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Unitary Representations on the Shilov Boundary of a Symmetric Tube Domain

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ABSTRACT. Let $\Omega = G/K$ be a symmetric tube domain where G is the universal cover of $\mathrm{Aut}(\Omega)$. Let χ be a line bundle on the Shilov boundary, and let $I(\chi)$ be the space of sections.

This paper determines (a) the composition series for $I(\chi)$ as a $(\mathfrak{g}, K)$-module, (b) the K-module structure of each constituent, (c) explicit formulas for possible invariant Hermitian forms on these constituents, and (d) the unitarizable constituents.

Introduction.

Let $\Omega = G/K$ be a symmetric tube domain of rank n and let G be the universal covering group of $\mathrm{Aut}(\Omega)$. The Shilov boundary of Ω is of the form G/P, where $P = LN$ is a maximal parabolic subgroup with abelian nilradical N. Let χ be a character of L such that the induced representation $I(\chi) = \mathrm{Ind}_P^G(\chi)$ has a non-trivial, invariant, Hermitian form.

This paper determines the composition series for such $I(\chi)$, describes the K-module structure of each constituent, and obtains explicit formulas for the invariant Hermitian forms on the constituents. In particular, this leads to a complete determination of the unitarizable constituents of the $I(\chi)$.

The main idea is the following: After suitable normalization, the Hermitian form has a *rational* dependence on the parameter. Moreover, if χ is suitably integral, then the Hermitian form is given by an equivariant differential operator and, as shown in [S], the Capelli identity of [KS] gives an explicit formula for this form at these points. In view of the rationality, this allows one to calculate Hermitian forms for all χ, and everything else follows. This technique should perhaps be considered an "algebraic" continuation of the Capelli identity.

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Our results extend those of [**W**], [**RV**], [**FK**], and those of [**KV**], [**G**], [**J**] among others.

The first three references deal with holomorphic representations on Ω. These imbed in the Shilov boundary through the boundary value map, and occur as certain constituents of $I(\chi)$. But $I(\chi)$ has other constituents as well, which should, perhaps, be related to other G-orbits on $G_{\mathbf{c}}/P_{\mathbf{c}}$.

[**KV**] studies $I(\chi)$ for $Sp(n,\mathbf{R})$ and $U(n,n)$, but only for the trivial character χ. [**G**] is in the setting of our paper, but describes only those constituents of $I(\chi)$ which have one-dimensional K-types. Finally, [**J**] answers the same questions as this paper but for a different class of groups, and uses completely different methods. (The group $U(n,n)$ is the only one common to both our classes.)

The organization of this paper is as follows. The necessary notation is introduced in §0. In §1 an explicit formula is obtained for possible Hermitian forms on $I(\chi)$. The Jantzen filtration is determined in §2, and the irreducible constituents are calculated in §3. In §4 the asymptotic supports (in the sense of [**KV1**]) of the various constituents are determined. These turn out to be either certain cones $T_{p,q}$ with $p+q \leq n$, or else the union of two such cones $T_{p,n-p} \cup T_{p+1,n-p-1}$.

§5 determines which of these constituents are unitary. Of particular interest is Theorem 5.C, which shows that for each pair of integers p, q with $p+q < n$ there is a *unique* "small" unitary representation $V_{p,q}$ with support $T_{p,q}$, whose K-types are given by formula (7). For $p = 0$ or $q = 0$, these are the representations obtained by Wallach in [**W**]. They are all unipotent in the sense of [**V1**].

In [**S1**], we show that each of these small representations is naturally realized on the L^2-space of an L orbit in N. In particular, this implies that these representations are irreducible upon restriction to P. This should be compared with the results of [**SS**].

The results of this paper also seem to have some relevance to the questions raised in [**KR**], which were in turn motivated by number theoretic considerations. In particular, one can show that if G is classical, and $I(\chi)$ descends to a representation of the corresponding linear group (or of its metaplectic double cover), then all the composition factors can be "obtained" via the oscillator correspondence. The details will appear elsewhere.

§0. Notation.

We start by recalling some notation from [**S**].

In what follows, all Lie algebras will be real unless complexified with a subscript "**c**".

Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible Hermitian symmetric pair of tube type. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and choose $\mathfrak{a}^s \subseteq \mathfrak{p}$ and $\mathfrak{t}^s \subseteq \mathfrak{k}$ so that $\mathfrak{h}^s = \mathfrak{a}^s \oplus \mathfrak{t}^s$ is a *maximally* split Cartan subalgebra of $\mathfrak{g}$.

It is known that the *restricted* root system is of type C_n where $n = \dim(\mathfrak{a}^s)$. Thus we may choose a basis $\varepsilon_1, \cdots, \varepsilon_n$ for $(\mathfrak{a}^s)^*$ such that $\Sigma(\mathfrak{g}, \mathfrak{a}^s) = \{\pm\varepsilon_i \pm \varepsilon_j\} \cup \{\pm 2\varepsilon_j\}$.

The root spaces for $\pm\varepsilon_i \pm \varepsilon_j$ have a common dimension which we will denote by d.

The root spaces for $\pm 2\varepsilon_j$ are one-dimensional. Thus $\pm 2\varepsilon_j$ may be regarded as roots of $\mathfrak{h}^s$ in $\mathfrak{g}$, vanishing on $\mathfrak{t}^s$. To each $2\varepsilon_j$ we may attach in a standard manner, an S-triple $\{h_j, e_j, f_j\}$, contained in $\mathfrak{g}$. These S-triples commute, and $\{h = \sum_{j=1}^n h_j, e = \sum_{j=1}^n e_j, f = \sum_{j=1}^n f_j\}$ is also an S-triple.

The eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{g}$ are -2, 0 and 2. Let us write $\bar{\mathfrak{n}}$, $\mathfrak{l}$ and $\mathfrak{n}$ for the corresponding eigenspaces. Then $\mathfrak{q} = \mathfrak{l} + \bar{\mathfrak{n}}$ is a maximal parabolic subalgebra.

The Cayley transform is the element $c = \exp\frac{\pi i}{4}(e+f)$ in $\mathrm{Ad}(\mathfrak{g}_{\mathbf{C}})$. Let $\mathfrak{t} = ic(\mathfrak{a}^s)$, then $\mathfrak{h} = \mathfrak{t} + \mathfrak{t}^s$ is a *compact* Cartan subalgebra for $\mathfrak{g}$ (and $\mathfrak{k}$); and $\{\gamma_i \mid \gamma_i = c \circ (2\varepsilon_i), i = 1, \cdots, n\}$ is a maximal set of (Harish-Chandra) strongly orthogonal roots in $\Sigma(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$.

Let us write ν for $\varepsilon_1 + \cdots + \varepsilon_n$ and μ for $\gamma_1 + \cdots + \gamma_n$.

Let G' be the adjoint group of $\mathfrak{g}$ and let G be its universal covering group. Let K' and K be the analytic subgroups corresponding to $\mathfrak{k}$, and let $P = L\overline{N}$ be the maximal parabolic subgroup of G corresponding to $\mathfrak{q}$. Then K' is a maximal compact subgroup of G', and $\Omega = G/K = G'/K'$ is a symmetric tube domain of rank n and G/P is its Shilov boundary.

We describe next the characters of L. First of all, ν is (the differential of) a positive character of L; and for each $t \in \mathbf{C}$, ν^t is a character of L. Similarly, μ is (the differential of) a unitary character of K; and for each $\varepsilon \in \mathbf{C}$, μ^ε is a character of K. Now μ^ε restricts to a character of the component group of $L \cap K$, which is the same as the component group of L; hence this restriction extends to a unique character of L, which we denote by μ^ε as well. Finally, the most general character of L is of the form $\mu^\varepsilon \otimes \nu^t$.

We will write $(\pi_{\varepsilon,t}, I(\varepsilon,t))$ for the induced representation $\mathrm{Ind}_P^G(\mu^\varepsilon \otimes \nu^t)$ (K-finite, normalized induction).

It is known that the highest weights of K-types in $I(\varepsilon,t)$ are of the form $\alpha + \varepsilon\mu$ with

$$\alpha = \sum_{i=1}^n a_i \gamma_i \tag{1}$$

where the a_i are integers with $a_1 \geq \cdots \geq a_n$, and each such K-type occurs *exactly* once.

We are interested in the situation when $I(\varepsilon,t)$ has a $\mathfrak{g}$-invariant, K-unitary Hermitian form. This happens if and only if ε is real and t is either real or purely imaginary. Furthermore, if t is imaginary and non-zero then $I(\varepsilon,t)$ is irreducible and unitary; if t is real then $I(\varepsilon,-t)$ is the Hermitian dual of $I(\varepsilon,t)$; and finally, $I(\varepsilon+1,t) \approx I(\varepsilon,t)$. Consequently, for the rest of this paper we assume

$$t \in \mathbf{R}^+ \text{ and } \varepsilon \in [0,1) \tag{2}$$

§1. Rationality.

It is convenient to realize all of the $\pi_{\varepsilon,t}$ on a fixed space. For this, let $(L\cap K)'$ be the (compact) image of $L\cap K$ under the projection from G to G', and write $V = L^2(K'/(L\cap K)')_{K'-\text{finite}}$. If $\varepsilon = 0$, then restriction to K gives an isomorphism of $I(0,t)$ with V. For general ε, multiplication by the function $\mu^{-\varepsilon}$ gives an isomorphism of $I(\varepsilon,t)$ with V. In view of this, when ε is understood, we will simply write α for the K'-type with highest weight $\alpha + \varepsilon\mu$.

Now, let t, ε be as in (2) and let $\langle\ ,\ \rangle_{\varepsilon,t}$ be a $\pi_{\varepsilon,t}(\mathfrak{g})$ invariant Hermitian form on $I(\varepsilon,t)$. On each K-type, $\langle\ ,\ \rangle_{\varepsilon,t}$ is a constant multiple of the standard (positive definite) form $\langle\ ,\ \rangle$ on V coming from $L^2(K')$ through the above identification.

Fix K-types α and β and choose $\langle\ ,\ \rangle$-unit vectors v and w in V_α and V_β respectively, and let

$$q_{\beta,\alpha}(\varepsilon,t) = \frac{\langle w,w\rangle_{\varepsilon,t}}{\langle v,v\rangle_{\varepsilon,t}}.$$

It is well known that $\pi_{\varepsilon,t}$ is irreducible for most ε and t, and thus $\langle\ ,\ \rangle_{\varepsilon,t}$ is unique up to a scalar and $q_{\beta,\alpha}(\varepsilon,t)$ is unique, period! Our first theorem is a formula for $q_{\beta,\alpha}(\varepsilon,t)$. An obvious multiplicative property implies that it suffices to consider the case where

$$(3) \qquad \beta = \alpha + \gamma_i \quad \text{for some } i \text{ such that } a_{i-1} \geq a_i + 1.$$

THEOREM. *For α as in* (1), *β as in* (3), *we have $q_{\beta,\alpha}(\varepsilon,t) = a_i^+/a_i^-$, where*

$$(4) \qquad a_i^\pm = a_i^\pm(\alpha,\varepsilon,t) = a_i + \varepsilon + d(n-2i+1)/4 + 1/2 \pm t/2.$$

PROOF. For each X in $\mathfrak{g}$, $(\varepsilon,t) \mapsto \pi_{\varepsilon,t}(X)$ is an affine map. Arguing as in (Theorem 4.11(c) of [**V1**]) we conclude that $q_{\beta,\alpha}(\varepsilon,t)$ is a *rational* function of ε and t. Thus it suffices to prove the theorem for the special case where t is a positive integer. For such t, the theorem follows from Theorem 1 of [**S**], once we note that the argument in the appendix of that paper shows that $\pi_{\varepsilon,t}$ is irreducible for *most* ε. □

§2. Reducibility.

For this section, let us fix ε, t as in (2), and write $V_0 = V = I(\varepsilon,t)$. Let $s \in \mathbf{R}$ be variable, then Theorem 1 shows that, after multiplying by a suitable power of $(s-t)$, we may assume that $\langle\ ,\ \rangle_{\varepsilon,s}$ is non-zero at t, and non-degenerate *elsewhere* in a small interval $(t-\delta, t+\delta)$.

Let V_k consist of those vectors v in V for which there exists a polynomial function f, defined on $(t-\delta,t+\delta)$ and taking values in a fixed, finite dimensional subspace of V, with the following two properties: (a) $f(t) = v$; (b) for all w in V, the function $\langle f(s), w\rangle_{\varepsilon,s}$ vanishes to order at least k at $s = t$.

DEFINITION([**V**]). The filtration $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k \supseteq \cdots$, is called the *Jantzen filtration* of $I(\varepsilon,t)$ and $Q_k \equiv V_k/V_{k+1}$ are called the *Jantzen subquotients.*

It is easy to check that each V_k is π_t-invariant. Furthermore, if $v, v' \in V_k$ and we choose f, f' as above, then $\lim_{s\to t} \frac{1}{(s-t)^k} \langle f(s), f'(s)\rangle_{\varepsilon,s}$ depends only on v, v' and defines a Hermitian form $\langle\ ,\ \rangle_t^k$ on V_k whose radical is exactly V_{k+1}. Thus the Jantzen subquotients are $(\mathfrak{g}, K)$ modules with non-degenerate Hermitian forms.

In our situation (K-multiplicity one), these subquotients admit a simpler description.

LEMMA. *In the present situation, $v \in V_k$ if and only if $\langle v, v\rangle_{\varepsilon,s}$ vanishes to order at least k at $s = t$.*

PROOF. Suppose v satisfies the condition of the Lemma, and define $f(s) \equiv v$. Since K-multiplicities are 1, $\langle\ ,\ \rangle_{\varepsilon,s}$ is definite on each K-type (for $s \neq t$) and we can write $V = \mathbf{C}v \oplus (\mathbf{C}v)^\perp$, with a *fixed*, orthogonal decomposition *for all* $\langle\ ,\ \rangle_{\varepsilon,s}$. If $w \in V$, we may write $w = cv + v^\perp$. Then $\langle f(s), w\rangle_{\varepsilon,s} = \langle v, cv\rangle_{\varepsilon,s}$ vanishes to order at least k at $s = t$. Thus v belongs to V_k.

Conversely, suppose $v \in V_k$ and $f(s)$ is as in the definition of V_k. Write $V = \mathbf{C}v \oplus (\mathbf{C}v)^\perp$ as above and decompose $f(s) = c(s)v + g(s)$ where $c(s)$ is a (scalar valued) polynomial, and $g(s)$ takes values in $(\mathbf{C}v)^\perp$. Then $\langle f(s), v\rangle_{\varepsilon,s} = c(s)\langle v, v\rangle_{\varepsilon,s}$. At $t = s$, the left side vanishes to order at least k, and since $c(t) = 1$, so does $\langle v, v\rangle_{\varepsilon,s}$. □

COROLLARY. *The Jantzen subquotient Q_k consists of the K-types for which $\langle\ ,\ \rangle_{\varepsilon,t}$ vanishes to order exactly k. Furthermore, if we fix $\alpha \in Q_k$ and normalize $\langle\ ,\ \rangle_{\varepsilon,t}^k$ to be equal to $\langle\ ,\ \rangle$ on V_α, then for each $\beta \in Q_k$, $\langle\ ,\ \rangle_{\varepsilon,t}^k = q_{\beta,\alpha}(\varepsilon, t)\langle\ ,\ \rangle$ on V_β.*

§3. Irreducibility.

In this section we decompose $I(\varepsilon, t)$ into its irreducible components.

LEMMA. *Let $\alpha, \beta, t, a_i^\pm$ be as in* (1) – (4). *Then*

(a) *$V_\beta \subseteq \pi_{\varepsilon,t}(\mathfrak{g})V_\alpha$ if and only if $a_i^- \neq 0$.*
(b) *$V_\alpha \subseteq \pi_{\varepsilon,t}(\mathfrak{g})V_\beta$ if and only if $a_i^+ \neq 0$*
(c) *β and α belong to the same irreducible component of $I(\varepsilon, t)$ if and only if $a_i^\pm \neq 0$.*

PROOF. Let $t_o = t_o(\alpha, i, \varepsilon) = 2(a_i + \varepsilon + \frac{d}{4}(n-2i+1) + \frac{1}{2})$, so that $a_i^\pm = \frac{1}{2}(t_o \pm t)$. Then Theorem 1 shows that $q_{\beta,\alpha}(\varepsilon, t) = \frac{t_o+t}{t_o-t}$.

Suppose $a_i^- = 0$, that is $t = t_o$. Now if $t_o \neq 0$, $q_{\beta,\alpha}(\varepsilon, t)$ has a pole at t_o. Thus in the Jantzen filtration of V at t_o, V_α occurs in lower degree than V_β. In particular $V_\beta \not\subseteq \pi_{\varepsilon,t_o}(\mathfrak{g})V_\alpha$. If $t_o = 0$ then $q_{\beta,\alpha}(\varepsilon, t) = -1$ is negative. However the representation $\pi_{\varepsilon,0}$ is unitary (and completely reducible), so α and β must lie in different summands; thus once again $V_\beta \not\subseteq \pi_{\varepsilon,t_o}(\mathfrak{g})V_\alpha$. This proves the "only if" part of (a).

For the other direction, fix $v \in V_\alpha$ and $X \in \mathfrak{g}$, and let $P(\varepsilon,t) = P_{X,v}(\varepsilon,t)$ be the orthogonal projection of $\pi_{\varepsilon,t}(X)v$ to V_β. Since $(\varepsilon,t) \mapsto \pi_{\varepsilon,t}(X)$ is affine, there are w, w_0 and w_1 in V_β such that $P(\varepsilon,t) = tw + \varepsilon w_0 + w_1$. The first part of the proof shows that $P(\varepsilon, t_o(\alpha,i,\varepsilon)) = 0$, whence $P(\varepsilon,t) = (2a_i^-)w$.

In the appendix it is shown that one can find $X \in \mathfrak{g}$ and $v \in V_\alpha$ such that $P_{X,v}(\varepsilon,t) \neq 0$ for *some* (ε,t). It follows that $w \neq 0$, and so $P(\varepsilon,t) \neq 0$ for *all* $t \neq t_o(\alpha,i,\varepsilon)$. This shows that $V_\alpha \subseteq \pi_{\varepsilon,t}(\mathfrak{g})V_\beta$ if $a_i^- \neq 0$. This completes the proof of (a); (b) is similar, and (c) follows. □

In the situation of the Lemma, we will say α and β are *linked* if (c) holds. Two K-types of $I(\varepsilon,t)$ will be called *equivalent* if they can be connected through a chain of linked K-types.

Let (π, W) be an irreducible subquotient of $I(\varepsilon,t)$, and α be a K-type of W. Clearly all the K-types in the equivalence class of α occur in W. On the other hand, as observed in Lemma 3.1 of [**S**], the K-types of $\pi(\mathfrak{g})(W_\alpha)$ are contained in the set $\{\alpha, \alpha \pm \gamma_j \mid j = 1, \cdots, n\}$. This shows that W *coincides* with the equivalence class of α.

The preceding remarks show that $I(\varepsilon,t)$ is reducible if and only if there is a K-type α as in (1), and an index i such that one of $a_i^\pm$ is an integer.

DEFINITION. We say that α is of *type* (p,q) if p is the smallest index such that one of $a_{p+1}^\pm$ is a *non-positive* integer; and q is the smallest index such that one of $a_{n-q}^\pm$ is a *positive* integer. Finally, we will write $V_{p,q}$ for the subspace spanned by the K-types of type (p,q).

(Not all possible $V_{p,q}$'s need occur for a given ε and t.)

THEOREM. *The irreducible constituents of $I(\varepsilon,t)$ are exactly the non-zero $V_{p,q}$'s.*

PROOF. It suffices to show that the K-types of $V_{p,q}$ form a single equivalence class. Suppose α as in (1) is of type (p,q) and β is as in (3). Now β is also of type (p,q), unless one of the following holds:

(a) $i \leq p$ and one of $a_i^\pm + 1$ is a non-positive integer;
(b) $i = p+1$ and neither of $a_i^\pm + 1$ is a non-positive integer;
(c) $i = n-q$ and neither of $a_i^\pm + 1$ is a positive integer;
(d) $i \geq n-q+1$ and one of $a_i^\pm + 1$ is a positive integer.

From the definition of p and q, it is clear that (a) and (c) are impossible, and that (b) or (d) can occur if and only if the appropriate $a_i^\pm$ equals *zero.* In view of the Lemma, we see that if $i \leq p+1$ or $i \geq n-q$, then $\beta = \alpha + \gamma_i$ is linked to α if and only if it is of the same type.

It remains to show that if $p+1 > i > n-q$, then β is linked to α. Suppose not, then one of $a_i^\pm$, say a_i^+, must equal 0. Then $a_{n-q}^+ < a_i^+ = 0 < a_{p+1}^+$, and now from the definition of p and q, it follows that a_{n-q}^- is a positive integer, and a_{p+1}^- is a non-positive integer. However, since $a_{n-q}^- < a_{p+1}^-$, this is impossible. This finishes the proof. □

§4. Supports.

Let $\mathcal{C}$ be the set $\{\sum_i r_i\gamma_i \mid r_1 \geq \cdots \geq r_n \in \mathbf{R}\}$.

DEFINITION. ([**KV**]; 6.1) If W is a subquotient of some $I(\varepsilon, t)$, we say that $\lambda \in \mathcal{C}$ is in the *asymptotic support* $T(W)$ of W, if there is a sequence $t_k \to 0$ of positive real numbers, and a sequence of α_k of K-types of W, such that $t_k\alpha_k \to \lambda$.

We now describe the asymptotic supports of the irreducible components of $I(\varepsilon, t)$. Clearly, if $I(\varepsilon, t)$ is irreducible, its asymptotic support is all of $\mathcal{C}$, and there is nothing more to say. If $I(\varepsilon, t)$ is reducible, then $T(V_{p,q})$ is determined by p and q. The next Lemma provides some control over the possible pairs (p, q) which can occur.

LEMMA. *Suppose $I(\varepsilon, t)$ is reducible and $V_{p,q}$ is an irreducible subquotient, then $p + q \leq n + 1$. Furthermore unless d is odd and $t - \frac{1}{2}$ is not an integer, we have $p + q \leq n$.*

PROOF. Let α be of type (p, q). Clearly $q \leq n$, so if $p = 0, 1$, then $p+q \leq n+1$. Assume therefore that $p \geq 2$. Since $I(\varepsilon, t)$ is reducible, one of the four numbers $a_p^{\pm}, a_{p-1}^{\pm}$ is an integer. By the definition of p, this integer must be positive. On the other hand if $n - q < p - 1$, this integer would be less than the corresponding $a_{n-q}^{\pm}$ which, by the definition of q, is non-positive. This shows that $n - q \geq p - 1$ which proves the first part of the Lemma.

Moreover, if d is even, or if $t - \frac{1}{2} \in \mathbf{Z}$, then one of $a_p^{\pm}$ is an integer. Arguing as before, we see that in this case $n - q \geq p$. □

For $p + q \leq n$, let $T_{p,q}$ be the cone $\{\sum_i r_i\gamma_i \in \mathcal{C} \mid r_{p+1} = \cdots = r_{n-q} = 0\}$

THEOREM. *Let $V_{p,q}$ be as in Theorem 3. Then if $p + q \leq n$, $T(V_{p,q}) = T_{p,q}$; and if $p + q = n + 1$, $T(V_{p,q}) = T_{p,q-1} \cup T_{p-1,q}$.*

PROOF. Fix a K-type α in $V_{p,q}$. Then the proof of Theorem 3 shows that if $c_1 \geq c_2 \geq \cdots \geq c_p \geq 0$ is any decreasing sequence of nonnegative integers, then the K-type $\alpha + \sum_{i=1}^{p} c_i\gamma_i$ also belongs to $V_{p,q}$. Similarly if $0 \geq c_{n-q+1} \geq \cdots \geq c_n$ is a decreasing sequence of non-positive integers then $\alpha + \sum_{i=n-q+1}^{n} c_i\gamma_i$ belongs to $V_{p,q}$.

Definition 3 implies that the first $n - q$ coefficients of possible K-types in $V_{p,q}$ are bounded below, and the last $n - p$ coefficients are bounded above. The theorem follows. □

§5. Unitarity.

Let W be an irreducible constituent of some $I(\varepsilon, t)$.

LEMMA. *W is unitarizable if and only if for each K-type α of W as in* (1) *and for each index $i = 1, \cdots, n$, satisfying $a_i^{\pm} \neq 0$ and $a_{i-1} > a_i$, we have*

$$t < 2|a_i + \varepsilon + \frac{d}{4}(n - 2i + 1) + \frac{1}{2}|. \tag{5}$$

PROOF. In view of Corollary 2 it suffices to show that for each pair of K-types α and β in W, $q_{\beta,\alpha}(\varepsilon, t)$ is positive. The discussion following Lemma 3 shows that it suffices to check this for linked pairs, and so we may assume that β is as in (3) and $a_i^{\pm} \neq 0$. Theorem 1 shows that $q_{\beta,\alpha}(\varepsilon, t)$ is positive if and only if $a_i^{\pm}$ have the same sign. Since $t \geq 0$, this happens if and only if either $a_i^- > 0$ or $a_i^+ < 0$. Rewriting this, we get (5). □

We now give explicit descriptions of the unitarizable constituents. For convenience, we divide the discussion into three cases:

(A) (Complementary series) $W = I(\varepsilon, t)$
(B) (Large Constituents) $W = V_{p,q}$ with $p + q = n, n + 1$
(C) (Small Constituents) $W = V_{p,q}$ with $p + q \leq n - 1$

Also let us abbreviate the frequently occurring constant $\varepsilon + \frac{d}{4}(n + 1) + \frac{1}{2}$ by $\eta(n, d, \varepsilon)$ or, simply, η. Thus $a_i^{\pm} = a_i + \eta - \frac{di}{2} \pm \frac{t}{2}$.

THEOREM A. *Let* $t_1 \equiv t_1(n, d, \varepsilon) = min\ \{2|a + \eta|, 2|a + \eta + \frac{d}{2}| \mid a \in \mathbf{Z}\}$. *Then* $I(\varepsilon, t)$ *is irreducible and unitary if and only if* $t < t_1$.

PROOF. The Lemma shows that $I(\varepsilon, t)$ is both irreducible and unitary if and only if (5) holds for all $i = 1, \cdots, n$ and all integers a_i. It is easy to see that the minimum value of right side of (5) is t_1. □

REMARK. We can make Theorem A still more explicit as follows: Let $\delta = \frac{1}{2} - |2\varepsilon - 1|$. If d is even, then $t_1 = \frac{1}{2} \pm \delta$ accordingly as $\frac{d}{2}(n + 1)$ is odd or even; if d is odd, then $t_1 = |\delta|$ or $\frac{1}{2} - |\delta|$ accordingly as n is even or odd.

The remark follows from an easy calculation once we note that $\varepsilon \in [0, 1)$.

THEOREM B. *Suppose d is even and $p + q = n$. $I(\varepsilon, t)$ has a constituent of type $V_{p,q}$ if and only if*

(0) *one of* $\eta \pm \frac{t}{2}$ *is an integer.*

This constituent is unitarizable if and only if at least one of the following conditions holds:

(1) t *is an integer;*
(2) $t = t_1$ *or, equivalently* $t < 1$;
(3) $p = 0$ *and* $\eta + \frac{t}{2}$ *is an integer.*
(4) $q = 0$ *and* $\eta - \frac{t}{2}$ *is an integer.*

PROOF. First consider the situation when both p and q are positive.

If t is an integer, and α is a K-type of $V_{p,q}$, then (0) implies that the four numbers $a_{n-q}^{\pm}$, $a_{p+1}^{\pm}$ are all integers. Definition 3 implies that $a_{n-q}^- \geq 0$ and $a_{p+1}^+ \leq 0$. Thus $a_i^{\pm} \geq 0$ for $i \leq p$ and $a_i^{\pm} \leq 0$ for $i > p$, and so the unitarity follows from the Lemma.

If t is not an integer, then exactly one of $a_i^{\pm}$, say a_i^-, is an integer for all i. Now as discussed in the proof of Theorem 4, we can find a K-type in $V_{p,q}$ whose first p coefficients are large and positive. Adding an integral multiple of γ_{p+1} we obtain a K-type α in $V_{p,q}$ such that $a_{p+1}^- = -1$. Now $a_{p+1}^+ = a_{p+1}^- + t = t - 1$. If

$t > 1$, this number is positive, and the Lemma shows that $V_{p,q}$ is not unitary. On the other hand if $t < 1$, one can check that (0) implies that $t = t_1$, and another application of the Lemma proves the unitarity. The argument is similar if a_i^+ is an integer for all i.

Now suppose that $p = 0$, and $t > 1$. Arguing as above, we see that $V_{p,q}$ is unitary if and only if $a_i^\pm \leq 0$ for all i and for every K-type α of $V_{p,q}$. This happens if and only a_1^+ is a (non-positive) integer for all α. This leads to (3). The proof of (4) is similar. □

REMARK. We note that the representations in (1) were discussed in [**S**]; those in (2) correspond to the endpoints of complementary series; and those in (3) and (4) are (limits of) holomorphic and anti-holomorphic discrete series.

THEOREM B′. *Suppose d is odd and $p + q = n$. $I(\varepsilon, t)$ has a constituent of type $V_{p,q}$ if and only if*

(0) *One of $\eta \pm \frac{t}{2}$ is an integer, and $2t$ is an odd integer.*

This constituent is unitarizable if and only if at least one of the following conditions holds:

(1) $\eta - \frac{dp}{2} - \frac{t}{2}$ *is an integer;*
(2) $t = t_1(n, d, \varepsilon) = \frac{1}{2}$*;*
(3) $p = 0$ *and* $\eta - \frac{d}{2} + \frac{t}{2}$ *is an integer;*
(4) $q = 0$ *and* $\eta - \frac{nd}{2} - \frac{t}{2}$ *is an integer.*

PROOF. First assume that p and q are positive, and let α be a K-type of $V_{p,q}$. Since d is odd, a_i^+ and a_{i+1}^+ cannot both be integers. Thus exactly one of the following two situations must hold:

(a) a_p^- and a_{p+1}^+ are integers.
(b) a_p^+ and a_{p+1}^- are integers.

These conditions are easily seen to be equivalent to (0) of the Theorem.

Arguing as in the proof of Theorem B, we have unitarity in case (a), which corresponds to (1) of the present Theorem; and non-unitarity in case (b), unless $t < 1$. However, since $2t$ is an odd integer, the last condition implies $t = \frac{1}{2}$, which corresponds to (2) of the Theorem.

The argument for cases (3) and (4) is the same as in Theorem B. □

THEOREM B″. *Suppose d is odd and $p + q = n + 1$. $I(\varepsilon, t)$ has a constituent of type $V_{p,q}$ if and only if*

(0) *One of $\eta - \frac{d(p+1)}{2} \pm \frac{t}{2}$ is an integer, and $2t$ is not an odd integer.*

This constituent is unitarizable if and only if at least one of the following conditions holds:

(1) $t < \frac{1}{2}$ *or, equivalently* $t = t_1$*;*
(2) $p = 0$ *and* $\eta - \frac{d}{2} + \frac{t}{2}$ *is an integer;*
(3) $q = 0$ *and* $\eta - \frac{nd}{2} - \frac{t}{2}$ *is an integer.*

PROOF. From Definition 3, it follows that a constituent of type $V_{p,q}$ occurs exactly when one of $a_{p+1}^{\pm}$ is an integer, and $a_{p+2}^{\pm}$ are both non-integral. Rewriting this gives (0).

The argument for (2) and (3) is the same as before. So we assume that p and q are positive. Let α be a K-type of $V_{p,q}$. For $i = p+1$, one of $a_i^{\pm}$ is an integer, say a_i^+. Since d is odd, a_p^+ is in $\mathbf{Z} + \frac{1}{2}$.

Now Theorem 4 shows that a_p can assume arbitrary integral values. In particular, we can arrange to have $a_p^+ = \frac{1}{2}$. For $V_{p,q}$ to be unitary, $a_p^- = \frac{1}{2} - t$ must also be positive. Conversely if $t < \frac{1}{2}$, then (0) implies that t must equal t_1, and the unitarity follows. □

THEOREM C. *Suppose $p + q < n$. Then $I(\varepsilon, t)$ has a constituent of type $V_{p,q}$ if and only if $\eta - \frac{d}{2}(p+1) - \frac{t}{2}$ is an integer and $t - \frac{d}{2}(n-1-p-q)$ is a positive integer.*

This constituent is unitarizable if and only if

$$t = 1 + \frac{d}{2}(n - 1 - p - q). \tag{6}$$

PROOF. Let α be a K-type of $V_{p,q}$. By the definition of p and q, one of $a_{p+1}^{\pm}$ is a non-positive integer and one of $a_{n-q}^{\pm}$ is a positive integer. Since $p+1 \geq n-q$, the only way this can happen is if a_{p+1}^- is the non-positive integer and a_{n-q}^+ is the positive integer. Rewriting this, we obtain the the first part of the Theorem.

Suppose $V_{p,q}$ is unitary. Now $a_{p+1}^+ \geq a_{n-q}^+$ must be positive. Since a_p can be arbitrarily large, the Lemma shows that a_{p+1}^- cannot be negative, so it must be 0. Arguing similarly, a_{n-q}^+ must be equal to 1. Thus $1 = a_{p+1}^+ - a_{n-q}^- = \frac{d}{4}(-2(p+1) + 2(n-q)) + t$, which yields (6).

On the other hand if (6) holds, then an easy calculation shows $0 \geq a_{p+1}^- = a_{p+1} + \varepsilon - \frac{d}{4}(p-q)$. Similarly, $1 \leq a_{n-q}^+ = a_{n-q} + \varepsilon - \frac{d}{4}(p-q) + 1$. Thus we see that $a_i + \varepsilon \geq \frac{d}{4}(p-q)$ for $i \leq n-q$ and that $a_i + \varepsilon \leq \frac{d}{4}(p-q)$ for $i \geq p+1$. Consequently, the K-types of the representation are

$$\{\frac{d}{4}(p-q)\mu + \sum_{i=1}^{n} c_i\gamma_i \mid c_1 \geq \cdots \geq c_n \in \mathbf{Z};\ c_{p+1} = \cdots = c_{n-q} = 0\}. \tag{7}$$

The unitarity of $V_{p,q}$ follows easily from the Lemma. □

Appendix.

In this appendix we complete the proof of Lemma 3 using some elementary facts from the theory of generalized Verma modules. A convenient reference for this section is [**FK**].

Let c be an integer and let ε be in $[0,1)$ as usual. Write ψ for the 1-dimensional $\mathfrak{k}$-type $\psi = c\mu + \varepsilon\mu$, let $M(\psi) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}+\mathfrak{p}^+)} \mathbf{C}_\psi$ be the corresponding generalized Verma module, and let $L(\psi)$ be its unique irreducible quotient.

LEMMA A. *If* $t = t(c, \varepsilon) \equiv 2(c + \varepsilon + \frac{d(n-1)}{4} + \frac{1}{2})$, *then* $L(\psi)$ *occurs as a subquotient of* $I(\varepsilon, t)$.

PROOF. It suffices to show that $\pi_{\varepsilon,t}(\mathfrak{p}^+)V_{c\mu} = 0$. Clearly $\pi_{\varepsilon,t}(\mathfrak{p}^+)V_{c\mu} \subseteq V_{c\mu+\gamma_1}$, and arguing as in the proof of the first part of Lemma 3, we see that for $t = t(c, \varepsilon)$, $c\mu$ is lower than $c\mu + \gamma_1$ in the Jantzen filtration. Consequently $\pi_{\varepsilon,t}(\mathfrak{p}^+)V_{c\mu} = 0$. □

Let c be fixed. Then it is well known that for "most" ε, $M(\psi) = M(c\mu + \varepsilon\mu)$ is irreducible. (For an exact statement see Theorem 5.3 in [**FK**].) On the other hand, if α, β and $t_o(\alpha, i, \varepsilon)$ are as in the proof of Lemma 3, then by suitably choosing ε and $c >> a_1$ we may arrange to have

$$V_\alpha, V_\beta \subseteq M(\psi) = L(\psi) \subseteq I(\varepsilon, t(c, \varepsilon))$$

and at the same time ensure that $t(c, \varepsilon) \neq \pm t_o(\alpha, i, \varepsilon)$.

This reduces the proof of Lemma 3 to a corresponding result about Verma modules, which we shall prove after introducing some necessary notation.

As in [**FK**], we may realize $M(\psi)$ on the space $\mathcal{P}$ of polynomials on $\mathfrak{p}^+$. (This is related via the Cayley transform (as in [**S**]) to the realization inside $I(\varepsilon, t)$.) The Lie algebra $\mathfrak{p}^+$ acts by constant coefficient vector fields.

The $\mathfrak{t}$-weights of $\mathfrak{p}^+$ are $\frac{1}{2}(\gamma_i + \gamma_j)$ and γ_i, and the latter have 1-dimensional weight spaces. Choose weight vectors X_i in the γ_i-weight spaces and extend these to a basis of $\mathfrak{p}^+$ consisting of $\mathfrak{t}$-weight vectors. If z_i is the coordinate function for the γ_i-weight vector, then X_i acts on $\mathcal{P}$ by $\partial_i \equiv \partial/\partial z_i$.

Let φ_i be the polynomial functions described in [**KS**] (and denoted by Δ_i in [**FK**]). Then polynomials of the form

$$p = \varphi_1^{d_1} \cdots \varphi_n^{d_n} \tag{8}$$

constitute the totality of *lowest* weight vectors in the various $\mathfrak{k}$-types of $M(\psi)$. Such a p has $\mathfrak{t}$-weight

$$(c + \varepsilon)\mu - [(d_1 + \cdots + d_n)\gamma_1 + (d_2 + \cdots + d_n)\gamma_2 + \cdots + d_n\gamma_n].$$

The *highest* weight vector in that $\mathfrak{k}$-type has weight

$$(c + \varepsilon)\mu - [d_n\gamma_1 + (d_{n-1} + d_n)\gamma_2 + \cdots + (d_1 + \cdots + d_n)\gamma_n].$$

Let $\alpha = a_1\gamma_1 + \cdots + a_n\gamma_n$ and β be as in (1), (3) and Lemma 3. Since $c >> a_1$ and $a_{i-1} > a_i$ (see (3)), we can choose integers $d_1, \ldots, d_n$ so that if p is as in (8), then $v = \varphi_i p$ and $w = \varphi_{i-1}p$ are lowest weight vectors in $V_\alpha = M(\psi)_{\alpha+\varepsilon}$ and $V_\beta = M(\psi)_{\beta+\varepsilon}$ respectively.

We recall next the positive definite, K-invariant, *Fischer* inner product on $\mathcal{P} = M(\psi)$ defined as in (§3 of [**FK**]) by $(p\,|\,q) = (\partial(p)\cdot\widetilde{q})|_{z=0}$, where $\widetilde{q}(z) = \overline{q(\overline{z})}$.

We leave it to the reader to verify the following simple facts:

(i) $\widetilde{\varphi_i} = \varphi_i$

(ii) $\partial_i \cdot \varphi_i = \varphi_{i-1}$ and $\partial_i \cdot \varphi_k = 0$ for $k < i$.

(iii) $(\partial(r) \cdot p\,|\,q) = (p\,|\,\widetilde{r}q)$ for all p, q, r in $\mathcal{P}$.

LEMMA B. $(\partial_i \cdot (\varphi_i p) \mid \varphi_{i-1} p) \neq 0$.

PROOF. If $D_j(m)$ is the differential operator $\partial(\varphi_j)^m \circ \varphi_j^m$ and q is a lowest weight vector of the form $\varphi_1^{e_1} \cdots \varphi_k^{e_k}$ with $k \leq j$, then $D_j(m) \cdot q$ is a multiple of q. Moreover, from (i) and (iii) above, and the definiteness of the form, it follows that this multiple is not zero.

Applying this remark and (iii) repeatedly, we see that $(\partial_i \cdot (\varphi_i p) \mid \varphi_{i-1} p)$ is a non-zero multiple of $(\partial_i \cdot (\varphi_i p') \mid \varphi_{i-1} p')$ where $p' = \varphi_1^{d_1} \cdots \varphi_{i-1}^{d_{i-1}}$. Now (ii) shows that $\partial_i \cdot (\varphi_i p') = \varphi_{i-1} p'$, and the Lemma follows by the definiteness of the form. □

PROOF OF LEMMA 3. Choose c, ε and $t = t(c, \varepsilon)$ as above. Also let $v = \varphi_i p$ in V_α and $w = \varphi_{i-1} p$ in V_β be as before, and set $X = X_i$ in $\mathfrak{p}^+$. Then Lemma B shows that $\pi_{\varepsilon,t}(X)v$ is not orthogonal to w. In particular, $P_{X,v}(\varepsilon, t(c, \varepsilon)) \neq 0$. □

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