J. reine angew. Math. **462** (1995), 1—18

Journal für die reine und angewandte Mathematik

Jordan algebras and degenerate principal series

By *Siddhartha Sahi* at New Brunswick*)

Introduction

Let G be a simple Lie group with Lie algebra $\mathfrak{g}$, and let K be the maximal compact subgroup corresponding to a Cartan involution θ. Suppose G has a parabolic subgroup $P = LN$ such that (i) the nilradical N is abelian, and (ii) P is conjugate to $\bar{P} = \theta(P)$.

The spherical (degenerate) $\bar{P}$-principal series representations of G are obtained by starting with a positive real character of L, extending trivially to $\bar{P}$, and inducing up to G. For such a representation, (i) implies that the K-types have multiplicity 1 and (ii) implies that each irreducible constituent has an invariant Hermitian form.

In this paper we provide a rather detailed analysis of these representations. We explicitly determine the K-types of their irreducible constituents and the signature of the Hermitian form on each K-type.

A direct consequence of this is a determination of the unitarizable constituents of the series. In particular, for each such group G, we obtain a finite set of "small" unitary representations which includes the "minimal" representation. (See Theorems 4B and 5A.)

In the sequel to this paper we will prove that these representations have natural realizations on appropriate L^2-spaces of L-orbits in N. (Thereby extending the results of [SS] and [S3].) A still later paper will finally reveal our ultimate objective which is to obtain an extension of Howe's theory of reductive dual pairs.

In the spirit of [HT], the present paper relies on elementary algebraic methods of the kind pioneered by [B]. Our results extend and generalize those of [W], [G], [BSS], [S1], [S2], [J1], [J2] and several others.

---

*) This work was supported by NSF grants at Princeton University and Rutgers University and carried out in part while the author was visiting Universitè de Nancy.

As outlined in [BSS], one can also compute the composition series using very general results of Barbasch and Vogan on the coherent continuation representation. However, this method will not yield sharp results on K-types and signatures, and without additional ideas, it is likely to require considerable case-by-case calculations.

We now describe the present situation and our techniques in more detail.

Parabolic subgroups satisfying (i) and (ii) arise naturally in the context of Jordan algebras. Indeed the nilradical of such a parabolic subgroup has a natural structure of a simple real Jordan algebra, which is unique up to isotopy.

Conversely, given a simple real Jordan algebra N, following the ideas of Tits [T], Kantor [Ka], and Koecher [Ko], one recovers L as the *Lorentz* group, i.e. the subgroup of $GL(N)$ which preserves the Jordan norm up to a scalar multiple. The semidirect product $P = LN$ is the *Poincaré* group, and G is the *conformal* group of rational transformations of N, generated by P and the *conformal inversion* $\iota : x \mapsto -x^{-1}$.

This means that some of our *intermediate* results (especially Lemma 1D) can be proved, perhaps more naturally, by Jordan algebra methods. However, this approach involves some basic results in the structure theory of non-Euclidean Jordan algebras for which there does not seem to be a convenient reference. Rather than invent (or reinvent) the necessary Jordan theory, we have provided Lie-theoretic proofs which, though clumsier, require fewer prerequisites.

The key to our approach is to use the structure of *finite*-dimensional constituents to study *infinite*-dimensional constituents. This was also the main idea in the proof of the Capelli identity of [KS].

Here is the overall strategy: Let $\mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. We first prove a nondegeneracy result (Lemma 1D) for the action of $\mathfrak{p}$ on K-types. This enables us to use the Capelli identity to obtain explicit formulas (Theorem 1) for the $\mathfrak{p}$-action, generalizing those for the "creation" and "annihilation" operators in [B].

Everything that we wish to know about the principal series follows from these simple formulas. Indeed most of the subsequent effort is devoted to organizing this information in a useful and succinct fashion.

The formula for the Hermitian form is obtained in Theorem 2. Theorem 3A determines the points of reducibility, and Theorems 3B and 3C describe the constituents and their K-types in terms of their "rank". Lemma 4 establishes a sharp criterion for unitarity and Theorems 4A–4E and 5A–5C determine all the unitarizable constituents.

The unitary representations described in Theorems 4B and 5A are of particular interest since they are all "unipotent" (in the sense of [V1]) and correspond to some of the smallest nilpotent coadjoint orbits of the group G. We expect that these representations (and their analogs over other fields) will have some interesting applications.

0. Roots and weights

In this section we recall some basic facts about the situation described in the introduction. We will denote the complexified Lie algebra of a group by the corresponding lower-case gothic letter. Our primary source is [KS], to which we direct the reader for proofs and further references to the literature.

As noted in [KS], parabolic subgroups satisfying (i) and (ii) can be determined rather simply from the restricted root system of $\mathfrak{g}$. Every maximal parabolic corresponds to a simple restricted root α. The condition (i) means that α has coefficient 1 in the highest root, and (ii) means that α is conjugate to $-\alpha$ by the long element of the restricted Weyl group. (This classification was first carried out by [KN]. See the appendix of the present paper for a list.)

Let $\mathfrak{t}$ be a maximal toral subalgebra in the orthogonal complement of $\mathfrak{l} \cap \mathfrak{k}$ in $\mathfrak{k}$, and write n for the dimension of $\mathfrak{t}$. Then it is known that $\Sigma(\mathfrak{t}, \mathfrak{g})$ is a root system of type C_n, and there are precisely *three* possibilities for the subsystem $\Sigma(\mathfrak{t}, \mathfrak{k})$, namely C_n, D_n, or A_{n-1}. (See for example [L].)

The last case occurs precisely when G/K is a symmetric tube domain. Since this case was dealt with in [S2], *in the rest of this paper we confine our attention to the first two situations, referring to them as cases* C *and* D *respectively.*

We now fix a positive root system in $\Sigma(\mathfrak{t}, \mathfrak{k})$ and choose a basis $\{\gamma_1, \ldots, \gamma_n\}$ of $\mathfrak{t}^*$ such that the simple roots are $\{(\gamma_i - \gamma_{i+1})/2\}$ together with γ_n or $(\gamma_{n-1} + \gamma_n)/2$, in cases C and D, respectively.

Let $\varrho = \sum_i r_i \gamma_i$ denote the half sum of positive roots in $\Sigma(\mathfrak{t}, \mathfrak{k})$. The numbers r_i were listed in Table 1 in [KS], but since they are *crucial* in what follows, we relate them to the root multiplicities in $\Sigma(\mathfrak{t}, \mathfrak{k})$.

Except for case D_2, $\Sigma(\mathfrak{t}, \mathfrak{k})$ is irreducible. Thus the short roots $(\gamma_i \pm \gamma_j)/2$ have a common multiplicity d, and the long roots γ_i have a common multiplicity e. ($e = 0$ in case D.) An easy calculation shows that $2r_i = d(n - i) + e$.

The case $\Sigma(\mathfrak{t}, \mathfrak{k}) = D_2$ arises precisely when G is the group $O(p, q)$ (for $p \geqq q \geqq 3$) regarded as the conformal group of the quadratic form of signature $(p-1, q-1)$ on $\mathbb{R}^{p+q-2}$. The parabolic P is the *classical* Poincaré group attached to this quadratic form. With appropriate choices, the roots $(\gamma_1 + \gamma_2)/2$ and $(\gamma_1 - \gamma_2)/2$ have multiplicities $p-2$ and $q-2$ respectively, and it follows that $r_1 = (p+q-4)/4$, $r_2 = (p-q)/4$.

Put $r = 2r_1 + 1$ and let ν be the positive character of L such that ν^{2r} is the determinant of the adjoint action of L on $\mathfrak{n}$. (As pointed out in [KS], the differential of ν is the Cayley transform of $(\gamma_1 + \cdots + \gamma_n)/2$; and r is closely related to the dual Coxeter number of $\mathfrak{g}$.)

For $t \in \mathbb{R}$, let $(\pi_t, I(t))$ denote the (normalized) induced representation $\operatorname{Ind}_{\bar{P}}^{G}(\nu^t)$ (by right derivatives on the space of functions on G which are right K-finite and which transform on the left by the character ν^{t-r} of $\bar{P}$). By restriction to K we realize all the π_t on

$V = L^2(K/L \cap K)_{K\text{-finite}}$. In this picture, "multiplication" is a K-equivariant map from $V \otimes V$ to V, and if X is in $\mathfrak{g}$, and u, v are in V, then we have

$$\pi_{t+t'}(X)(uv) = (\pi_t(X)u)v + u(\pi_{t'+r}(X)v)\,. \tag{0}$$

Since $K/L \cap K$ is a symmetric space, by the Cartan-Helgason theorem [He 2], the K-types of V have multiplicity 1 and the set $\mathcal{S}$ of their highest weights consists of $\sum_i a_i \gamma_i$ where the a_i's are integers which satisfy $a_i \geqq a_{i+1}$, and the additional conditions $a_n \geqq 0$ and $a_{n-1} \geqq |a_n|$ in cases C and D respectively.

Finally we recall Corollary 2.1 of [KS], which shows that for each nonnegative integer k, $I(2k+r)$ has an irreducible, spherical, finite dimensional subrepresentation F_k whose K-types have highest weights $\{\sum a_i \gamma_i \in \mathcal{S} : |a_i| \leqq k\}$. For F_1, these weights are $\mu_j = \gamma_1 + \gamma_2 + \cdots + \gamma_j$ in case C, while in case D we also have $\mu' = \gamma_1 + \cdots + \gamma_{n-1} - \gamma_n$.

This has the following easy, but important, consequence.

Lemma 0. *If a weight $\sum a_i \gamma_i$ in $\mathfrak{t}^*$ has some coefficient a_i strictly larger than k, then it is not a weight in F_k.* □

1. The action of $\mathfrak{p}$

For $\alpha \in \mathcal{S}$, let us write V_α for the corresponding K-type in V. We wish to understand the K-types "which can be reached by *one* application of $\mathfrak{g}$ to V_α."

Let $\pi_t(\mathfrak{g}) V_\alpha$ denote the linear span of vectors of the form $\pi_t(X)v$ for $X \in \mathfrak{g}$ and $v \in V_\alpha$. Since $\mathfrak{k}$ stabilizes V_α, it follows that $X \otimes v \mapsto \pi_t(X)v$ is a K-equivariant projection from $\mathfrak{p} \otimes V_\alpha$ *onto* $(\pi_t(\mathfrak{g}_\alpha) V_\alpha \setminus V_\alpha)$.

Now the $\mathfrak{t}$-weights of $\mathfrak{p}$ are $\{\pm\gamma_j, (\pm\gamma_i \pm \gamma_j)/2\}$, and so the *possible* K-types of $\pi_t(\mathfrak{g}) V_\alpha$ are contained in the set $\{\alpha, \alpha \pm \gamma_j\} \cap \mathcal{S}$.

Suppose now that α and $\alpha + \gamma_i$ are both in $\mathcal{S}$. Since γ_i is an extreme weight, $\mathfrak{p}_{\gamma_i}$ is one dimensional and so $\mathfrak{p} \otimes V_\alpha$ contains a *unique* submodule isomorphic to $V_{\alpha+\gamma_i}$. Moreover, if we fix a nonzero element X_i in $\mathfrak{p}_{\gamma_i}$, then $X_i \otimes v_\alpha$ has a nonzero component in this submodule.

This means that if we choose highest weight vectors v_α and $v_{\alpha+\gamma_i}$ in V_α and $V_{\alpha+\gamma_i}$, then there is a scalar $c_i(\alpha, t)$ depending on i, α, and t, such that the K-orthogonal projection of $\pi_t(X_i)v_\alpha$ to $V_{\alpha+\gamma_i}$ is $c_i(\alpha, t)v_{\alpha+\gamma_i}$, and $V_{\alpha+\gamma_i}$ occurs in $\pi_t(\mathfrak{g}) V_\alpha$ if and only if $c_i(\alpha, t) \neq 0$.

Similarly, if $\bar{X}_i \in \mathfrak{p}_{-\gamma_i}$ is the complex conjugate of X_i, then there is a scalar $d_i(\alpha, t)$ such that the projection of $\pi_t(\bar{X}_i)v_{\alpha+\gamma_i}$ to V_α is $d_i(\alpha, t)v_\alpha$, and V_α occurs in $\pi_t(\mathfrak{g}) V_{\alpha+\gamma_i}$ if and only if $d_i(\alpha, t) \neq 0$.

(The operators X_i and $\bar{X}_i$ are the obvious generalizations of the "creation" and "annihilation" operators studied by Bargmann in [B].)

Since for $X \in \mathfrak{g}$, $\pi_t(X)$ depends affinely on t, it follows that $c_i(\alpha, t)$ and $d_i(\alpha, t)$ are affine functions of t, i.e. of the form $a + bt$. The main result in this section is the following explicit formula for these functions.

Theorem 1. *Suppose $\alpha, \alpha + \gamma_i$ are in $\mathscr{S}$, with $\alpha = \sum_j a_j \gamma_j$. Then $c_i(\alpha, t)$ and $d_i(\alpha, t)$ are nonzero multiples of $a_i + r_i + 1/2 - t/2$, and $a_i + r_i + 1/2 + t/2$ respectively.*

(As mentioned earlier, we restrict ourselves to cases C and D. See [S2] for case A.)

It is nontrivial to show that $c_i(\alpha, t)$ is not *identically* zero as a function of t! This we do through a sequence of four lemmas.

Lemma 1A. *Suppose $\alpha, \alpha + \gamma_i$ are in $\mathscr{S}$. If $i \leqq j$, then $c_i(\alpha + \mu_j, t + 2)$ is a nonzero multiple of $c_i(\alpha, t)$. In case D, if $i < n$ then $c_i(\alpha + \mu', t + 2)$ is a nonzero multiple of $c_i(\alpha, t)$.*

Proof. Since $j \geqq i$, $\mu_j + \gamma_i$ has i-th coefficient 2 and so by Lemma 0 it is not a weight in F_1. Consequently, $\pi_{2+r}(X_i) v_{\mu_j} = 0$, and specializing formula (0), we get

$$\pi_{t+2}(X_i)(v_\alpha v_{\mu_j}) = \big(\pi_t(X_i) v_\alpha\big) v_{\mu_j}.$$

We calculate the projection of both sides of this identity to $V_{\alpha + \mu_j + \gamma_i}$ using the fact that the product of two highest weight vectors in V is a highest weight vector.

On the left side we get a nonzero multiple of $c_i(\alpha + \mu_j, t + 2) v_{\alpha + \mu_j + \gamma_i}$.

The behavior on the right is somewhat more subtle. First of all, by weight considerations, $\pi_t(X_i) v_\alpha$ is contained in the sum of the K-types $V_{\alpha + \gamma_k}$ for $k \leqq i$. However, if $k \neq i$, then $\mu_j + \gamma_i - \gamma_k$ has its i-th coefficient equal to 2 and so, by Lemma 0, it is not a weight in V_{μ_j}. It follows that $V_{\alpha + \mu_j + \gamma_i}$ does not occur in $V_{\alpha + \gamma_k} \otimes V_{\mu_j}$.

Thus the projection of the right side to $V_{\alpha + \mu_j + \gamma_i}$, may be accomplished by first projecting $\pi_t(X_i) v_\alpha$ to $V_{\alpha + \gamma_i}$, and then multiplying by v_{μ_j}. This implies that the right side is a nonzero multiple of $c_i(\alpha, t) v_{\alpha + \mu_j + \gamma_i}$. Comparing the two sides, we get the result for μ_j.

The argument for μ' in case D is similar, the key point being that if $k \neq i$ *and* $i < n$, then $\mu' + \gamma_i - \gamma_k$ is not a weight in $V_{\mu'}$. □

We now observe that since γ_i and γ_j are strongly orthogonal roots, X_i and X_j commute. This leads to the following useful result.

Lemma 1B. *Suppose $i > j$ and $\alpha, \alpha + \gamma_i$ and $\alpha + \gamma_j$ are all in $\mathscr{S}$.*

If $c_j(\alpha, t)$, $c_j(\alpha + \gamma_i, t)$, and $c_i(\alpha, t)$ are nonzero, then $c_i(\alpha + \gamma_j, t)$ is nonzero.

Proof. The projection of $\pi_t(X_j) \pi_t(X_i) v_\alpha = \pi_t(X_i) \pi_t(X_j) v_\alpha$ to $V_{\alpha + \gamma_i + \gamma_j}$ is a scalar multiple of $v_{\alpha + \gamma_i + \gamma_j}$, and we calculate this scalar in two different ways.

On the left, $\pi_t(X_i)v_\alpha$ is contained in the sum of $V_{\alpha+\gamma_k}$ for $k \leqq i$. But from the weights of $\mathfrak{p}$, one sees that $\pi_t(\mathfrak{g})V_{\alpha+v_k}$ can contain $V_{\alpha+\gamma_i+\gamma_j}$ *only* if k equals i or j. Thus the projection of the left side can be accomplished by projecting $\pi_t(X_i)v_\alpha$ to $V_{\alpha+\gamma_i} \oplus V_{\alpha+\gamma_j}$, applying $\pi_t(X_j)$ and then projecting to $V_{\alpha+\gamma_i+\gamma_j}$. This shows that the scalar is a *nonzero* multiple of $c_i(\alpha, t)c_j(\alpha+\gamma_i, t)$ plus some multiple of $c_j(\alpha, t)c_i(\alpha+\gamma_j, t)$.

Now since $j < i$, the projection of $\pi_t(X_j)v_\alpha$ to $V_{\alpha+\gamma_i}$ is zero. Thus a similar argument on the right side of the identity shows that the scalar is a multiple of $c_j(\alpha, t)c_i(\alpha+\gamma_j, t)$.

Comparing the two sides we conclude that $c_j(\alpha+\gamma_i, t)c_i(\alpha, t)$ is a multiple of $c_i(\alpha+\gamma_j, t)c_j(\alpha, t)$, and the lemma follows. □

We now prove the nontriviality of $c_i(\alpha, t)$ for some special values of α.

Lemma 1 C. (a) *$c_i(\mu_{i-1}, t)$ is not identically zero; and if k is a positive integer then in case* D, (b) *$c_n(k\mu', t)$ is not identically zero.*

Proof. Statement (a) follows from the K-structure of F_1. For if $c_i(\mu_{i-1}, 2+r)$ were 0, then this would mean that V_{μ_i} could not occur in $\pi_{2+r}(\mathfrak{g})(V_{\mu_{i-1}})$. But then it would follow that the subspace $\sum\{V_{\mu_j} : 0 \leqq j \leqq i-1\}$ is $\mathfrak{g}$-stable, which would contradict the irreducibility of F_1.

Statement (b) follows similarly from the K-structure of F_k. Indeed the only possible K-types of $\pi_{2k+r}(\mathfrak{g})V_{k\mu'}$ are $V_{k\mu'}$ and $V_{k\mu'+\gamma_n}$. By the irreducibility of F_k, we get

$$c_n(k\mu', 2k+r) \neq 0 . \quad \square$$

Let us remark that since $c_i(\alpha, t)$ is of the form $a + bt$, if it is nonzero for some t, then it is nonzero for all but *possibly* one value of t.

Lemma 1 D. *If α and $\alpha+\gamma_i$ are in $\mathcal{S}$, then $c_i(\alpha, t)$ is not identically zero as a function of t.*

Proof. For $i = 1$, the result follows trivially from the formula for $c_1(\alpha, t)$ in Lemma 2.2 of [KS]. (Or directly from Lemma 3.4 of [V].) We now proceed by induction, assuming the result for all indices less than i.

Suppose that for some β in $\mathcal{S}$, $c_i(\beta, t)$ is not identically zero. Then, for *all* j, $c_i(\beta+\mu_j, t)$ is not identically zero. For $j \geqq i$ this follows from Lemma 1 A, and for $j < i$ it is a consequence of Lemma 1 B (together with the inductive hypothesis). Lemma 1 A also shows that in case D, if $i < n$ then $c_i(\beta+\mu', t)$ is not identically zero.

Now if i is less than n, then the dominance of α and $\alpha+\gamma_i$ means α is equal to μ_{i-1} plus a nonnegative integral combination of the μ_j (and possibly μ' in case D). Thus the present lemma follows by repeatedly applying the results of the previous paragraph to part (a) of Lemma 1 C.

If i is equal to n, then either α equals μ_{n-1} plus a nonnegative integral combination of the μ_j (but *not* μ'); or else we are in case D, and α is of the form $k\mu'$ plus a combination of μ_1 through μ_{n-1} (but *not* μ_n!). In the two cases, respectively using parts (a) and (b) of Lemma 1C, and arguing as before, we get the result. □

(We mention in passing that it is possible, and quite entertaining, to give a different proof of Lemma 1D using explicit Jordan-theoretic expressions for the highest weight vectors v_α.)

Proof of the theorem. We recall from [KS] that for each positive integer m there is a differential operator D_m, which intertwines $I(m)$ and $I(-m)$ in case D, and $I(2m)$ and $I(-2m)$ in case C.

The main theorem (Capelli identity) in [KS] shows that if $\alpha = \sum a_j \gamma_j$ then $D_1 v_\alpha$ is equal to $\prod_{j=1}^{n} (a_j + r_j)$ in case D, and to $\prod_{j=1}^{n} (a_j + r_j - 1/2)(a_j + r_j + 1/2)$ in case C.

For definiteness, we treat case D – case C is similar. To establish the formula, we may assume $a_j + r_j$ is strictly positive for all j. If it is not, we can consider α plus a suitable multiple of μ_n, and use Lemma 1A!

Consider the identity $\pi_{-1}(X_i) D_1 v_\alpha = D_1 \pi_1(X_i) v_\alpha$ and project both sides to $V_{\alpha+\gamma_i}$. After cancelling the (nonzero) factors $(a_j + r_j)$ for $j \neq i$, we get the equation

$$(a_i + r_i)\, c_i(\alpha, -1) = c_i(\alpha, 1)(a_i + r_i + 1)\,.$$

By Lemma 1D, $c_i(\alpha, 1)$ and $c_i(\alpha, -1)$ are not *both* zero, and, in view of the above equation, $c_i(\alpha, t)$ cannot be a *nonzero* constant! Thus we may write $c_i(\alpha, t)$ as a scalar times $a - t/2$, and then the previous equation becomes $(a + 1/2)(a_i + r_i) = (a - 1/2)(a_i + r_i + 1)$. This gives $a = a_i + r_i + 1/2$ proving the first part of the theorem.

Next if $\langle , \rangle$ denotes the $L^2(K)$-inner product on V, then the representations π_{-t} and π_t are Hermitian duals with respect to $\langle , \rangle$. This implies that

$$\langle \pi_{-t}(X_i) v_\alpha, v_{\alpha+\gamma_i} \rangle = \langle v_\alpha, \pi_t(\bar{X}_i) v_{\alpha+\gamma_i} \rangle\,.$$

From the first part of the theorem the left side is a multiple of $a_i + r_i + 1/2 + t/2$. Considering the right side, we get the formula for $d_i(\alpha, t)$. □

Remark. We can be a little more precise about the implied constants in the theorem. Let us choose v_α and $v_{\alpha+\gamma_i}$ so that $c_i(\alpha, t)$ *equals* $(a_i + r_i + 1/2 - t/2)$. The proof of the theorem then shows that $d_i(\alpha, t) = (a_i + r_i + 1/2 + t/2)\langle v_{\alpha+\gamma_i}, v_{\alpha+\gamma_i} \rangle / \langle v_\alpha, v_\alpha \rangle$.

2. The Hermitian form

The conjugacy of P and $\bar{P}$ implies that $I(t)$ has an invariant Hermitian form $\langle , \rangle_t$. Since the K-types have multiplicity 1, on each K-type $\langle , \rangle_t$ is a scalar multiple of $\langle , \rangle$. If α

and $\alpha + \gamma_i$ are in $\mathscr{S}$, let $q_i(\alpha, t)$ be the ratio of these scalars for the corresponding K-types. In other words, put

$$q_i(\alpha, t) = (\langle v_{\alpha+\gamma_i}, v_{\alpha+\gamma_i}\rangle_t / \langle v_{\alpha+\gamma_i}, v_{\alpha+\gamma_i}\rangle) / (\langle v_\alpha, v_\alpha\rangle_t / \langle v_\alpha, v_\alpha\rangle) .$$

Theorem 1 easily implies that $I(t)$ is irreducible for generic t. For such t, $\langle , \rangle_t$ is nondegenerate and unique up to a scalar multiple. Consequently $q_i(\alpha, t)$ is well-defined. A simple argument (see Theorem 4.11(c) of [V]) shows that $q_i(\alpha, t)$ is a *rational* function of t.

Theorem 2. *For* $\alpha = \sum_j \alpha_j \gamma_j$, $q_i(\alpha, t) = (a_i + r_i + 1/2 + t/2)/(a_i + r_i + 1/2 - t/2)$.

Proof. The invariant form satisfies $\langle \pi_t(X_i) v_\alpha, v_{\alpha+\gamma_i}\rangle_t = \langle v_\alpha, \pi_t(\bar{X}_i) v_{\alpha+\gamma_i}\rangle_t$. Choosing $v_\alpha, v_{\alpha+\gamma_i}$ as in the remark following Theorem 1 we get the result. □

Now proceeding *exactly* as in [S2], we describe the Jantzen filtration of $I(t)$.

Fix t and let $s \in \mathbb{R}$ be variable. Then multiplying by a suitable power of $(s - t)$, we may assume that $\langle , \rangle_s$ is nonzero at t, and nondegenerate *elsewhere* in a small interval

$$(t - \delta, t + \delta) .$$

Definition. Let V_k consist of those vectors v in V for which $\langle v, v\rangle_s$ vanishes to order at least k at $s = t$. Then $V = V_0 \supseteqq V_1 \supseteqq \cdots \supseteqq V_k \supseteqq \cdots$ is called the Jantzen filtration of $I(t)$.

The usual definition of the Jantzen filtration is slightly more involved, but Lemma 2 of [S2] shows that it is equivalent to the above for K-multiplicity 1. In particular the above filtration is $\pi_t(\mathfrak{g}, K)$-stable.

It can be easily seen that in the above setting, $\lim_{s \to t} \langle , \rangle_s / (s - t)^k$ gives an invariant form on V_k whose radical is precisely V_{k+1}. The k-th *Jantzen subquotient* of $I(t)$ is $Q_k \equiv V_k / V_{k+1}$. It consists of the K-types for which $\langle , \rangle_t$ vanishes to order *exactly* k, and the above formula gives a nondegenerate, invariant Hermitian form on Q_k.

From Theorem 2 it is rather easy to calculate the Jantzen filtration of $I(t)$. In fact we can get a sharper result.

Corollary 2. *The K-types $\alpha = \sum_j a_j \gamma_j$ and $\alpha + \gamma_i$ belong to the same irreducible constituent of $I(t)$ if and only if t is not equal to $|2a_i + 2r_i + 1|$.*

Proof. Consider the two numbers $a_i + r_i + 1/2 \pm t/2$. If *both* numbers are nonzero then by Theorem 1, $V_{\alpha+\gamma_i} \subseteqq \pi_t(\mathfrak{g}) V_\alpha$ and $V_\alpha \subseteqq \pi_t(\mathfrak{g}) V_{\alpha+\gamma_i}$.

If *exactly one* of the numbers is zero, then $q_i(\alpha, s)$ has either a zero or a pole at $s = t$, which means that α and $\alpha + \gamma_i$ lie in different Jantzen subquotients.

Finally, suppose *both* numbers are zero. This forces $a_i + r_i + 1/2 = 0$ and $t = 0$. By the theorem, the first condition means that $q_i(\alpha, t) \equiv -1$ as a function of t. However at $t = 0$, the representation $I(t)$ is unitary. The negativity of $q_i(\alpha, t)$ means that α and $\alpha + \gamma_i$ must belong to different constituents. □

3. The irreducible constituents

We now determine the irreducible subquotients of $I(t)$. Since $I(t)$ and $I(-t)$ have the same constituents, in the rest of the paper we may, and do, assume that t is *nonnegative*.

Definition. Suppose α and $\alpha + \gamma_i$ are in $\mathcal{S}$ with $\alpha = \sum_j a_j \gamma_j$. The corresponding K-types will be called linked in $I(t)$ if $t \neq |2a_i + 2r_i + 1|$.

Lemma 3. *Two K-types belong to the same irreducible constituent of $I(t)$ if and only if they can be connected by a sequence of K-types in which successive pairs are linked.*

Proof. The "if" part is clear by Corollary 2. Conversely, if two K-types belong to the same component, then successive applications of $\pi_t(\mathfrak{p})$ to the first K-type will eventually yield the second K-type. Thus the two can be connected by a sequence in which each K-type is contained in $\pi_t(\mathfrak{g})$ applied to the previous K-type.

In particular, successive pairs in this sequence are of the form α, $\alpha + \gamma_i$ or $\alpha + \gamma_i$, α, and *one* of the two numbers $a_i' + r_i + 1/2 \pm t/2$ is nonzero. Now if the other number were zero, then by Theorem 2 these two K-types would belong to different levels in the Jantzen filtration. However since the filtration is $\mathfrak{g}$-stable the level can only decrease along the sequence, and since the first and last term belong to the same constituent, the level must actually be constant along the sequence. This shows that both numbers must be nonzero and so the successive pairs must be linked. □

The lemma has several important consequences.

Theorem 3 A. *The representation $I(t)$ is reducible if and only if one of the following two conditions holds*:

(a) *$r_i + 1/2 - t/2$ is a nonpositive integer for some i.*

(b) *We are in case* D, *and one of $r_n + 1/2 + t/2$, $r_n + 1/2 - t/2$ is an integer.*

Proof. If neither condition holds, then $a_i + r_i + 1/2 \pm t/2$ can never be zero for any $\alpha = \sum_j a_j \gamma_j$ in $\mathcal{S}$. The irreducibility of $I(t)$ is now immediate from the lemma.

Conversely, suppose (a) holds and that $r_i + 1/2 - t/2$ is a nonpositive integer $-a_i$, say. Now if α is any K-type with i-th coefficient a_i, then α and $\alpha + \gamma_i$ belong to different constituents of $I(t)$. The argument for (b) is similar. □

Recalling the formulas for r_i in terms of the root multiplicities, we observe two things. First, since $r_i - r_{i+2} = d$ is a positive integer, it is enough to check condition (a) for $i = n - 1$

and $i = n$. Secondly, if $\Sigma(\mathfrak{t}, \mathfrak{k}) \neq D_2$ then $r_n = 0$ from which it follows rather easily that condition (b) is a special case of (a) and may be dropped altogether!

Indeed we may restate the theorem as follows:

Theorem 3 A bis. (a) *Suppose $\Sigma(\mathfrak{t}, \mathfrak{k}) \neq D_2$. Then $I(t)$ is reducible if and only if t lies in $\{e+1, e+3, e+5, \ldots\} \cup \{e+d+1, e+d+3, e+d+5, \ldots\}$. (Of course if d is even, the second set is contained in the first.)*

(b) *Suppose $\Sigma(\mathfrak{t}, \mathfrak{k}) = D_2$ and $\mathfrak{g} = \mathfrak{o}(p, q)$. Then $I(t)$ is reducible if and only if one of the following three holds:*

(i) *p and q have different parities and $t - 1/2$ is an integer.*

(ii) *$(p-q)/2$ is an integer and t is an integer with the opposite parity.*

(iii) *p and q are both odd and $t - (p+q-4)/2$ is an odd positive integer.*

Proof. Part (a) follows from the remarks above and the formulas $r_n = e/2$ and $r_{n-1} = e/2 + d/2$.

For part (b), Theorem 3 A shows that we have reducibility if (1)

$$(p+q-4)/4 + 1/2 - t/2$$

is a nonpositive integer, or if (2) one of $(p-q)/4 + 1/2 \pm t/2$ is an integer. Condition (2) leads to (i) and (ii), while (1) is a special case of (2) *unless* p and q are both odd, in which case we get (iii). □

We now describe the K-types of the irreducible constituents in terms of a notion of "rank". (The analog for case A is Definition 3 of [S 2].)

Definition. If $\alpha = \sum_j a_j \gamma_j$ is a K-type of $I(t)$, let l be the smallest index, if any, for which $a_l + r_l + 1/2 - t/2$ is a nonpositive integer (i.e. less than or equal to zero).

In case C, if there exists such an index l, we define the rank of α to be $l - 1$, otherwise we say that α has rank n.

In case D, if l is strictly less than n we define the rank to be $l - 1$. Otherwise, consider the two numbers $a_n + r_n + 1/2 - t/2$ and $a_n + r_n + 1/2 + t/2$. If neither is an integer we define the rank to be n. If both are integers, then the rank is n^-, $n-1$, n^+ accordingly as both, one, or neither, of these is nonpositive. If only one is an integer we define the rank of α to be n^- if this integer is nonpositive, and to be n^+ if it is positive.

(The last sentence of the definition is applicable only to $O(p, q)$ when p and q have *different* parities.)

Here is the *raison d'être* of this complicated definition.

Theorem 3 B. *Two K-types belong to the same irreducible subquotient of $I(t)$ if and only if they have the same rank.*

Proof. By Lemma 3, it suffices to show that $\alpha = \sum a_j \gamma_j$ and $\alpha + \gamma_i$ are linked if and only if they have the same rank.

First consider case C. Since a_i, r_i and t are all nonnegative it follows that

$$(a_i + r_i + 1/2 + t/2)$$

is positive for all i. Thus α and $\alpha + \gamma_i$ are linked unless $a_i + r_i + 1/2 - t/2 = 0$.

By the definition of rank, it is easy to see that α and $\alpha + \gamma_i$ have the same rank, unless the rank of α is $i - 1$ and $a_i + r_i + 1/2 - t/2 = 0$.

Finally note that if α has rank $l - 1$ (with $l \leqq n + 1$), then $(a_i + r_i + 1/2 - t/2)$ cannot be zero unless $i = l$. For if $i < l$ then $(a_i + r_i + 1/2 - t/2)$ is not a nonpositive integer, while if $i > l$ then $(a_i + r_i + 1/2 - t/2)$ is strictly smaller than the nonpositive integer

$$(a_l + r_l + 1/2 - t/2).$$

This completes the proof for case C.

Case D is a bit more involved – chiefly because $a_n + r_n + 1/2 + t/2$ can be negative in this case.

First of all, suppose α has rank $l - 1$ with l strictly less than n. If $i \leqq n - 1$, then the above arguments work. For $i = n$, α and $\alpha + \gamma_n$ have the same rank, and as before $a_n + r_n + 1 - t/2 \neq 0$, but we also need to check that $a_n + r_n + 1/2 + t/2$ is nonzero. But if it were, then combining this with $a_l + r_l + 1/2 - t/2 \leqq 0$, we would get

$$(a_l + a_n) + (r_l + r_n) + 1 \leqq 0.$$

However by the dominance of α and ϱ, $a_l + a_n$ and $r_l + r_n$ are both nonnegative, which gives a contradiction.

Now suppose the rank of α is $n - 1$, n^+, n^- or n. Once again if $i \leqq n - 1$ then arguing as in case C we see that α, $\alpha + \gamma_i$ are linked and have the same rank. If $i = n$, then $\alpha + \gamma_n$ is linked to α unless one of $a_n + r_n + 1/2 - t/2$ and $a_n + r_n + 1/2 + t/2$ is zero, and in this case it is easy to see that the two K-types will have different ranks. The converse is equally straightforward, and this completes the proof of the theorem. □

In view of the above theorem, we define the *rank* of an irreducible constituent of $I(t)$ to be the rank of any one of its K-types. (In the sequel we will relate this definition to that of [Ho].)

Our description of the irreducible constituents of $I(t)$ is completed by the following theorem.

Theorem 3C. *In case C, $I(t)$ always has a constituent of rank n. It has a constituent of rank $l \leqq n-1$ if and only if $r_{l+1} + 1/2 - t/2$ is a nonpositive integer.*

In case D, $I(t)$ has a constituent of rank $l \leqq n-2$ if and only if $r_{l+1} + 1/2 - t/2$ is a nonpositive integer. There is a constituent of rank $n-1$ if and only if $r_n + 1/2 + t/2$ and $r_n + 1/2 - t/2$ are both integers and $t \neq 0$; there is a rank n constituent if and only if neither is an integer; and there are constituents of rank $n^{\pm}$ if and only if at least one of these two numbers is an integer.

Proof. This follows immediately from the definition of rank, and the fact that, except for a_n in case D, all the a_i's are positive integers. □

4. The unitary subquotients

As before, we assume that t is nonnegative.

Lemma 4. *An irreducible constituent U of $I(t)$ is unitarizable if and only if for each K-type $\alpha = \sum a_j \gamma_j$ of U, and for each i such that $\alpha + \gamma_i$ is in $\mathcal{S}$, we have the inequality $t \leqq |2a_i + 2r_i + 1|$.*

Proof. On each K-type of the constituent, the invariant form is a multiple of the standard $(L^2(K))$ form. Evidently the representation is unitarizable if and only if all of these multiples have the same sign. By Lemma 3 it suffices to check this for pairs of linked K-types. Then Theorem 2 implies that a constituent is *nonunitary* if and only if we can find a K-type α and an index i such that $\alpha + \gamma_i$ is in $\mathcal{S}$, and

$$a_i + r_i + 1/2 - t/2 \quad \text{and} \quad a_i + r_i + 1/2 + t/2$$

are both nonzero and have different signs. This clearly implies the present lemma. □

The lemma has a "picturesque" reformulation which we state as a corollary.

Corollary 4. *An irreducible constituent of $I(t)$ is nonunitary if and only if it has a pair of K-types α and $\alpha + \gamma_i$ which belong to different constituents in some $I(s)$ for $0 \leqq s < t$.*

Proof. As observed in the proof of the lemma above, a constituent is nonunitary if and only if it has a pair of K-types α and $\alpha + \gamma_i$ such that the functions $(a_i + r_i + 1/2 + s/2)$ and $(a_i + r_i + 1/2 - s/2)$ have opposite signs at $s = t$. But this can happen if and only if one of the functions vanishes for some $0 \leqq s < t$, which implies the result. □

The corollary immediately implies that there is no complementary series beyond the first reducibility point. In other words, if t_0 is the smallest value of t for which the representation $I(t)$ is reducible, then

Theorem 4A. *$I(t)$ is both irreducible and unitary if and only if $t < t_0$.* □

(Theorem 3 A bis shows that if $\Sigma(\mathfrak{t}, \mathfrak{k}) \neq D_2$, then $t_0 = e + 1$ while if $\Sigma(\mathfrak{t}, \mathfrak{k}) = D_2$ and $\mathfrak{g} = \mathfrak{o}(p, q)$ then t_0 equals (a) 0 if $(p - q)/2$ is an odd integer, (b) 1 if $(p - q)/2$ is an even integer, and (c) $1/2$ if $(p - q)/2$ is not an integer.)

The remaining results depend on the nature of the numbers r_i, so some case-by-case discussion is inevitable. For the rest of this section we specifically exclude the case D_2.

Next we consider submodules of "small" rank.

Theorem 4 B. *Assume $\Sigma(\mathfrak{t}, \mathfrak{k})$ is not of type D_2. Then $I(t)$ has a unitarizable constituent of rank $i - 1 < n$, if and only if $t = 2r_i + 1$.*

Moreover, for $t = 2r_i + 1$, this constituent is actually a submodule and its K-types are $\{\alpha = \sum_j a_j \gamma_j \mid a_i = a_{i+1} = \cdots = a_n = 0\}$.

Proof. By Theorem 3 C, the rank $i - 1$ constituents occur for $t = 2r_i + 1 + 2m$ where m is a nonnegative integer. (In case D, for $i = n$, we need to use the fact that $r_n = 0$.)

First suppose that $m = 0$, so that $t = 2r_i + 1$, and let U_{i-1} be the subspace of V spanned by the given K-types. If α is in U_{i-1}, all the K-types of $\pi_t(\mathfrak{g})(V_\alpha)$ are also in U_{i-1}, with the possible exceptions of $\alpha + \gamma_i$, and of $\alpha - \gamma_n$ in case D if $i = n$.

However, $c_i(\alpha, 2r_i + 1) = a_i + r_i + 1/2 - t/2 = 0 + r_i + 1/2 - (2r_i + 1)/2 = 0$. So by Theorem 1, $\alpha + \gamma_i$ does not occur in $\pi_{2r_i+1}(\mathfrak{g}) V_\alpha$. Also in case D, a similar calculation using Theorem 1 and the fact that $r_n = 0$ shows that $\alpha - \gamma_n$ does not occur in $\pi_1(\mathfrak{g}) V_\alpha$.

This shows that in all cases, U_{i-1} is a submodule of $I(2r_i + 1)$. It is easy to check that all its K-types have rank $i - 1$, thus by Theorem 3 B, U_{i-1} is irreducible, and is precisely the rank $i - 1$ constituent. By the lemma above, the unitarity follows from the trivial inequality $2r_i + 1 \leqq |2a_j + 2r_j + 1|$ for $j \leqq i$.

Finally, if m is positive, let $\alpha = m(\gamma_1 + \cdots + \gamma_{i-1})$. Then it is easy to check that α and $\alpha + \gamma_i$ both have rank $i - 1$ in $I(2r_i + 1 + 2m)$. But $2r_i + 1 + 2m > 2r_i + 1 = |2a_i + 2r_i + 1|$. So by the lemma, the constituent is nonunitary. □

The unitary constituents in the above theorem are the most interesting constituents of $I(t)$. They are all unipotent in the sense of [V1], and for rank 1, we get the "minimal" representations considered in [V], for example.

We now consider the large constituents in case C.

Theorem 4 C. *Assume $\Sigma(\mathfrak{t}, \mathfrak{k})$ is of type C_n. If $I(t)$ is reducible, then it has a unitarizable constituent of rank n if and only if $t - 2r_n - 1$ is a nonnegative even integer.*

Proof. First of all if $t - 2r_n - 1$ is a nonnegative even integer, then $I(t)$ is reducible. Moreover, if $\alpha = \sum_j a_j \gamma_j$ is a K-type then $a_n + r_n + 1/2 - t/2$ is an integer, which must be positive if α has rank n. Thus for any i, $a_i + r_i + 1/2 - t/2 \geqq a_n + r_n + 1/ - t/2 > 0$, which implies the unitarity of the rank n constituent.

Now if d is even, $I(t)$ is reducible for precisely the above set of values of t and so there is nothing more to prove. If d is odd, $I(t)$ is also reducible if $t - 2r_{n-1} - 1$ is a nonnegative even integer, say $2m$. However if we put $\alpha = m(\gamma_1 + \cdots + \gamma_{n-1})$ then α has rank n, but $t > 2r_n + 1 = 2a_n + 2r_n + 1$, so by the lemma, the constituent is nonunitary. □

Next we deal with constituents of rank n^+ and n^- in case D.

Theorem 4 D. *Assume $\Sigma(\mathfrak{t}, \mathfrak{k})$ is of type D_n, with $n \neq 2$. Then $I(t)$ has irreducible constituents of type n^+ and n^- if and only if t is an odd integer. These constituents are unitarizable, and are quotients of $I(t)$.*

Proof. The first assertion follows from Theorem 3C together with the fact that $r_n = 0$. So let us suppose t is an odd integer.

Now if $\alpha = \sum_j a_j \gamma_j$ is a K-type of rank n^+, then $a_n + r_n + 1/2 - t/2$ is an integer, which must be positive. It follows that $a_i + r_i + 1/2 \pm t/2$ is positive for all i, which in turn implies the unitarity of the constituent.

Similarly, if α has rank n^-, then $a_n + r_n + 1/2 \pm t/2$ are integers, which must be nonpositive. For i less than n, $a_i \geqq |a_n|$ and so $a_i + r_i + 1/2 - t/2 > -(a_n + r_n + 1/2 + t/2) \geqq 0$ and once again the unitarity follows by the lemma.

Finally from Theorem 1 it is easy to see that the set of K-types for which

$$a_n + r_n + 1/2 - t/2$$

is a nonpositive integer is $\mathfrak{g}$-stable, and the corresponding quotient is precisely the rank n^+ constituent. The argument for the rank n^- constituent is similar. □

Finally we deal with the rank n constituents in case D.

Theorem 4 E. *Assume $\Sigma(\mathfrak{t}, \mathfrak{k})$ is of type D_n, with $n \neq 2$. Then if $I(t)$ is reducible, it has a constituent of type n if and only if d is odd and $t - 2r_{n-1} - 1$ is a nonnegative even integer. This constituent is a nonunitary quotient.*

Proof. $I(t)$ has a constituent of rank n if $1/2 + t/2$ and $1/2 - t/2$ are not integers. If $I(t)$ is also reducible, then *some* $r_i + 1/2 - t/2$ must be a negative integer. Since $r_i - r_{i-1} = d/2$, the first statement follows easily.

Suppose now that $t = 2m + 2r_{n-1} + 1$, then the nonunitarity follows by calculating $q_n(\alpha, t)$ for $\alpha = (m+1)(\gamma_1 + \cdots + \gamma_{n-1})$.

Finally, it is easy to see that the set of K-types of rank strictly less than n form a submodule, whose quotient is the rank n constituent. □

5. The D_2 case

In this section we deal with case where $\Sigma(\mathfrak{t},\mathfrak{k}) = D_2$. This root system arises for the group $G = O(p,q)$ with $p \geqq q \geqq 3$ and the parabolic $P = LN$, where

$$L = O(p-1, q-1) \times GL_1(\mathbb{R}) \quad \text{and} \quad N = \mathbb{R}^{p-1,q-1}.$$

The principal series attached to this parabolic has been studied in [HT] and in the references mentioned therein, so the results of this section are certainly not new. We include them merely for the sake of completeness.

As before we assume that t is nonnegative. Theorem 3C describes the constituents of $I(t)$ and we now determine which of these are unitary. First of all, the rank 0 constituents are finite dimensional, and among these only the trivial representation, which occurs for $t = 2r_1 + 1 = (p+q-2)/2$, is unitary.

For rank 1 constituents we have the following anomalous situation.

Theorem 5A. *$I(t)$ has a rank 1 unitary representation if and only if $p \equiv q \bmod (4)$, and $t = 1$. In this case its K-types are* $\left\{a_1\gamma_1 - \dfrac{(p-q)}{4}\gamma_2 \,|\, a_1 \geqq \dfrac{(p-q)}{4}\right\}$.

Proof. By Theorem 3C, $I(t)$ has a rank 1 constituent if and only if $r_2 + 1/2 \pm t/2$ are both integers and $t \neq 0$. Since $r_2 = (p-q)/4$, this means either (a) $(p-q)/2$ is an even integer and t is an odd integer, or (b) $(p-q)/2$ is an odd integer and t is a positive even integer.

Arguing as in the proof of Theorem 4B, using Corollary 4 one sees that all of these constituents are nonunitary except perhaps for $t = 1$ in case (a), and $t = 2$ in case (b). The first is easily checked to be unitary and has the K-types described above. To see that the second is nonunitary, put $a = (p-q)/4 - 1/2$, then the K-type $a\gamma_1 - a\gamma_2$ has rank 1 in $I(2)$, but the inequality of Lemma 4 fails for $i = 2$. □

(One may similarly study the "nonspherical" principal series whose K-types have highest weights $a_1\gamma_1 + a_2\gamma_2$ where a_1 and a_2 are half integers with $a_1 \geqq |a_2|$. If $p - q \equiv 2 \bmod (4)$, then for $t = 1$ there is a unitary constituent of rank 1 in the nonspherical principal series, whose K-types are those with $a_2 = (p-q)/4$. However if p and q have different parities then there are *no* rank 1 constituents in either series. (See Theorem 2.13 in [V].))

We consider next the unitarizable constituents of rank 2^+ and 2^-, which occur when one of the numbers $(p-q)/4 + 1/2 \pm t/2$ is an integer. (See Theorem 3C.)

Theorem 5B. *If $(p-q)/2$ is an integer then $I(t)$ has constituents of rank 2^+ and 2^- when t is an integer with the same parity as $1 + (p-q)/2$. These are always unitarizable and are quotients of $I(t)$.*

If $(p-q)/2$ is not an integer, then there are constituents of rank 2^+ and 2^- when $t-1/2$ is an integer. For $t=1/2$ both constituents are unitarizable, while for $t>1/2$ the rank 2^+ (resp. 2^-) constituent is unitarizable if and only if $(p-q)/4+1/2-t/2$ (resp. $(p-q)/4+1/2+t/2$) is an integer.

Proof. If $(p-q)/2$ is an integer, the proof proceeds exactly as in Theorem 4D.

If $(p-q)/2$ is not an integer, then from part (b) of Theorem 3A we see that $t_0 = 1/2$ is the first point of reducibility. $I(1/2)$ has two constituents of rank 2^+ and 2^-, which are unitarizable by Corollary 4.

Now constituents of rank $2^{\pm}$ occur if and only if one of the two numbers

$$(p-q)/4+1/2-t/2 \quad \text{and} \quad (p-q)/4+1/2+t/2$$

is an integer. Since their sum is $1+(p-q)/2$, they cannot both be integers. We now consider the case where $(p-q)/4+1/2-t/2$ is an integer.

If $a_1\gamma_1 + a_2\gamma_2$ is a K-type of rank 2^+, then $a_2+(p-q)/4+1/2-t/2$ is a positive integer. This implies that all four of the numbers $a_i+r_i+1/2 \pm t/2$ are positive, and by Lemma 4 the rank 2^+ constituent is unitary.

We now show that for $t \neq 1/2$, the constituent of rank 2^- is nonunitary. Put $-a_2 = 1+[(p-q)/4+1/2-t/2]$, and put $a_1 = |a_2|+1$. Then the K-type $a_1\gamma_1+a_2\gamma_2$ has rank 2^-, and $a_2+(p-q)/4+1/2-t/2$ equals -1, which is negative. On the other hand, by the third paragraph of the proof, $a_2+(p-q)/4+1/2+t/2$ is a half-integer. Being larger than -1, it is either $-1/2$, which corresponds to $t=1/2$, or else it is positive, in which case the nonunitarity follows from Lemma 4.

The proof is similar when $(p-q)/4+1/2+t/2$ is an integer. □

Finally we consider the rank 2 constituents. In view of Theorem 4A, we may assume that $I(t)$ is reducible.

Theorem 5C. *If $I(t)$ is reducible then it has a rank 2 constituent if and only if p and q are both odd and $t-(p+q)/4$ is a nonnegative even integer. This representation is a nonunitary quotient.*

Proof. If $I(t)$ is reducible with a rank 2 constituent, then by Theorem 3C,

$$r_2 + 1/2 \pm t/2$$

must both be nonintegers, while $r_1+1/2-t/2$ must be a nonpositive integer. The first statement follows easily from this.

The second statement is proved exactly as in Theorem 4E. □

Appendix

The parabolic subgroups of simple real groups which satisfy conditions (i) and (ii) of the introduction are given in Table 1 in [KS], together with the symmetric space $K/(L \cap K)$, and its root system $\Sigma = \Sigma(\mathfrak{t}, \mathfrak{k})$. The table also lists the numbers r_i but without explaining how they were calculated.

If $K'/(L \cap K)$ is the noncompact dual of $K/(L \cap K)$, then Σ is precisely the restricted root system of K', and the root multiplicities e and d may be read off from standard tables such as those in [He1].

Except for the D_2 case discussed earlier, the numbers r_i can be determined by the formula $r_i = d(n-i)/2 + e/2$ in cases C and D, while in case A one has the analogous formula $r_i = d(n-2i+1)/4$. (Here $n = \dim(\mathfrak{t})$ is the *rank* of N as a Jordan algebra.)

For the convenience of the reader, we paraphrase here the salient features of the table for cases C and D, describing the group G the parabolic $P = LN$ and the Jordan algebra structure on N. (See [S2] for case A.)

Case D: $(e = 0)$

(a) $G = SL(2n, \mathbb{R})$, $L = S(GL(n, \mathbb{R}) \times GL(n, \mathbb{R}))$ and N is the algebra of $n \times n$ real matrices. In this case the rank is n and $d = 1$, so that $r_i = (n-i)/2$.

(b) $G = O(p, q)$, with $p \geqq q \geqq 3$. This is the D_2 case which was discussed in the previous section.

(c) $G = O(2n, 2n)$, $L = GL(2n, \mathbb{R})$ and N is the algebra of $2n \times 2n$ real skew symmetric matrices. The rank is n and $d = 2$, so that $r_i = (n-i)$.

(d) $G = E_7(7)$, $L = E_6(6) \times \mathbb{R}^\times$ and N is the split 27-dimensional exceptional Jordan algebra. The rank is 3 and $d = 4$, so that $r_i = 2(3-i)$.

Case C (complex): $(e = 1)$

(a) $G = SL(2n, \mathbb{C})$, $L = S(GL(n, \mathbb{C}) \times GL(n, \mathbb{C}))$ and N is the algebra of $n \times n$ complex matrices. The rank is n and $d = 2$, so that $r_i = (n-i) + 1/2$.

(b) $G = O(p, \mathbb{C})$, $p \geqq 5$, $L = O(p-2, \mathbb{C}) \times \mathbb{C}^\times$ and N is $\mathbb{C}^{p-2}$. The rank is 2 and $d = p-4$, so that $r_1 = (p-3)/2$ while $r_2 = 1/2$.

(c) $G = Sp(n, \mathbb{C})$, $L = GL(n, \mathbb{C})$ and N is the algebra of $n \times n$ complex symmetric matrices. The rank is n and $d = 1$, so that $r_i = (n-i)/2 + 1/2$.

(d) $G = O(4n, \mathbb{C})$, $L = GL(2n, \mathbb{C})$ and N is the algebra of $2n \times 2n$ complex skew symmetric matrices. The rank is n and $d = 4$, so that $r_i = 2(n-i) + 1/2$.

(e) $G = E_7(\mathbb{C})$, $L = E_6(\mathbb{C}) \times \mathbb{C}^\times$ and N is the complex exceptional Jordan algebra. The rank is 3 and $d = 8$, so that $r_i = 4(3-i) + 1/2$.

Case C (non complex): $(e > 1)$

(a) $G = SL(2n, \mathbb{H})$, $L = (GL(n, \mathbb{H}) \times GL(n, \mathbb{H}))$ and N is the algebra of $n \times n$ quaternionic matrices. The rank is n, $d = 4$, and $e = 3$, so that $r_i = 2(n-i) + 3/2$.

(b) $G = O(p, 1)$ with $p \geqq 4$, $L = O(p-1) \times \mathbb{R}^\times$, and N is the Jordan division algebra $\mathbb{R}^{p-1}$. In this case the rank is 1, $e = p - 2$ and $d = 0$, so that $r_1 = (p-2)/2$.

(c) $G = Sp(n, n)$, $L = GL(n, \mathbb{H})$ and N is the algebra of those $n \times n$ quaternionic matrices which are symmetric when regarded as complex $2n \times 2n$ matrices. The rank is $n, d = 2$, and $e = 2$, so that $r_i = n - i + 1$.

References

[BSS] *D. Barbasch, S. Sahi, B. Speh*, Degenerate series representations for $GL(2n, \mathbb{R})$ and Fourier analysis, In: Cantoni, V. (ed.), Indecomposable representations of Lie groups and their physical applications, Symp. Math. **XXXI** (1989), 45–69, Academic Press, London–New York.

[B] *V. Bargmann*, Irreducible unitary representations of the Lorentz group, Ann. Math. **48** (1947), 568–640.

[G] *A. Guillemonat*, On some semi-spherical representations of a Hermitian symmetric pair of tubular type, Math. Ann. **246** (1980), 93–116.

[He1] *S. Helgason*, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York 1978.

[He2] *S. Helgason*, Groups and Geometric Analysis, Academic Press, New York 1984.

[Ho] *R. Howe*, On a notion of rank for unitary representations of the classical groups, In: Harmonic Analysis and Group Representations, CIME II (1980), 223–332, Liguori editore, Naples 1982.

[HT] *R. Howe, E.-C. Tan*, Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations, Bull. Am. Math. Soc. **28** (1) (1993), 1–74.

[J1] *K. Johnson*, Degenerate principal series and compact groups, Math. Ann. **287** (1990), 703–718.

[J2] *K. Johnson*, Degenerate principal series on tube type domains, In: Richards, D. (ed.), Hypergeometric functions, domains of positivity, Jack polynomials, and applications, Contemp. Math. **138** (1992), Am. Math. Soc., Providence.

[Ka] *I. L. Kantor*, Nonlinear transformation groups defined by general norms of Jordan algebras, Dokl. Acad. Nauk SSSR **172** (1967), 779–782 [= Sov. Math. Dokl. **8** (1967), 176–180].

[KN] *S. Kobayashi, T. Nagano*, On filtered Lie algebras and geometric structures, J. Math. Mech. **13** (1964), 875–907.

[Ko] *M. Koecher*, Über eine Gruppe von rationalen Abbildungen, Invent. Math **3** (1967), 136–171.

[KS] *B. Kostant, S. Sahi*, Jordan algebras and Capelli identities, Invent. Math. **112** (1993), 657–664.

[L] *O. Loos*, Bounded symmetric domains and Jordan pairs, Mathematical Lectures, University of California, Irvine 1977.

[S1] *S. Sahi*, The Capelli identity and unitary representations, Compos. Math. **81** (1992), 247–260.

[S2] *S. Sahi*, Unitary representations on the Shilov boundary of a symmetric tube domain, In: Representations of Groups and Algebras, Contemp. Math. **145** (1993), 275–286, Am. Math. Soc., Providence.

[S3] *S. Sahi*, Explicit Hilbert spaces for certain unipotent representations, Invent. Math. **110** (1992), 409–418.

[SS] *S. Sahi, E. M. Stein*, Analysis in matrix space and Speh's representations, Invent. Math. **101** (1990), 379–393.

[T] *J. Tits*, Une classe d'algèbres de Lie en relation avec les algèbres de Jordan, Indag. Math. **24** [= Proc. Ned. Akad. Wet. (A) **65** (1962), 530–535.

[V] *D. Vogan*, Singular unitary representations, In: Non-Commutative Harmonic Analysis and Lie groups, Lect. Notes Math. **880** (1980), 506–535, Springer, Berlin–Heidelberg–New York.

[V1] *D. Vogan*, Unitary representations of reductive groups, Ann. Math. Stud. **118**, Princeton University Press, Princeton 1987.

[W] *N. Wallach*, The analytic continuation of the discrete series II, Trans. Am. Math. Soc. **251** (1979), 19–37.

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Eingegangen 27. September 1993