsilon_k^{\mathrm{L}}$ form a $\mathbf{C}$-basis for $\mathfrak{h}^{\mathrm{L}}$ and the roots of $\mathfrak{h}^{\mathrm{L}}$ are $\{\varepsilon_i^{\mathrm{L}} \pm \varepsilon_j^{\mathrm{L}}\}$. (Similarly for $\varepsilon_k^{\mathrm{R}}$.) Under the inclusion map ι of $\mathfrak{a}^*$ into $\mathfrak{h}_c^*$, $\iota(\varepsilon_k) = \frac{1}{2}(\varepsilon_k^{\mathrm{L}}, \varepsilon_k^{\mathrm{R}})$.

Now $W \approx S_n$, and $W_c \approx S_n \times S_n$ acting on each factor. Since G is quasi-split, we have $\rho_o = 0$ in (0.15), so $I_o(\mathfrak{h}_c) \approx I(\mathfrak{h}_c)$. Also, in this case, the positive restricted system $\Delta(\mathfrak{n}) = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ determines a unique positive root system for $\mathfrak{h}_c$ in $\mathfrak{g}_c$. Its projection onto each component determines positive root systems for $\mathfrak{g}^{\mathrm{L}}$ and $\mathfrak{g}^{\mathrm{R}}$. Write ρ^{L} and ρ^{R} for half the sums of positive roots in each case. Then

$$\rho^{\mathrm{L}} = \frac{1}{2} \sum_k (n - 2k + 1)\, \varepsilon_k^{\mathrm{L}}, \qquad \rho^{\mathrm{R}} = \frac{1}{2} \sum_k (n - 2k + 1)\, \varepsilon_k^{\mathrm{R}}. \tag{5.1}$$

Define the polynomial q_m on $\mathfrak{h}_c = \mathfrak{h}^L \oplus \mathfrak{h}^R$ by

$$q_m(\mu, \nu) = \prod_{k=1}^{n} \prod_{j=0}^{m-1} \left(\nu_k + \frac{n-1}{2} - j\right), \tag{5.2}$$

for $\mu = \sum_j \mu_j \varepsilon_j^L$ and $\nu = \sum_j \nu_j \varepsilon_j^R$. Then clearly $q_m \in I(\mathfrak{h}_c) = I_o(\mathfrak{h}_c)$ and

$$\begin{aligned} q_m(\iota(\lambda)) &= q_m\left(\sum_k \frac{\lambda_k}{2}\,\varepsilon_k^L, \sum_k \frac{\lambda_k}{2}\,\varepsilon_k^R\right) \\ &= \prod_{k=1}^{n} \prod_{j=0}^{m-1} \left(\frac{\lambda_k}{2} + \frac{n-1}{2} - j\right) = p_m(\lambda). \end{aligned}$$

Thus by (0.16), $D_m^c = \pi^c(Z_m)$, where $Z_m = \psi^{-1}(q_m)$. From the definition of q_m it is clear that Z_m belongs to the center of $\mathscr{U}(\mathfrak{g}^R)$, and it may be calculated by explicitly inverting ψ, as described in Section 23.3 of [Hm] for example.

In the case $m=1$, let us just check that Z_1 is indeed the (holomorphic extension of the) left side of (0.1). An easy argument (see [Ho]) shows that the left side is a central operator. In its expansion as in (0.3), all terms except the diagonal term have a last factor of the form $\pi^c(E_{ij})$ with $i<j$. This implies that these terms kill every highest weight vector in $\mathscr{P}(\mathfrak{u}_c)$. Finally, if f is a highest weight vector for $\mathfrak{g}^R$ with weight $\nu = \sum_k \nu_k \varepsilon_k^R$, then the diagonal term multiplies it by

$$\prod_{k=1}^{n} (\nu_k + n - k). \tag{5.3}$$

On the other hand,

$$Z_1 f = q_1(\nu + \rho^R) f = \prod_k \left(\nu_k + \frac{n-2k+1}{2} + \frac{n-1}{2}\right) f. \tag{5.4}$$

Comparing (5.3) and (5.4) we obtain the Capelli identity.

Type B: The group G is $O(p-1,1) \times \mathbf{R}_+$, and the cone C is the set of all vectors $(x_1, x_2, ..., x_p) \in \mathbf{R}^p$ such that $x_1 > 0$ and $x_1^2 > x_2^2 + \cdots + x_p^2$. The functions φ_1 is simply x_1 and $\varphi_2(x) = x_1^2 - (x_2^2 + \cdots + x_p^2)$. The root multiplicities are $d = p-1$.

Theorem 1, in this case, specializes to a formula for the N-radial part of the wave operator, and (0.16) leads to a well known identity for powers of the Laplace operator.

Type C. The group G is $GL(n, \mathbf{R})$, the cone C is the set of all $n \times n$ positive definite symmetric matrices, and the functions φ_k are the principal $k \times k$ minors. G acts on C by $g \cdot x = gxg^t$, where g^t is the transpose of g. The root multiplicities are $d = 1$.

In this case the identity (0.16), with $m = 1$, was first proved by [T]; see also [G].

Type D. The group G is $GL(n, \mathbf{H})$, the cone C is the set of all $n \times n$ positive definite Hermitian quaternionic matrices, and the functions φ_k are the principal $k \times k$ minors, which happen to be real valued and, in fact, positive for such matrices. G acts on C by $g \cdot x = gxg^{\dagger}$, where $g^{\dagger}$ is the quaternionic conjugate transpose of g. The root multiplicities are $d = 4$.

After complexification, the space $\mathfrak{u}$ becomes the space of $2n \times 2n$ complex *skew*-symmetric matrices, and the various φ_k become the Pfaffians of the leading $2k \times 2k$ submatrices. An identity corresponding to (0.16), again with $m = 1$, is briefly referred to in [T], although details are not given.

Type E. The group G is $G' \times \mathbf{R}_+$, where G' is a real form of E_6 with split rank 2. The space $\mathfrak{u}$ may be identified with the set of all 3×3 Hermitian matrices over the Cayley numbers. The root multiplicities are $d = 8$. The cone C is the set of all positive definite matrices, and the stabilizer of a point on the cone is a compact form of the exceptional group F_4.

The functions φ_k are the principal $k \times k$ minors, which happen to be real valued for Hermitian matrices. Suitable care must be taken in defining these minors since the entries of the matrices belong to a non-commutative, indeed non-associative, ring. This can be done for 3×3 matrices; see [F]. The function φ is sometimes called the Freudenthal determinant.

This example is different from the others in that we cannot assume $Z'_m = 1$ in (0.16). However, a recent result of Helgason [H3] shows that one can always take Z'_m to a polynomial in the Casimir. The identity (0.16) seems to be completely new in this case.

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Note added in proof. Using algebraic techniques, Wallach [Wa] has extended the results of this paper to a larger class of differential operators.

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