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Jordan algebras and Capelli identities*

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Introduction

The purpose of this paper is to establish a connection between semisimple Jordan algebras and certain invariant differential operators on symmetric spaces; and to prove an identity for such operators which generalizes the classical Capelli identity.

The norm function on a simple real Jordan algebra gives rise to invariant differential operators D_m on a certain symmetric space which is a natural "conformal compactification" of the Jordan algebra. If $\mathfrak{t}$ is the Lie algebra of a maximal torus of the symmetric space, and γ is the Harish–Chandra isomorphism, then $q_m = \gamma(D_m)$ is a polynomial on $\mathfrak{t}^*$, and our generalized Capelli identity is an explicit formula for q_m which we now describe.

It turns out that the restricted root system of the symmetric space is one of three possible types — A_{n-1}, D_n, or C_n where $n = \dim(\mathfrak{t})$. We shall call these cases A, D, and C, respectively, and choose a basis $\gamma_1, \cdots, \gamma_n$ for $\mathfrak{t}^*$ such that the root system is $\{\pm(\gamma_i - \gamma_j)/2\}$, $\{\pm(\gamma_i \pm \gamma_j)/2\}$, or $\{\pm(\gamma_i \pm \gamma_j)/2, \pm\gamma_i\}$ in the three cases. Let p_m be the polynomial on $\mathfrak{t}^*$ given by

$$p_m\left(\sum a_i\gamma_i\right) = \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a_i + (m-2j+1)/2\right)\right).$$

Theorem. *Up to a scalar multiple, q_m equals p_m in cases AD and p_{2m} in case C.*

In case A, the Jordan algebra is formally real, and its compactification is the Shilov boundary of a symmetric tube domain. Here the result was proved earlier in [KS] (see also [W2]). As pointed out in [S1, S2], in this case the identity is closely related to the results of [W1, G, FK]. Additionally, for the Jordan algebras of skew-Hermitian matrices over **R**, **C**, and **H**, the identity is implicit in [J].

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The argument in [KS] depends on the holomorphic structure on the tube domain and makes essential use of the Laplace transform — a technique which is not available for cases C and D.

To obtain the result in its present generality, we exploit the connection between the Capelli identity and the representation theory of a certain semisimple Lie group, the "conformal group" of the Jordan algebra. This group acts on sections of line bundles on the symmetric space, and the operators D_m are intertwining operators for appropriate line bundles. Now, using some information about the K-spectrum of finite-dimensional subrepresentations, we deduce that q_m must have certain factors, and then the explicit formula for q_m follows from its Weyl group invariance.

The classical Capelli identity expresses the holomorphic "Cayley" differential operator $\det(x_{ij})\det(\partial/\partial x_{ij})$ on $n \times n$ complex matrices, in terms of a "determinant" of n^2 non-commuting vector fields (the polarization operators), and we now explain the precise sense in which our main theorem generalizes this identity.

By restricting the Cayley operator to the (non-compact) symmetric space of positive definite Hermitian matrices, and applying the Cayley transform, one gets an invariant differential operator — the "spherical" operator on the (compact) symmetric space $S(U_n \times U_n)/SU_n$. As observed in [KS] and [S1], the Capelli identity follows easily from an explicit formula for the radial component of this operator. This formula in turn is a very special case of our result, and corresponds to the Jordan algebra of Hermitian matrices (with $m = 1$).

Similar identities can be obtained for the other cases, and in the appendix, by way of example and *dessert*, we deduce such an identity for the Jordan algebra of all $n \times n$ real matrices, which expresses the operator $\det(1 + x^t x)\det(\partial/\partial x_{ij})$ in terms of a certain "Pfaffian" of vector fields!

In a subsequent paper we will return to the connection with representation theory and obtain results analogous to [S1, S2, S3] for the general case. These will include explicit constructions of certain small unitary representations of conformal groups of Jordan algebras in a form which may be useful for certain physical and number-theoretic considerations.

1. Preliminaries

1.1 Generalities

With the possible exception of certain neologisms, the material of this subsection is well-known. (See [BK], [Sat], and [Sp] for background on Jordan algebras.)

Let N be a real semisimple Jordan algebra with unit e and norm ϕ. The trace form on N gives an isomorphism ∂ between the algebra of polynomial functions on N and the algebra of constant coefficient differential operators. We define the (generalized) *Cayley operator* D to be $\partial(\phi)$.

Motivated by the special case of Minkowski space (as in [JV]), we define the following Lie groups associated to N: the *Lorentz* group L is the subgroup of $GL(N)$ which preserves ϕ up to a scalar multiple, the *Poincaré* group is the semidirect product $P = LN$, and the *conformal* group is the group G of rational transformations generated by P and the *conformal inversion* $\iota : x \mapsto -x^{-1}$.

Then, as shown in [K1] (see also [Sp, §2]), G is a semisimple Lie group and $P = LN$ is the Levi decomposition of a parabolic subgroup such that (i) N is abelian, and (ii) P is conjugate to its opposite $\overline{P}$. (Note the happy notational coincidences!) We define the *conformal compactification* of N to be the flag space $G/\overline{P}$ into which N embeds as an open and dense subset. If K is a maximal compact subgroup of G, and $M = L \cap K$, then $G/\overline{P} = K/M$ is a symmetric space for K.

(In the literature on Jordan algebras, K is called the *automorphism group* and L is called the *structure group*, or the *norm-preserving group* of N. The Lie algebra $\mathfrak{g}$ = Lie(G), is the Koecher-Tits algebra of N (see [Ti] and [K2]). The conformal compactification $G/\overline{P}$ is called a *symmetric R-space* in [T], [N], [L], and elsewhere.)

Conversely, every parabolic subgroup $P = LN$ (in a simple group G) which satisfies (i) and (ii) above, arises in the indicated manner from a (unique up to isotopy) Jordan algebra structure on N. From the viewpoint of Lie theory, such parabolics are easily classified, and we give a complete list in the appendix A.2.

1.2 Structure theory

Let $\mathfrak{t}$ be a maximal toral subalgebra in the orthogonal complement of $\mathfrak{m}(=\text{Lie}(M))$ in $\mathfrak{k}(=\text{Lie}(K))$, and let Σ be the restricted root system of $\mathfrak{t}$ in $\mathfrak{k}$. From appendix A.2, we see that there are only three possibilities for Σ, namely A_{n-1}, D_n, and C_n, where $n = \dim(\mathfrak{t})$. Case A occurs if and only if K has a one-dimensional center, while case C occurs if and only if the degree of ϕ is $2n$. (The degree is n in the other two cases!).

Thus there is a basis $\{\gamma_1, \cdots, \gamma_n\}$ of $\mathfrak{t}^*$ of the form described in the introduction, and we choose positive root systems so that the simple roots of $\mathfrak{t}$ in $\mathfrak{k}$ are $\{(\gamma_i - \gamma_{i+1})/2\}$ in case A, together with the additional roots $(\gamma_{n-1} + \gamma_n)/2$ and γ_n in cases D and C respectively.

Now the roots of $\mathfrak{t}$ in $\mathfrak{g}$ are $\{\pm(\gamma_i \pm \gamma_j)/2, \pm\gamma_i\}$ in *all* cases. (For case A this is proved in [M], and for cases CD it follows easily by passing to the complexification.) Moreover if $\mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, then the extreme $\mathfrak{t}$-weights of $\mathfrak{p}_c$ are $\gamma_{\pm i}$, and the Cartan–Helgason theorem implies that the root spaces $\mathfrak{p}_{\pm\gamma_i}$ are 1 dimensional.

Fix non–zero elements X_i in $\mathfrak{p}_{\gamma_i}$, and put $X_{-i} = \overline{X}_i$, $Z_i = [X_i, X_{-i}]$, $H_i = X_i + X_{-i}$; then $\{X_i, X_{-i}, Z_i\}$ are standard bases for n commuting S-triples. The Z_i's span $\mathfrak{t}$ while the H_i's span an **R**-split toral subalgebra $\mathfrak{a}$ of $\mathfrak{l}$ and $\mathfrak{g}$. The *Cayley transform* is the element c in Ad($\mathfrak{g}$) defined by $c = \prod_i \exp\{\pi(X_i - X_{-i})/4\}$ and we have $c(\mathfrak{t}) = \sqrt{-1}\mathfrak{a}$.

Let $\nu = c(\gamma_1 + \cdots + \gamma_n)/2 \in \mathfrak{a}^*$, then ν extends to a character of $\mathfrak{l}$ which, in turn, is the differential of a positive (multiplicative) character of L that we continue to denote by ν. Write 2δ for the trace of the adjoint action of $\mathfrak{l}$ on $\mathfrak{n}$ and let r be such that $\delta = r\nu$.

(As observed in the appendix, r is closely related to the dual Coxeter number of $\mathfrak{g}_c$.)

Now let $\rho = \sum_i r_i\gamma_i$ be the half sum of positive roots in Σ. Then the following crucial fact can easily be checked from appendix A.2: $\quad r = 2r_1 + 1$.

1.3 Representation Theory

Consider for each real s, the (normalized) induced representation $(\pi_s, I(s)) = \mathrm{Ind}_{\overline{P}}^{G}(\nu^s)$ regarded as a $(\mathfrak{g}, K)$-module. By restricting to N we get the "noncompact picture" which is a realization of $I(s)$ as a subspace of $C^\infty(N)$. If ∂ is as in 1.1 and 'ad' is the adjoint representation of $\mathfrak{l}$ on the space of functions on $\mathfrak{n}$, then we have $\pi_s(x) = \partial(x)$ and $\pi_s(u) = (s-r)\nu(u) + \mathrm{ad}(u)$ for $x \in \mathfrak{n}$ and $u \in \mathfrak{l}$.

Alternatively, restricting to K, we may realize all the $I(s)$ on the *fixed* space of K-finite functions in $C^\infty(K/M)$, which is called the "compact picture". The K-types have multiplicity 1 and if $\mathscr{S}$ is the set of their highest weights, then by the Cartan–Helgason theorem, $\mathscr{S}$ consists of $\sum_i a_i\gamma_i$ where the a_i's are integers which satisfy $a_i \geq a_{i+1}$ in all cases, and which satisfy the additional conditions $a_{n-1} \geq |a_n|$ and $a_n \geq 0$ in cases D and C respectively.

It follows from a result of [B] (see also [JV]) that the m-th power of the Cayley operator (see 1.1) intertwines the noncompact pictures of $I(m)$ and $I(-m)$ in cases AD, and $I(2m)$ and $I(-2m)$ in case C. Thus in the compact picture D^m becomes an operator D_m in the algebra $\mathbf{D}(K/M)$ of K-invariant differential operators on the symmetric space K/M.

Via the Harish–Chandra homomorphism γ, D_m determines a polynomial $q_m = \gamma(D_m)$ on $\mathfrak{t}^*$, which is invariant under the Weyl group of K/M. (See [H] for details.) For cases AD, q_m has degree mn, while in case C it has degree $2mn$.

2. Proofs

2.1 Finite dimensional representations

It follows from [K], for example, that for each non-negative integer k, $I(2k+r)$ has a finite dimensional spherical subrepresentation F_k.

Lemma. *The K-types of F_1 have highest weights $\{\sum a_i\gamma_i \in \mathscr{S} : |a_i| \leq 1\}$.*

Proof. For case A, this is a special case of Theorem 3 of [S2] which computes the K-structure of *all* the constituents of $I(s)$. For cases C and D, we argue as follows:

The highest $\mathfrak{a}$-weight of F_1 is 2ν and applying the Cayley transform we see that $\gamma_1 + \cdots + \gamma_n$ is the highest $\mathfrak{t}$-weight of F_1. This means that the corresponding K-type occurs in F_1, and the possible highest weights for the other K-types are $\{\gamma_1 + \cdots + \gamma_i \mid i = 1, \cdots, n-1\}$ in case C, and $\{\gamma_1 + \cdots + \gamma_i\} \cup \{\gamma_1 + \cdots + \gamma_{n-1} - \gamma_n\}$ in case D. It remains only to show that each of these K-types actually occurs in F_1.

If V_μ is a K-type of F_1 with highest weight μ, then from the $\mathfrak{t}$-weights of $\mathfrak{g}$ it follows that $\pi_s(\mathfrak{g})V_\mu$ is contained in the sum $\sum\{V_\lambda \mid \lambda = \mu, \mu \pm \gamma_j\}$. So if one of the K-types, say $\gamma_1 + \cdots + \gamma_i$, were not to occur in F_1, then the subspace $\sum\{V_{\gamma_1+\cdots+\gamma_j} \mid 0 \leq j < i\}$ would be a $\mathfrak{g}$-submodule — a contradiction. This proves the result for case C.

Since P and $\overline{P}$ are conjugate, it follows that F_1 admits a $(\mathfrak{g}, K)$-invariant Hermitian form. But F_1 is the complexification of a *real* representation — the real valued functions in F_1. This implies that F_1 is self contragredient. In case D, the contragredient of the K-type $\gamma_1 + \cdots + \gamma_n$ is $\gamma_1 + \cdots + \gamma_{n-1} - \gamma_n$, so it too must occur. $\square$

Corollary. *The K-types of F_k have highest weights $\{\sum a_i\gamma_i \in \mathscr{S} : |a_i| \leq k\}$.*

Proof. . The highest $\mathfrak{t}$-weight of F_k is $k(\gamma_1 + \cdots + \gamma_n)$, so the K-types of F_k are contained in the indicated set. Also, regarded as a vector space of functions on $\mathfrak{n}$, F_k is spanned by k-fold products of functions in F_1. It follows that if $\lambda_1, \cdots, \lambda_k$ are K-types in F_1, then $\lambda_1 + \cdots + \lambda_k$ is a K-type of F_k. This proves the reverse containment. □

2.2 The main result

Let $\alpha = \sum a_i\gamma_i$ and $\beta = \alpha + \gamma_1$ be K-types of $I(s)$ with corresponding highest weight vectors v and w, and let $X_1 \in \mathfrak{p}_{\gamma_1}$ be as in 1.2.

Lemma. $\pi_s(X_1)v = c(s - 2a_1 - r)w$, *where c is a non-zero number, independent of s.*

Proof. Considering the compact picture we see that $\pi_s(X_1)v$ is a multiple of w, and that it is an *affine* function of the parameter s. For generic s, $I(s)$ is irreducible and so is not a highest weight module, and then Lemma 3.4 of [V] shows that $\pi_s(X_1)$ is injective. In particular $\pi_s(X_1)v \not\equiv 0$. On the other hand, Corollary 2.1 implies that α is a K-type of F_{a_1} while β is not. Thus $\pi_s(X_1)v = 0$ for $s = 2a_1 + r$, and the Lemma follows. □

Proof. (*Of the main theorem*) By definition, $D_m v = q_m(\alpha + \rho)v$ and $D_m w = q_m(\beta + \rho)w$, while 1.3 shows that in cases AD, $D_m\pi_m(X_1)v = \pi_{-m}(X_1)D_m v$. Combining these with the Lemma, and using the fact that $r = 2r_1+1$, we get that $q_m(\beta+\rho)/q_m(\alpha+\rho) = \left(a_1 + r_1 + 1/2 + m/2\right) / \left(a_1 + r_1 + 1/2 - m/2\right)$.

So if $\eta_m(z) = \prod_{j=1}^{m} \left(z - (m - 2j + 1)/2\right)$ then we get $q_m(\beta+\rho)/\eta_m(a_1 + r_1 + 1) = q_m(\alpha + \rho)/\eta_m(a_1 + r_1)$. Since $q_m(\alpha) = q_m(a_1, \cdots, a_n)$ is a polynomial in a_1 it follows that $q_m(z, a_2 + r_2, \cdots, a_n + r_n)/\eta_m(z)$ is independent of z. Thus $\eta_m(a_1)$ divides $q_m(\alpha)$.

Now the Weyl groups of A_{n-1} and D_n contain the symmetric group S_n, so the invariance of q_m implies it must be divisible by $p_m = \prod_{i=1}^{n} \eta_m(a_i)$. Since both polynomials have degree mn, the result follows. In case C, $D_m\pi_{2m}(X_1)v = \pi_{-2m}(X_1)D_m v$, and arguing as above, we conclude that q_m must be a scalar multiple of p_{2m}. □

Appendix

A.1 Applications

We illustrate our main result by specializing it to the Jordan algebra of $n \times n$ real matrices, whose conformal compactification is $K/M = SO_{2n}/S(O_n \times O_n)$.

Let $\mathscr{Z}$ be the center of the enveloping algebra of $\mathfrak{k} = \mathfrak{so}_{2n}$. Since rank$(K)$ = rank(K/M), the natural map $\pi : \mathscr{Z} \to \mathbf{D}(K/M)$ is an isomorphism; moreover, if $\xi : \mathscr{Z} \to \mathscr{S}(\mathfrak{t})^W$ is the usual Harish–Chandra isomorphism ([Hu, §23]) then $\gamma \circ \pi = \xi$.

$\mathscr{Z}$ contains a distinguished element ff of degree n, which is obtained from the Pfaffian by the usual procedure of dualizing and symmetrizing, and is characterized as the unique element in $\mathcal{Z}$ of degree $\leq n$ which transforms by sign(det) under the adjoint action of O_{2n}.

From this it follows easily that $\xi(ff)(\sum_i a_i\gamma_i) = \prod_i a_i$, and since, by our main theorem, this is the same as $\gamma(D_1)$, we get the following:

Corollary. *Up to a scalar multiple,* $D_1 = \pi(ff)$.

(If n is odd, this is obvious since then ff is the unique element of degree n in $\mathcal{Z}$!)

We now make this explicit as follows: Write a typical $n \times n$ matrix as $\sum x_{ij}E_{ij}$ where E_{ij} is the ij-th elementary matrix, and write ∂_{ij} for $\partial/\partial x_{ij}$. Then $X_{ij} = \begin{pmatrix} E_{ij}-E_{ji} & 0 \\ 0 & 0 \end{pmatrix}$, $X_{i,n+j} = -X_{n+j,i} = \begin{pmatrix} 0 & E_{ij} \\ -E_{ji} & 0 \end{pmatrix}$, and $X_{n+i,n+j} = \begin{pmatrix} 0 & 0 \\ 0 & E_{ij}-E_{ji} \end{pmatrix}$ form a basis for $\mathfrak{so}_{2n}$.

Writing $g \in G = SL_{2n}(\mathbf{R})$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are $n \times n$ matrices, the action in the noncompact picture is $\pi_s(g)f(x) = |\det(a + xc)|^{s-n} f\left((a + xc)^{-1}(b + xd)\right)$.

Differentiating this we get $\pi_s(X_{ij}) = \sum_q(x_{iq}\partial_{jq} - x_{jq}\partial_{iq})$, $\pi_s(X_{n+i,n+j}) = \sum_p(x_{pi}\partial_{pj} - x_{pj}\partial_{pi})$, and $\pi_s(X_{i,n+j}) = -\pi_s(X_{n+j,i}) = (n-s)x_{ij} + \partial_{ij} + \sum_{p,q} x_{pj}x_{iq}\partial_{pq}$.

A typical skew symmetric matrix may be written as $\sum z_{kl}X_{kl}$; its Pfaffian is a polynomial of degree n in z_{kl}, and the element ff in $\mathcal{Z}$ is obtained from this polynomial by replacing each z_{kl} by the corresponding X_{kl}. (The X_{kl}'s occurring in each term of the polynomial commute amongst themselves, and so there is "no need to symmetrize" !)

It is easily checked that $\psi = \det(1+x^tx)$ is the K-fixed vector in F_1 (see 2.1). As in [S1], it follows that the map $f \mapsto \psi^t f$ is a K-isomorphism between $I(s)$ and $I(s+2t)$; and that in the non-compact picture at $I(s)$, $D_1 = \det(1 + x^tx)^{(1+s)/2}\det(\partial_{ij})\det(1 + x^tx)^{(1-s)/2}$. The Corollary says that this operator is a scalar multiple of the "Pfaffian" of the $\pi_s(X_{kl})$!

A.2 Classification

The parabolics P of 1.1 can be determined easily from the restricted root system of G. If G is simple, then P must be a maximal parabolic corresponding to a simple root α which has coefficient 1 in the highest root, and which is mapped to $-\alpha$ by the long element of the Weyl group. In the usual indexing, the possible roots are $\alpha_n \in A_{2n-1}$, $\alpha_1 \in B_n$ or D_n, $\alpha_n \in C_n$, α_{2n-1} or $\alpha_{2n} \in D_{2n}$ and $\alpha_7 \in E_7$.

We list below all simple groups with these restricted root systems organizing them according to the *complex* form of the (implicit) parabolics. In each case, we also write down the corresponding symmetric spaces K/M, their restricted root systems Σ, and the numbers r and r_i (see 1.2). (X_p denotes the symmetric space $SO_p/S(O_{p-1} \times O_1)$.)

We conclude with an intriguing observation. Recall that the *dual Coxeter number* of an irreducible root system is defined to be 1 plus the sum of the coefficients of the expression of the highest short root of the dual system in terms of its simple roots. From the table we observe that in cases AD, the dual Coxeter number of $\mathfrak{g}_c$ is $2r$, while in case C the dual Coxeter number of each irreducible component of $\mathfrak{g}_c$ is r.

Table 1. Conformal groups of simple real Jordan algebras.

G	K/M	Σ	r, r_i
$SU_{n,n}$	$S(U_n \times U_n)/SU_n$	A_{n-1}	$n, (n-2i+1)/2$
$SL_{2n}(\mathbf{R})$	$SO_{2n}/S(O_n \times O_n)$	D_n	$n, (n-i)/2$
$SL_{2n}(\mathbf{C})$	$SU_{2n}/S(U_n \times U_n)$	C_n	$2n, n-i+1/2$
$SL_{2n}(\mathbf{H})$	$Sp_{2n}/(Sp_n \times Sp_n)$	C_n	$4n, 2(n-i)+3/2$
$SO^o_{p,2}$	$X_p \times X_2$	A_1	$p/2, r_1 = r_2 = (p-2)/4$
$SO^o_{p,q}$ $p \geq q \geq 3$	$X_p \times X_q$	$D_2 = A_1 \times A_1$	$(p+q)/2 - 1, r_1 = (p+q)/4 - 1,$ $r_2 = (p-q)/4$
$SO_p(\mathbf{C})$	$SO_p/(SO_{p-2} \times U_1)$	C_2	$p-2, r_1 = (p-3)/2, r_2 = 1/2$
$SO^o_{p,1}$	X_p	$C_1 = A_1$	$p-1, r_1 = (p-2)/2$
$Sp_n(\mathbf{R})$	U_n/O_n	A_{n-1}	$(n+1)/2, (n-2i+1)/4$
$Sp_n(\mathbf{C})$	Sp_n/U_n	C_n	$n+1, (n-i+1)/2$
$Sp_{n,n}$	$(Sp_n \times Sp_n)/Sp_n$	C_n	$2n+1, n-i+1$
SO^*_{4n}	U_{2n}/Sp_n	A_{n-1}	$2n-1. n-2i+1$
$SO^o_{2n,2n}$	$(SO_{2n} \times SO_{2n})/SO_{2n}$	D_n	$2n-1, n-i$
$SO_{4n}(\mathbf{C})$	SO_{4n}/U_{2n}	C_n	$4n-2, 2(n-i)+1/2$
$E_7(-25)$	$(E_6 \times SO_2)/F_4$	A_2	$9, 2(3-2i+1)$
$E_7(7)$	SU_8/Sp_4	$D_3 = A_3$	$9, 2(3-i)$
$E_7(\mathbf{C})$	$E_7/(E_6 \times SO_2)$	C_3	$18, 4(3-i)+1/2$

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