Jean-Luc Brylinski
Department of Mathematics
Penn State University
University Park, PA 16802

Ranee Brylinski
Department of Mathematics
Penn State University
University Park, PA 16802

Victor Guillemin
Department of Mathematics
MIT
Cambridge, MA 02139

Victor Kac
Department of Mathematics
MIT
Cambridge, MA 02139

Library of Congress Cataloging In-Publication Data
Lie theory and geometry : in honor of Bertram Kostant / Jean-Luc Brylinski . . . [et al.], editors.
p. cm. -- (Progress in mathematics ; v. 123)
Invited papers, some originated at a symposium held at MIT in May 1993.
Includes bibliographical references.
ISBN 978-1-4612-6685-3 ISBN 978-1-4612-0261-5 (eBook)
DOI 10.1007/978-1-4612-0261-5
1. Lie groups. 2. Geometry. I. Kostant, Bertram.
II. Brylinski, J.-L. (Jean-Luc) III. Series: Progress in mathematics : vol. 123.
QA387.L54 1994 94-32297
512'55--dc20 CIP

Printed on acid-free paper

Originally published by Birkhäuser Boston in 1994
Softcover reprint of the hardcover 1st edition 1994

ISBN 978-1-4612-6685-3

Text prepared by TeXniques, Inc. from disks.

9 8 7 6 5 4 3 2 1

The Spectrum of Certain Invariant Differential Operators Associated to a Hermitian Symmetric Space*

Siddhartha Sahi

This paper is dedicated to Prof. Bertram Kostant with affection and admiration on the occasion of his sixty-fifth birthday

Let G/K be an irreducible Hermitian symmetric space of rank n and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_+ + \mathfrak{p}_-$ be the usual decomposition of $\mathfrak{g} = \text{Lie}(G)_{\mathbf{C}}$. Let us write $\mathcal{P}$, $\mathcal{D}$, and $\mathcal{W} = \mathcal{P} \otimes \mathcal{D}$ respectively, for the algebra of holomorphic polynomials, the algebra of constant coefficient holomorphic differential operators, and the "Weyl algebra" of polynomial coefficient holomorphic differential operators on $\mathfrak{p}_-$, and regard all three as K-modules in the usual way.

Let $\mathcal{I} = \mathcal{W}^K$ be the algebra of K-invariant differential operators on $\mathfrak{p}_-$, and let Λ be the set $\{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}$.

Then, as we show in the next section, Λ naturally parametrizes both the irreducible K-submodules $\mathcal{P}_\lambda$ of $\mathcal{P}$, as well as a certain distinguished (vector space) basis $\{D_\lambda\}$ of $\mathcal{I}$. The problem we consider is to determine, for $\lambda, \mu \in \Lambda$, the scalar eigenvalue $c_\mu(\lambda)$ by which D_μ acts on $\mathcal{P}_\lambda$.

Our main result, proved in Section 1, is the following characterization of these eigenvalues. For $\lambda = (\lambda_1, \cdots, \lambda_n)$, let us write $|\lambda|$ for $\lambda_1 + \cdots + \lambda_n$; and set $\Lambda_m = \{\lambda \in \Lambda \mid |\lambda| \leq m\}$. Also, put $\rho = (\rho_1, \cdots, \rho_n)$ where $\rho_i = d(n - 2i + 1)/2$ and d is as in the next section.

Theorem 1. *There is a polynomial p_μ in n variables such that $c_\mu(\lambda) = p_\mu(\lambda + \rho)$, for all $\lambda \in \Lambda$. Moreover, up to a scalar multiple, p_μ is characterized by the following properties:*

(a) *p_μ is symmetric and has degree at most $|\mu|$.*

(b) *$p_\mu(\mu + \rho)$ is nonzero, while $p_\mu(\lambda + \rho)$ is zero for all other $\lambda \in \Lambda_{|\mu|}$.*

(This result was conjectured by B. Kostant during a conversation with the author).

---

* This research was partially supported by an NSF grant at Rutgers University.

To indicate the power of this result we give two applications. For each $k = 0, 1, \cdots, n$, let us put $\nu^k = (1, \cdots, 1, 0, \cdots, 0) \in \Lambda$ (with k 1's), and write $\nu = \nu^n$.

The first of our applications is a simple proof of the Capelli identity of [KS] which is the formula for $p_{m\nu}$ for $m \in \mathbf{Z}^+$.

Corollary 1. *$p_{m\nu}(x)$ equals $\prod_{i=1}^{n} \prod_{j=0}^{m-1} (x_i - \rho_n - j)$, where $\rho_n = d(-n + 1)/2$.*

Proof. $p_{m\nu}$ obviously satisfies part (a) of the theorem, while (b) follows by observing that $\lambda \in \Lambda_{|m\nu|}$ implies $\lambda_n \leq m$ (with equality if and only if $\lambda = m\nu$). ■

(We refer the reader to [KS] for the precise sense in which this formula generalizes the classical Capelli identity. Also see [S1, S2, BK] for some of the representation theoretic applications of this formula).

Our second application is an easy derivation of Wallach's formula [W] for p_{ν^k}. Let us define the symmetric polynomials σ_j and τ_j of degree j, in n variables, via the generating functions $\prod_{i=1}^{n}(1 + tx_i) = \sum_{j=0}^{n} t^j \sigma_j(x_1, \cdots, x_n)$ and $\prod_{i=1}^{n}(1 + tx_i)^{-1} = \sum_{j=0}^{\infty} t^j \tau_j(x_1, \cdots, x_n)$.

Corollary 2. *$p_{\nu^k}(x)$ equals $\sum_{i=0}^{k} \tau_{n-i}(0, \cdots, 0, \rho_{k+1}, \cdots, \rho_n)\sigma_i(x)$.*

Proof. The proposed polynomial is the coefficient of t^k in the power series expansion of $\prod_{i=1}^{n}(1 + tx_i) / \prod_{i=k+1}^{n}(1 + t\rho_i)$, and clearly satisfies part (a) of the theorem.

Now if $\lambda \in \Lambda_{|\nu^k|}$, $\lambda \neq \nu^k$, then the last $n - k$ coordinates of λ are zero. For such λ, if we put $x_i = \lambda_i + \rho_i$, then the above power series becomes a polynomial in t (of degree less than k), which implies the vanishing requirements of part (b). ■

Except when $n = 2$, there does not seem to a nice closed formula for the general p_μ. However, in Section 2, we describe an inductive procedure for calculating these polynomials.

The proofs are almost embarrassingly easy.

0. Preliminaries

0.1 Hermitian symmetric spaces. Let us choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ and $\mathfrak{g}$ and fix positive root systems such that $\Sigma^+(\mathfrak{h}, \mathfrak{g}) = \Sigma^+(\mathfrak{h}, \mathfrak{k}) \cup \Sigma(\mathfrak{h}, \mathfrak{p}_+)$. Let $\{\gamma_1, \cdots, \gamma_n\}$ be the Harish-Chandra strongly orthogonal noncompact roots, and let $\mathfrak{t}$ be the subalgebra of $\mathfrak{h}$ spanned by their coroots.

The weights of $\mathfrak{t}$ in $\mathfrak{g}$ form a root system of type C_n or BC_n, and the (restricted) roots are $\{\pm(\gamma_i \pm \gamma_j)/2, \pm\gamma_i\}$, and possibly $\{\pm\gamma_i/2\}$. The number d, referred to in the introduction, is the (common) multiplicity of the roots $\{\pm(\gamma_i \pm \gamma_j)/2\}$.

(For these and other standard facts about Hermitian symmetric spaces see [H1]).

As shown by Kostant–Joseph [J], and Schmid [Sc], the K-submodules of $\mathcal{P}$ occur with multiplicity 1, and their highest weights are of the form $\lambda_1\gamma_1 + \cdots + \lambda_n\gamma_n$, where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Lambda$. For each such λ, the corresponding submodule $\mathcal{P}_\lambda$ occurs in polynomials of degree $|\lambda|$.

Since $\mathcal{D}$ is isomorphic to $\mathcal{P}^*$, we have compatible decompositions $\mathcal{P} = \oplus_{\lambda\in\Lambda}\mathcal{P}_\lambda$ and $\mathcal{D} = \oplus_{\lambda\in\Lambda}\mathcal{D}_\lambda$, where $\mathcal{D}_\lambda \approx \mathcal{P}^*_\lambda$. For $\mu \in \Lambda$, choose a basis $\{\varphi_j\}$ of $\mathcal{P}_\mu$ and let $\{d_j\}$ be the dual basis of $\mathcal{D}_\mu$. Then $D_\mu = \sum_j \varphi_j \otimes d_j$ is an invariant differential operator of order $|\mu|$, which is independent of this choice. Moreover, the operators $\{D_\mu \mid \mu \in \Lambda\}$ form a (vector space) basis for $\mathcal{I}$. By Schur's Lemma, each D_μ preserves the decomposition of $\mathcal{P}$, and acts by a scalar $c_\mu(\lambda)$ on each $\mathcal{P}_\lambda$.

This completes the explanation of the undefined terms in the introduction.

0.2 Symmetric tube domains. The results of this section are due to Wallach [W].

It is well known that the Hermitian symmetric space G/K admits a realization as a generalized half plane (tube type domain) precisely when the root system $\Sigma(\mathfrak{t}, \mathfrak{g})$ is of type C_n (*i.e.* if there are no roots of the form $\pm\gamma_i/2$).

Now suppose $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair, not necessarily of tube type, and let $\widetilde{\mathfrak{g}}$ be the subalgebra generated by the root-spaces for $\{\pm(\gamma_i \pm \gamma_j)/2, \pm\gamma_i\}$. If we put $\widetilde{\mathfrak{k}} = \widetilde{\mathfrak{g}} \cap \mathfrak{k}$, and $\widetilde{\mathfrak{p}}_\pm = \widetilde{\mathfrak{g}} \cap \mathfrak{p}_\pm$, then $(\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{k}})$ is a Hermitian symmetric pair of tube type, and we have the decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}_+ + \widetilde{\mathfrak{p}}_-$.

(For example, if $\mathfrak{g} = \mathfrak{u}(m, n)$ with $m \geq n$, then $\widetilde{\mathfrak{g}} = \mathfrak{u}(n, n)$).

Let $\mathfrak{r}$ be the subspace of $\mathfrak{p}_-$ spanned by the root spaces for $\{-\gamma_i/2\}$, then we have the dual decompositions $\mathfrak{p}_- = \widetilde{\mathfrak{p}}_- + \mathfrak{r}$ and $\mathfrak{p}^*_- = \widetilde{\mathfrak{p}}^*_- + \mathfrak{r}^*$. Consequently, we get $\mathcal{P} = \widetilde{\mathcal{P}} + \mathfrak{r}^*\mathcal{P}$, where $\widetilde{\mathcal{P}}$ is the polynomial algebra on $\widetilde{\mathfrak{p}}_-$.

Now the $\widetilde{\mathfrak{k}}$ types of $\widetilde{\mathcal{P}}$ are parametrized by the same set Λ as the $\mathfrak{k}$-types of $\mathcal{P}$. Fix $\lambda \in \Lambda$ and let φ_λ be a highest weight vector in $\mathcal{P}_\lambda$.

Lemma A. *φ_λ belongs to $\widetilde{\mathcal{P}}$, and thus is a highest weight vector in $\widetilde{\mathcal{P}}_\lambda$.*

Proof. Choose a basis of $\mathfrak{p}_-$ consisting of $\mathfrak{t}$-weight vectors, and write φ_λ as a sum of monomials in the dual basis of $\mathfrak{p}^*_- = \widetilde{\mathfrak{p}}^*_- + \mathfrak{r}^*$. Then each of these monomials must have weight λ and degree $|\lambda|$.

For $\alpha = a_1\gamma_1 + \cdots + a_n\gamma_n \in \mathfrak{t}^*$, let us define the *total weight* of α to be $a_1 + \cdots + a_n$. Then the basis vectors of $\widetilde{\mathfrak{p}}^*_-$ have total weight 1, while those of $\mathfrak{r}^*$ have total weight 1/2. So if a monomial of degree $|\lambda|$ were to have one or more factors from $\mathfrak{r}^*$, then its total weight would be less than $|\lambda|$ and, in particular, its $\mathfrak{t}$-weight could not be λ. ■

We now show that the problem of computing the eigenvalues can be reduced to the tube case. Proceeding as in the introduction, we define, for the pair $(\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{k}})$, the invariant differential operators $\widetilde{D}_\mu$ and their eigenvalues $\widetilde{c}_\mu(\lambda)$.

Lemma B. *There is a nonzero scalar κ_μ such that $\widetilde{c}_\mu(\lambda)$ equals $\kappa_\mu c_\mu(\lambda)$ for all $\lambda \in \Lambda$.*

Proof. For $\mu \in \Lambda$ we have the dual decompositions $\mathcal{P}_\mu = \widetilde{\mathcal{P}}_\mu + \mathcal{P}'_\mu$ and $\mathcal{D}_\mu = \widetilde{\mathcal{D}}_\mu + \mathcal{D}'_\mu$, where $\mathcal{P}'_\mu \subset \mathfrak{r}^*\mathcal{P}$, and $\mathcal{D}'_\mu \subset \mathfrak{r}\mathcal{D}$. From the definition of D_μ it follows easily that D_μ is a nonzero scalar multiple of $\widetilde{D}_\mu + D'_\mu$, where $D'_\mu \in \mathcal{P}'_\mu \otimes \mathcal{D}'_\mu$.

If φ_λ is as above, then $D_\mu\varphi_\lambda = c_\mu(\lambda)\varphi_\lambda$ and $\widetilde{D}_\mu\varphi_\lambda = \widetilde{c}_\mu(\lambda)\varphi_\lambda$. By Lemma A, vector fields from $\mathfrak{r}$ annihilate φ_λ; thus $D'_\mu\varphi_\lambda = 0$ and the lemma follows easily.

1. The combinatorial characterization

We are now ready for the proof of Theorem 1. In view of Lemma B, we may assume that $(\mathfrak{g}, \mathfrak{k})$ is of tube type. The chief advantage of this assumption is the following:

Consider the Harish-Chandra imbedding of G/K as a bounded domain in $\mathfrak{p}_-$, and let $\mathfrak{S}$ be its Shilov boundary. If G/K is of tube type then $\mathfrak{S}$ is a *symmetric space* K/M for the action of K.

Moreover, we can choose M such that if $\mathfrak{k} = \mathfrak{m} + \mathfrak{s}$ is the corresponding Cartan decomposition of $\mathfrak{k}$, then $\mathfrak{t}$ (see section 0) is a maximal Cartan subspace of $\mathfrak{s}$. The roots of $\mathfrak{t}$ in $\mathfrak{k}$ are $(\gamma_i - \gamma_j)/2$, each with multiplicity d. Thus the half sum of positive roots is $\sum \rho_i\gamma_i$ where $\rho_i = d(n - 2i + 1)/2$ as in the introduction.

Now holomorphic functions on $\mathfrak{p}_-$ are determined by their restrictions to $\mathfrak{S}$, and, conversely, real analytic functions on $\mathfrak{S}$ can be uniquely extended to holomorphic functions in a neighborhood of $\mathfrak{S}$. Thus if D is a holomorphic differential operator on $\mathfrak{p}_-$, it has a unique restriction to $\mathfrak{S}$ which satisfies $D|_{\mathfrak{S}} f|_{\mathfrak{S}} = (Df)|_{\mathfrak{S}}$ for all functions f holomorphic in a neighborhood of $\mathfrak{S}$.

Proof of Theorem 1. As noted above, we assume that G/K is of tube type. Since $D_\mu \in \mathcal{I}$, its restriction $D_\mu|\mathfrak{S}$ is an invariant differential operator of order $|\mu|$ on the symmetric space K/M. By Theorem II.5.18 of [H2], we conclude that $c_\mu(\lambda) = p_\mu(\lambda + \rho)$ where p_μ is a polynomial of degree $|\mu|$ on $\mathfrak{t}^*$ which is invariant under the Weyl group of $\Sigma(\mathfrak{t}, \mathfrak{k})$. Since this Weyl group is simply the symmetric group acting on the basis $\{\gamma_1, \cdots, \gamma_n\}$, we get part (a) of the theorem.

The action of "differentiation" gives us a K-map from $\mathcal{D} \otimes \mathcal{P}$ to $\mathcal{P}$, where the image of $\mathcal{D}_\mu \otimes \mathcal{P}_\lambda$ is contained in polynomials of degree $|\lambda| - |\mu|$. Thus this image is zero if $|\lambda| < |\mu|$; while if $|\lambda| = |\mu|$, then the image is contained in $\mathbf{C}$ (the polynomials of degree zero), and thus we get a K-invariant pairing between $\mathcal{P}_\lambda$ and $\mathcal{D}_\mu$. It follows easily that in the latter case the image is nonzero if and only if $\lambda = \mu$.

Recalling the definition of D_μ, and applying these observations, we conclude that p_μ satisfies part (b). It remains only to show that parts (a) and (b) *characterize* p_μ.

Let $S_{|\mu|}$ be the space of symmetric polynomials (in n variables) of degree at most $|\mu|$. Then, with the notation as in the introduction, the map $\sum a_k \nu^k \mapsto \prod \sigma_k^{a_k}$ gives a bijection between the set $\Lambda_{|\mu|}$ and a basis of $S_{|\mu|}$. Thus the dimension of $S_{|\mu|}$ is the same as the cardinality N, say, of $\Lambda_{|\mu|}$.

We claim that every polynomial in $S_{|\mu|}$, in particular p_μ, is determined by its values on the set $\{\lambda + \rho \mid \lambda \in \Lambda_{|\mu|}\}$. To see this we argue as follows:

"Evaluation" at the points in this set gives a linear map Ev from $S_{|\mu|}$ to $\mathbf{C}^N$. Applying this map to $\{p_\lambda \mid \lambda \in \Lambda_{|\mu|}\}$, we get an $N \times N$ matrix whose (λ, λ')-th entry is $p_\lambda(\lambda' + \rho)$. If we arrange the rows and columns in order of increasing $|\lambda|$, then (b) implies that the matrix is triangular, with nonzero diagonal entries.

It follows easily that Ev is bijective, completing the proof of the theorem. ∎

2. The inductive formula

We now describe an inductive procedure for calculating p_μ which is valid in a more general combinatorial context. Let $\rho = (\rho_1, \cdots, \rho_n)$ be an *arbitrary* decreasing sequence of real numbers (*i.e.* ρ need not correspond to a root system).

Theorem 2. *For each $\mu \in \Lambda$ there is a polynomial p_μ^ρ in n variables, unique up to a scalar multiple, which satisfies:*

(a) *p_μ^ρ is symmetric and has degree at most $|\mu|$.*

(b) $p^\rho_\mu(\mu+\rho)$ *is nonzero, while* $p^\rho_\mu(\lambda+\rho)$ *is zero for all other* $\lambda \in \Lambda_{|\mu|}$.

Once the existence of the $\{p^\rho_\mu\}$ is established, the uniqueness follows from the bijectivity of the Ev map exactly as in the proof of Theorem 1. However, since these polynomials are not related to invariant differential operators, we no longer have an *a priori* existence result. We shall provide an alternative argument which has the added advantage of being mostly constructive. We preface the proof with two simple lemmas and a definition.

Shifting Lemma. *Let r be any real number. Suppose Theorem 2 holds for some (ρ,μ), then it holds for (ρ',μ) where $\rho' = (\rho_1+r,\cdots,\rho_n+r)$.*
Moreover, $p^{\rho'}_\mu(x_1,\cdots,x_n) = p^\rho_\mu(x_1-r,\cdots,x_n-r)$.

Proof. It is easy to see that $p^{\rho'}_\mu$ satisfies the conditions of theorem. ■

Factoring Lemma. *Let m be any positive integer. Suppose Theorem 2 holds for (ρ,μ), then it holds for (ρ,μ') where $\mu' = (\mu_1+m,\cdots,\mu_n+m)$. Moreover, $p^\rho_{\mu'}(x) = \left(\prod_{i=1}^n \prod_{j=1}^{\mu_n}(x_i-\rho_n-j+1)\right) p^\rho_\mu(x_1-m,\cdots,x_n-m)$.*

Proof. The proposed polynomial is easily seen to be symmetric and of degree $|\mu'|$. It remains only to check the vanishing conditions of part (b) for $\lambda' \in \Lambda_{|\mu'|}$.

If the last coordinate of λ' is less than m, then $(x_n-\rho_n-\lambda'_n)$ is a factor of $p^\rho_{\mu'}$, and so the polynomial vanishes at $x=\lambda'+\rho$. On the other hand, if $\lambda'_n \ge m$, then $\lambda' = \lambda + (m,\cdots,m)$, where $\lambda \in \Lambda_{|\mu|}$; and the vanishing results follow by applying part (b) to $p^\rho_\mu(\lambda)$. ■

Definition. Suppose p is a symmetric polynomial in $n-1$ variables, and k is a real number. We shall write $\mathrm{Symm}_k(p)$ for the polynomial in n variables given by

$$\begin{aligned}&\mathrm{Symm}_k(p)(x_1,\cdots,x_n)\\ &\quad = \sum_{j=0}^{n-1}(-1)^{n-1-j}\left(\sum_{1\le i_1<\cdots<i_j\le n} p(x_{i_1}-k,\cdots,x_{i_j}-k,0,\cdots,0)\right)\end{aligned}$$

where the inner sum runs over all j-tuples $1 \le i_1 < \cdots < i_j \le n$.

It is easy to check that $\mathrm{Symm}_k(p)$ is symmetric, has the same degree as p, and satisfies $\mathrm{Symm}_k(p)(x_1+k,\cdots,x_{n-1}+k,k) = p(x_1,\cdots,x_{n-1})$ for all $x_1,\cdots,x_{n-1}$.

We are now ready to prove Theorem 2.

Proof of Theorem 2. We proceed by induction on n. If $n=1$ and $\rho=r$ and $\mu=m$, then the polynomial is $p^\rho_\mu(x) = \prod_{j=0}^{m-1}(x-r-j)$. We now assume the result for $(n-1)$ variables and describe the construction in n variables. In view of the two lemmas, it suffices to construct p^ρ_μ for n-tuples ρ and μ whose last coordinates are zero.

We shall construct, by induction on k, polynomials p_k which satisfy part (a) of the theorem, and also satisfy part (b) for those λ in $\Lambda_{|\mu|}$ whose last coordinate is at most k. The desired polynomial will then be p_l where l is the greatest integer not exceeding $|\mu|/n$.

For $k=0$, the polynomial p_0 is obtained by applying Symm_0 to the symmetric polynomial $p^{(\rho_1,\dots,\rho_{n-1})}_{(\mu_1,\dots,\mu_{n-1})}$ of degree $|\mu|$ in $(n-1)$ variables. The latter polynomial exists in view of the inductive hypothesis on the number of variables, and using the remarks following the definition of Symm_k, it is easy to check that p_0 satisfies part (b) for $\lambda \in \Lambda_{|\mu|}$ with $\lambda_n = 0$.

Now given p_{k-1} with $k \le l$, if q_k is any symmetric polynomial of degree $|\mu| - nk$, then $p_k = p_{k-1} - \prod_{i=1}^n \prod_{j=0}^{k-1} (x_i - j) q_k$ still satisfies (a) and (b) for $\lambda \in \Lambda_{|\mu|}$ with $\lambda_n \le k-1$. We claim that there is a symmetric polynomial h_k of degree $|\mu| - nk$ in $n-1$ variables, such that if $q_k = \mathrm{Symm}_k(h_k)$ then p_k also satisfies (b) for $\lambda \in \Lambda_{|\mu|}$ with $\lambda_n = k$.

Rewriting these requirements, we see that h_k must satisfy

$$\begin{aligned} &h_k(\lambda_1+\rho_1,\cdots,\lambda_{n-1}+\rho_{n-1}) \\ &\qquad = p_{k-1}(\lambda_1+\rho_1,\cdots,\lambda_{n-1}+\rho_{n-1})/k! \prod_{i=1}^{n-1}\prod_{j=1}^{k}(\lambda_i+\rho_i+j) \end{aligned}$$

for all $(\lambda_1,\cdots,\lambda_{n-1}) \in \Lambda^-_{|\mu|-nk}$, where the sets Λ^- and Λ^-_m are defined just as Λ and Λ_m, but for $(n-1)$ variables.

As remarked above, the inductive hypothesis implies the bijectivity of the Ev map in $(n-1)$ variables. Consequently, we can find a (unique) polynomial h_k satisfying these requirements. This completes the proof of theorem. ■

Our argument is constructive, except for the definition of the h_k which involves inverting the Ev map. However, even this is not too bad, since, as noted in the proof of Theorem 1, the matrix of this map is triangular with respect to a natural basis.

Finally, observe that for each ρ, the polynomials $\{p^\rho_\mu \mid \mu \in \Lambda\}$ give a (vector space) basis for the space of symmetric polynomials in n variables. It would be interesting to express p^ρ_μ in terms of other bases such as $\{\prod \sigma_k^{a_k}\}$.

References

[BK] R. Brylinski and B. Kostant, Minimal representations of E_6, E_7 and E_8 and the generalized Capelli identity, *Proc. Natl. Acad. Sci. USA* **91** (1994), 2469–2472.

[H1] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York 1978.

[H2] S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York 1984.

[J] A. Joseph, A preparation theorem for the prime spectrum of a semisimple Lie algebra, *J. of Algebra* **48** (1977), 241–289.

[KS] B. Kostant and S. Sahi, The Capelli identity, tube domains and the generalized Laplace transform, *Adv. Math.* **87** (1991), 71–92.

[S1] S. Sahi, The Capelli identity and unitary representations, *Compos. Math.* **81** (1992), 247–260.

[S2] S. Sahi, Unitary representations on the Shilov boundary of a symmetric tube domain, in: Representations of Groups and Algebras, *Contemp. Math.* vol. 145, Am. Math. Soc., Providence 1993.

[Sc] W. Schmid, Die Randwerte Holomorpher Funktionen auf Hermitesch Symmetrischen Räumen, *Invent. Math.* **9** (1969), 61–80.

[W] N. Wallach, Polynomial differential operators associated with Hermitian symmetric spaces, in: *Representation Theory of Lie Groups and Lie Algebras*, World Scientific, 1992, pp. 76–94, Singapore.

Mathematics Department, Rutgers University, New Brunswick, NJ 08903

Received January 21, 1994